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Projective Γ-sets

By Tomoyuki YOSHIDA

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The purpose of this paper is to study projective functors from a small category to the category of sets. Our results are generalizations of semigroup cases.

1. Some basic definitions and properties

Let $\mathscr{S}$ be the category of sets and Γ a small category. We denote by $\mathscr{S}^{\Gamma}$ the functor category from Γ to $\mathscr{S}$. Then an object in $\mathscr{S}^{\Gamma}$ is called a (*right*) Γ*-set* and a morphism in $\mathscr{S}^{\Gamma}$ is called a Γ*-map*. We denote the hom-set from i to j in Γ by $\Gamma(i, j)$. Furthermore *throughout the paper*, we denote the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ by $fg: X \to Z$. So a (right) Γ-set X consists of sets X_i, $i \in \Gamma$, which are called *stalks* at i, together with maps

$$X_i \times \Gamma(i, j) \longrightarrow X_j : (x_i, \alpha) \longmapsto x_i \cdot \alpha$$

for $i, j \in \Gamma$ which satisfy the conditions:

(a) $x_i \cdot 1_i = x_i$ for $i \in \Gamma$, $x_i \in X_i$ and the identity 1_i;

(b) $(x_i \cdot \alpha) \cdot \beta = x_i \cdot (\alpha\beta)$ for $x_i \in X_i$, $\alpha \in \Gamma(i, j)$, $\beta \in \Gamma(j, k)$.

Furthermore a Γ-map $f: X \to Y$ between Γ-sets is a family of maps f_i: $X_i \to Y_i$, $i \in \Gamma$, satisfying the condition $(x_i \cdot \alpha) f_j = (x_i f_i) \cdot \alpha$ for $i, j \in \Gamma$, $x_i \in X_i$, $\alpha \in \Gamma(i, j)$. The set of all Γ-maps of X to Y is denoted by $\Gamma(X, Y)$. We define analogously *left* Γ-sets which can be regarded as contravariant functor from Γ to $\mathscr{S}$. The category of Γ-sets, $\mathscr{S}^{\Gamma}$, is complete and cocomplete. In fact, limits and colimits of Γ-sets are constructed pointwise. A Γ-set X is called a *finite* Γ-set provided each stalk X_i is a finite set. The full subcategory of finite Γ-sets in $\mathscr{S}^{\Gamma}$ is denoted by $\mathscr{S}^{\Gamma}_f$.

Any set A is regarded as a *constant* Γ-set defined by $A_i = A$ and $a \cdot \alpha = a$ for all $a \in A_i$, $\alpha \in \Gamma(i, j)$. For each $k \in \Gamma$, the hom-functor $H^k : i \mapsto \Gamma(k, i)$ is a Γ-set, which is called a *representable* Γ-set. Of course the map $H^k{}_i \times \Gamma(i, j) \to H^k{}_j : (\gamma, \alpha) \mapsto \gamma\alpha$ is defined by the compositions. If Γ is a finite category, that is, all morphisms in Γ makes a finite set, then H^k is a finite Γ-set. It is well-known as the Yoneda Lemma that Γ-maps of H^k to X are bijectively corresponding with elements of X_k. The Yoneda embedding Y: $\Gamma^{op} \to \mathscr{S}^{\Gamma} : k \mapsto H^k$ is fully-faithful, and furthermore Y preserves and reflects limits, monomorphisms and isomorphisms.

Some elementary concepts about sets are generalized to ones about Γ-sets. See Chapter 14 of [3]. For example, a *Γ-subset* A of a Γ-set X consists of subsets A_i of X_i, $i\in\Gamma$, such that $A_i\cdot\alpha\subseteq A_j$ for $\alpha\in\Gamma(i,j)$, i, $j\in\Gamma$. A non-empty Γ-set is called *indecomposable* if it is not decomposed to the disjoint union of two non-empty Γ-subsets. For example, each representable Γ-set is indecomposable. Furthermore, for each $\lambda:j\to k$ in Γ, the Γ-subset $\lambda H^k\subseteq H^j$ is also indecomposable. The following lemma is easily proved.

LEMMA 1. (1) *A Γ-set is uniquely decomposed to the disjoint union of minimal indecomposable Γ-subsets.*

(2) *A decomposition of a Γ-set to a disjoint union of Γ-subsets is preserved by the inverse image of a Γ-map.*

(3) *A Γ-set X is indecomposable if and only if every equivalence relation $\sim$ on $\coprod_{i\in\Gamma}X_i$ satisfying*

$$x_i\sim x_i\cdot\alpha \quad \textit{for} \quad x_i\in X_i,\ \alpha\in\Gamma(i,j)$$

has only one equivalence class.

A Γ-set P is called *projective* provided every Γ-epimorphism onto P is split, that is, whenever $f: A\to P$ is a Γ-epimorphism, then there is $g: P\to A$ such that $gf=1_P: P\to A\to P$. A Γ-set which is a disjoint union of representable Γ-subsets is called a *free* Γ-set

LEMMA 2. (1) *A Γ-set is projective if and only if each indecomposable direct summand of it is so.*

(2) *A Γ-set X is projective if and only if the hom-functor $H^X: \mathscr{S}^\Gamma\to \mathscr{S}: A\mapsto(X,A)_\Gamma$ preserves epimorphisms.*

(3) *A Γ-set which has a split epimorphism from a projective Γ-set is also projective.*

(4) *A Γ-set has a Γ-epimorphism from a free Γ-set.*

(5) *A free Γ-set is projective.*

The proof is easy, but the Axiom of Choise is necessary to prove the "if-part" of (1) and (5). For a Γ-set X, we can choose as a free Γ-set F in (4) the free Γ-set

$$F:=\coprod_{k\in\Gamma}X_k\times H^k$$

with a Γ-epimorphism $\pi: F\to X$ defined by $\pi_i:(x_k,\alpha)\mapsto x_k\cdot\alpha$.

2. Projective Γ-sets

In this section we characterize projective Γ-sets.

THEOREM A. *Let Γ be a small (resp. finite) category and X a Γ-set (resp. finite Γ-set). Then X is projective if and only if X is isomorphic to the direct sum of Γ-sets of the form εH^k for some $k\in\Gamma$ and some idempotent $\varepsilon^2=\varepsilon\in\Gamma(k, k)$.*

PROOF. We may assume that X is indecomposable. We will first show that εH^k, where $k\in\Gamma$ and $\varepsilon^2=\varepsilon\in\Gamma(k, k)$, is projective. Let $\pi: A\to\varepsilon H^k$ be a Γ-epimorphism. Then there is an element a_k of A_k such that $\pi_k: a_k\mapsto\varepsilon$. We define a Γ-map $\phi:\varepsilon H^k\to A$ by

$$\phi_i:\varepsilon\Gamma(k, i)\longrightarrow A_i:\varepsilon\alpha\longmapsto a_k\varepsilon\alpha\,.$$

Then ϕ is acturely a Γ-map and $\phi\pi=1$ on εH^k. Thus each epimorphism to εH^k is split, and hence εH^k is projective, as required. Next let X be an indecomposable and projective Γ-set. Take a Γ-epimorphism $\pi: F\to X$ from a free Γ-set F. Then there is a Γ-map $\phi: X\to F$ such that $\phi\pi=1_X$. By the indecomposability of X and Lemma 1 (2), we have that the image of ϕ is contained in a unique indecomposable direct summand, and so for some k, we have a commutative diagram

$$1_X=\phi\pi:\ \begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr] & F \arrow[r, "\pi"] & X\,. \\ & H^k \arrow[u] & \end{tikzcd}$$

Thus we may assume that $F=H^k$, so that

$$1_X=\phi\pi: X\xrightarrow{\ \phi\ } H^k\xrightarrow{\ \pi\ } X\,.$$

Set $\theta=\pi\phi: H^k\to H^k$. Then $\theta^2=\theta$ and $X\cong\operatorname{Im}\theta\subseteq H^k$. Define $\varepsilon\in\Gamma(k, k)$ by

$$\theta_k: H^k{}_k\longrightarrow H^k{}_k: 1_k\longmapsto\varepsilon\,.$$

Then for any α: $k\to i$ in Γ, we have a commutative diagram

$$\begin{tikzcd} k \arrow[d, "\alpha"] & \Gamma(k, k) \arrow[r, "\theta_k"] \arrow[d] & \Gamma(k, k) \arrow[d] & 1_k \arrow[r, mapsto] \arrow[d, mapsto] & \varepsilon \arrow[d, mapsto] \\ i & \Gamma(k, i) \arrow[r, "\theta_i"] & \Gamma(k, i) & \alpha \arrow[r, mapsto] & \varepsilon\alpha \end{tikzcd}$$

Thus θ_i sends each $\alpha\in\Gamma(k, i)$ to $\varepsilon\alpha$, and so $X\cong\operatorname{Im}\theta=\varepsilon H^k$. Furthermore, since θ is an idempotent, so is ε, as required. The theorem is proved.

LEMMA 3. *Let k, $l\in\Gamma$, $\varepsilon=\varepsilon^2\in\Gamma(k, k)$ and $\eta=\eta^2\in\Gamma(l, l)$. Then εH^k is isomorphic to ηH^l if and only if there exist κ: $k\to l$ and λ: $l\to k$ such that $\lambda=\eta\lambda\varepsilon$, $\kappa=\varepsilon\kappa\eta$, $\varepsilon=\kappa\lambda$, $\eta=\lambda\kappa$.*

$$\varepsilon \circlearrowleft k \underset{\lambda}{\overset{\kappa}{\rightleftarrows}} l \circlearrowright \eta .$$

This lemma is easily proved by the similar way as in the Yoneda Lemma.

COROLLARY A. 1. *Let Γ be a small (resp. finite) category in which every endomorphism is an automorphism. Then the Yoneda functor Y: $\Gamma^{op} \to \mathcal{S}^{\Gamma}$: $k \mapsto H^k$ yields an equivalence between Γ^{op} and the full subcategory of indecomposable and projective Γ-sets in $\mathcal{S}^{\Gamma}$ (resp. $\mathcal{S}^{\Gamma}_f$).*

COROLLARY A. 2. *Let Γ and Δ be small (resp. finite) category in which every endomorphism is an autonorphism. If $\mathcal{S}^{\Gamma}$ (resp. $\mathcal{S}^{\Gamma}_f$) and $\mathcal{S}^{\Delta}$ (resp. $\mathcal{S}^{\Delta}_f$) are equivalent, then Γ and Δ are equivalent, too.*

NOTE 1. Two small categories Γ and Δ are called *Morita equivalent* provided $\mathcal{S}^{\Gamma}$ and $\mathcal{S}^{\Delta}$ are equivalent. The problem when two categories are Morita equivalent is solved in monoidcase (Knauer [1]). Moreover, it is well-known that $\mathcal{S}^{\Gamma}_f$ and $\mathcal{S}^{\Delta}_f$ are equivalent for groups Γ and Δ if and only if the profinite completion $\hat{\Gamma}$ and $\hat{\Delta}$ are isomorphic.

NOTE 2. A Γ-set P is called *internally projective* if $H^k \times P$ is projective Γ-set for every k in Γ. The characterization of internally projective Γ-sets seems to be more troublesome excepting the case where Γ is a finite monoid.

3. Projective covers.

A Γ-epimorphism f: $X \to Y$ between Γ-sets is called *essential* if no proper Γ-subsets of X is mapped onto Y by f. A Γ-set X with projection π : $\tilde{X} \to X$ is called a *projective cover* of X provided $\tilde{X}$ is projective and π is an essential Γ-epimorphism. Differing from injective envelopes, projective covers does not always exist. Similarly as module-case, some finiteness conditions are absolutely necessary for the existence of projective covers. Let Γ be a small category and $k \in \Gamma$. Let $k \backslash \Gamma$ be the category of morphisms from k. For α: $k \to i$ and β: $k \to j$ in Γ, we write $\alpha \overset{k}{\sim} \beta$ if there exist λ: $\alpha \to \beta$ and μ: $\beta \to \alpha$ in $k \backslash \Gamma$ such that $a\lambda = \mu$, $\beta\mu = \alpha$.

$$\begin{tikzcd} & i \arrow[d, shift right, "\lambda"'] \\ k \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & j \arrow[u, shift right, "\mu"'] \\ & \end{tikzcd}$$

LEMMA 4. *Let Γ be a small category and $k \in \Gamma$. Then the representable Γ-set H^k possesses only a finite number of Γ-subsets if and only if $k \backslash \Gamma$*

has a finite number of equivalence classes with respect to $\overset{k}{\sim}$.

PROOF. For present, we call a Γ-set X *irreducible* if X has a Γ-epimorphism from a representable Γ-set. By Lemma 2 (4), every Γ-set is a union of set-indexed irreducible Γ-subsets. Thus the Γ-set H^k has only a finite number of Γ-subsets if and only if H^k has only a finite number of irreducible Γ-subsets. Let $\alpha\in\Gamma(k, i)$ and $\beta\in\Gamma(k, j)$. Suppose first the image of $\alpha^*: H^i\rightarrow H^k$ and $\beta^*: H^j\rightarrow H^k$ are coincident. Since α_i^* sends 1_i to α, there exists $\mu\in H^j{}_i=\Gamma(j, i)$ which is mapped to $\alpha=\beta\mu$ by β^*. Similarly, there exists $\lambda\in\Gamma(i, j)$ such that $\beta=\alpha\lambda$. Hence $\alpha\overset{k}{\sim}\beta$. Conversely, suppose $\alpha\overset{k}{\sim}\beta$, so that there exist $\lambda\in\Gamma(i, j)$ and $\mu\in\Gamma(j, i)$ such that $\beta=\alpha\lambda$ and $\alpha=\beta\mu$. Then $\alpha\Gamma(i, l)=\beta\Gamma(j, l)$ for each l in Γ, and hence $\mathrm{Im}\,\alpha^*=\mathrm{Im}\,\beta^*\subseteq H^k$. Thus we have that $\mathrm{Im}\,\alpha^*=\mathrm{Im}\,\beta^*$ if and only if $\alpha\overset{k}{\sim}\beta$. Now the lemma follows from Yoneda.

THEOREM B. *Let Γ be a small category, Δ a finite set of objects in Γ and X and finite Γ-set. Assume the following conditions*:

(a) *$d\backslash\Gamma$ has only finite $\overset{k}{\sim}$-equivalence classes for every d in Δ*;

(b) *$X_k=\bigcup_{d\in\Delta}X_d\cdot\Gamma(d, k)$ for every k in Γ.*

Then X possesses a projective cover $\tilde{X}\rightarrow X$ uniquely determined up to isomorphism on X. In particular, if Γ is a finite category and X is a finite Γ-set, then X has a projective cover.

PROOF. Take the Γ-map

$$\pi: P=\coprod_{d\in\Delta}X_d\times H^d\longrightarrow X: (x_d, \alpha)\longmapsto x_d\cdot\alpha\,.$$

Then by the assumption (b), π is a Γ-epimorphism. Furthermore by the assumption (a) and Lemma 4, P has only a finite number of Γ-subsets. Thus there exists a minimal Γ-subset $\tilde{X}$ of P such that $\pi_{|\tilde{X}}: \tilde{X}\rightarrow X$ is a Γ-epimorphism. Then the minimality implies that $\pi_{|\tilde{X}}: \tilde{X}\rightarrow X$ is an essential epimorphism. We will show that $\tilde{X}$ is projective. Let $i: \tilde{X}\rightarrow P$ be the injection. Since $i\pi: \tilde{X}\rightarrow X$ is an epimorphism and P is projective, there exists a Γ-map $\phi: P\rightarrow\tilde{X}$ such that $\pi=\phi i\pi$.

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \tilde{X} \arrow[r, "i"] \arrow[d, "i\pi"] & P \\ & X & \end{tikzcd}$$

Thus $\pi=\phi\cdot(i\pi)=\phi i\phi\cdot(i\pi)$. Since $\pi_{|\tilde{X}}=i\pi$ is essential and π is an epimorphism, we have that $\phi: P\rightarrow\tilde{X}$ and $\phi i\phi: P\rightarrow\tilde{X}$ are both epimorphisms, and so

$\phi(P)=\phi(\tilde{X})=\tilde{X}$. Thus it follows from the finiteness of $\tilde{X}$ that $i\phi: \tilde{X}\to\tilde{X}$ is a Γ-isomorphism. Set $\psi=(i\phi)^{-1}i: \tilde{X}\to P$. Then $\psi\phi=1_{\tilde{X}}$, that is, $\phi: P\to\tilde{X}$ is a split epimorphism, and hence X is projective. We proved that $i\pi$: $\tilde{X}\to X$ is a projective cover with $\tilde{X}$ finite. Next we will show the uniqueness. Let $p: P\to X$ and $q: Q\to X$ be two projective covers of X. We may assume that P is finite. Then there are Γ-maps $f: P\to Q$ and $g: Q\to P$ such that $fq=p$ and $gp=q$.

$$\begin{tikzcd} P \arrow[dr, "p"] \arrow[d, shift right, "f"'] & \\ Q \arrow[r, "q"'] \arrow[u, shift right, "g"'] & X \end{tikzcd}$$

Thus $fgp=p$ and $gfq=q$. Since p and q are essential, we have that fg and gf are epimorphisms. The finiteness of P implies that $fg: P\to P$ is an automorphism, and so f is not only a epimorphism but also a monomorphism. Thus f and also g are isomorphisms on X, as required. The theorem is proved.

The following lemma is useful to see whether the category Γ satisfies the assumption of the theorem.

LEMMA 5. *Let Γ be a small category satisfying the following conditions:*

(a) *Any morphism in Γ can be factored as an epi followed by a mono*;

(b) *Any object of Γ has only finite quotient objects*;

(c) *Any object of Γ is injective.*

Then for each $k\in\Gamma$, $k\backslash\Gamma$ has finite $\overset{k}{\sim}$-equivalence classes, that is, H^k has finite Γ-subsets.

PROOF. Let $\alpha: k\to i$ be a morphism in Γ and let $\alpha=\alpha'\alpha'': k\xrightarrow{\alpha'}k'\xrightarrow{\alpha''}i$ be an epi-mono factorization of α. Then by (c), we have that $\alpha\overset{k}{\sim}\alpha'$. Thus there is a bijective correspondence between $\overset{k}{\sim}$-equivalence classes and quotient objects of k.

EXAMPLES. (1) Let Γ be a monoid. Then Γ may be considered as a category with only one object. In this case, a projective Γ-set is a direct sum of cyclic Γ-sets $e\Gamma$ generated idempotents e of Γ (Knauer [1]). In particular, when Γ is a group, a Γ-set is projective if and only if it is semi-regular.

(2) Let Γ be the category with two objects and four morphisms as follows:

$$id \circlearrowleft 1 \rightrightarrows 0 \circlearrowright id\,.$$

Then a Γ-set is regarded as directed graph. An indecomposable and projective Γ-set is represented by one of the following graphs:

$$\circ, \quad \circ \longrightarrow \circ .$$

Furthermore in this case every Γ-set has unconditionally a projective cover.

(3) Let Δ be the category which has as objects the sets $[n] := \{0, 1, \cdots, n\}$, $n \geq 0$, and as morphisms all monotone functions. Set $\Gamma = \Delta^{op}$. Then Γ-sets are called simplicial sets (May [2]). In this case, an indecomposable and projective simplicial set is always representable, and so it is isomorphic to the simplicial set given by a finite simplex. The category Γ satisfies the conditions of Lemma 5. Thus by Theorem B, a simplicial set of finite dimensional (that is, with a finite dimensional CW complex as its geometric realization) has a projective cover. Contrary, infinite dimentional simplicial sets have no projective covers.

References

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Department of Mathematics
Hokkaido University