Hokkaido Mathematical Journal Vol. 17 (1988), p. 333~343

On pseudo-product graded Lie algebras

Tomoaki YATSUI
(Received April 13, 1987, Revised April 27, 1988)

Introduction.

For several years, N. Tanaka has worked on the geometry of pseudo-product manifolds in connection with the geometric study of systems of k-th order ordinary differential equations, where $k\geqq 2$. A study in this line can be found in his recent paper [6]. His theory shows that the geometry is closely related to the study of pseudo-product graded Lie algebras, which we will explain later on.

The main purpose of this paper is to prove structure theorems on some restricted types of pseudo-product graded Lie algebras.

Let $\mathfrak{m}=\oplus_{p<0}\mathfrak{g}_p$ be a graded Lie algebra with $0<\dim\mathfrak{m}<\infty$. Then $\mathfrak{m}$ is called a fundamental graded Lie algebra or simply an FGLA, if $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the triplet $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is called a pseudo-product FGLA if the following conditions are satisfied:

(1) $\mathfrak{m}$ is an FGLA.

(2) $\mathfrak{g}_{-1}=\mathfrak{e}\oplus\mathfrak{f}$ and $[\mathfrak{e}, \mathfrak{e}]=[\mathfrak{f}, \mathfrak{f}]=\{0\}$

A pseudo-product FGLA $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is called non-degenerate, if the condition "$x\in\mathfrak{g}_{-1}$ and $[x, \mathfrak{g}_{-1}]=\{0\}$" implies $x=0$.

Now let $\mathfrak{g}=\oplus_{p\in Z}\mathfrak{g}_p$ be a graded Lie algebra and let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Set $\mathfrak{m}=\oplus_{p<0}\mathfrak{g}_p$. Then $\mathfrak{g}$ (together with $\mathfrak{e}$ and $\mathfrak{f}$) is called a pseudo-product graded Lie algebra if the following conditins are satisfied:

(1) $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is a pseudo-product FGLA.

(2) $\mathfrak{g}$ is transitive, i.e. the condition "$p\geqq 0$, $x\in\mathfrak{g}_p$ and $[x, \mathfrak{g}_{-1}]=\{0\}$" implies $x=0$.

(3) $[\mathfrak{g}_0, \mathfrak{e}]\subset\mathfrak{e}$ and $[\mathfrak{g}_0, \mathfrak{f}]\subset\mathfrak{f}$

Let $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ be an FGLA and $\mathfrak{g}_0$ be its derivations of the graded Lie algebra $\mathfrak{m}$ leaving both $\mathfrak{e}$ and $\mathfrak{f}$ invariant. Then the prolongation $\check{\mathfrak{g}}=\oplus_{p\in Z}\check{\mathfrak{g}}_p$ of the pair $(\mathfrak{m}; \mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ (see [4] and [6]), which may be characterized as the maximum pseudo-product graded Lie algebra $\mathfrak{g}=\oplus_{p\in Z}\mathfrak{g}_p$ such that $\oplus_{p\leqq 0}\mathfrak{g}_p=\mathfrak{m}\oplus\check{\mathfrak{g}}_0$ (as graded Lie algebras). It is known that if $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is non-degenerate, then $\check{\mathfrak{g}}$ is of finite dimension (see

N. Tanaka [6], page 292).

These being prepared, our main theorems (Theorem 3.2 and 3.3) together may be stated as follows: let $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ be a pseudo-product graded Lie algebra over the field $\boldsymbol{C}$ of complex numbers or the field $\boldsymbol{R}$ of real numbers. Assume that the natural representations of $\mathfrak{g}_0$ on both $\mathfrak{e}$ and $\mathfrak{f}$ are irreducible and that the pseudo-product FGLA $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ is non-degenerate. If $\mathfrak{g}_2\neq\{0\}$, the Lie algebra $\mathfrak{g}$ is of finite dimension and simple.

Following N. Tanaka (see [5] and [6]), we will explain how the geometry of pseudo-product manifolds is related to the study of pseudo-product graded Lie algebras, as we promised. Let R be a manifold, and E and F be two differential systems on R. Then the triplet $(R\,;E, F)$ is called a pseudo-product manifold, if both E and F are completely integrable, and $E\cap F=\{0\}$. Let $(R\,;E, F)$ be a pseudo-product manifold. Assuming that the differential system $D=E+F$ is regular, let us consider the symbol algebra $(\mathfrak{m}(x)\,;E(x), F(x))$ of $(R\,;E,F)$ at each point $x\in R$, which is a pseudo-product FGLA. Note that $\mathfrak{m}(x)$ is the symbol algebra of D at x, and $\mathfrak{g}_{-1}=D(x)$. Given a pseudo-product FGLA, $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$, the pseudo-product manifold $(R\,;E,F)$ is called of type $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$, if D is regular, the symbol algebra $(\mathfrak{m}(x)\,;E(x), F(x))$ of $(R\,;E,F)$ at each $x\in R$ is isomorphic with the given $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$, and $\dim R=\dim\mathfrak{m}$.

Now, let $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ be a non-degenerate pseudo-product FGLA, and let $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ be its prolongation. Then N. Tanaka showed that to every pseudo-product manifold $(R\,;E,F)$ of type $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ there is associated, in canonical manner, a manifold (P,ω) with absolutely parallelism satisfying the following conditions: 1) $\dim P=\dim\mathfrak{g}$ 2) P is a fibred manifold over M, and 3) ω is a $\mathfrak{g}$-valued 1-form on P, and gives the absolutely parallelism. In particular, it follows that the Lie algebra $\mathfrak{a}$ of all infinitesimal automorphisms of $(R\,;E,F)$ is of finite dimension, and $\dim\mathfrak{a}\leqq\dim\mathfrak{g}$. Futhermore, he showed that if $\mathfrak{g}$ is simple, to every pseudo-product manifold $(R\,;E,F)$ of type $(\mathfrak{m}\,;\mathfrak{e},\mathfrak{f})$ there is associated a connection of type $\mathfrak{g}$ on R in natural manner. Recently he has generalized this fact to the case where $\mathfrak{g}$ is not semisimple (and satisfies certain conditions), and has applied the result to the geometric study of systems of k-th order ordinary differential equations, where $k\geqq 3$.

We have thus seen that our main theorems are applicable to the geometry of pseudo-product manifolds. It should be remarked that our main theorems are likewise applicable to the geometry of pseudo-complex manifolds, which is based on N. Tanaka's work [4] and the fact that the complexification of a pseudo-complex FGLA becomes naturally a pseudo-product FGLA (see also [6]).

We will now give a brief description of the varoius sections. Following V. G. Kac [1], we first give basic definitions on graded Lie algebras and minimal graded Lie algebras. In Section 2, we consider a finite dimensional transitive graded Lie algebra $\mathfrak{g}=\oplus_{p\in \boldsymbol{Z}}\mathfrak{g}_p$ over $\boldsymbol{C}$ for which the natural representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is completely reducible. Our main task in this section is to determine the structure of the local part $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ of $\mathfrak{g}$, and discuss conditions for $\mathfrak{g}$ to be semisimple (Corollary 2.5). To do these, we apply the reasonings, due to V. G. Kac [1], in the realization of graded Lie algebras, and use the fundamental representation theory of finite dimensional Lie algebras. In Section 3 we prove the main theorems by using the finite dimensionality of the pseudo-product graded Lie algebras and by applying the results in Section 2.

Finally I warmly thank Professor N. Tanaka for his kind suggestion of the problem and thank Dr. Yamaguchi for his invaluable help.

§ 1. Preliminaries

In this section, the ground field K is assumed to be of characteristic zero. In fact in our applications K will be the field $\boldsymbol{C}$ of complex numbers or the field $\boldsymbol{R}$ of real numbers.

1.1. Graded Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. If $\boldsymbol{Z}$ is the ring of integers, a $\boldsymbol{Z}$-gradation of $\mathfrak{g}$ is, by definition, a direct decomposition

$$\mathfrak{g}=\oplus_{i\in\boldsymbol{Z}}\mathfrak{g}_i \text{ such that } [\mathfrak{g}_i, \mathfrak{g}_j]=\mathfrak{g}_{i+j},\ \dim\mathfrak{g}_i<\infty\ (i\in\boldsymbol{Z})$$

We will call a Lie algebra $\mathfrak{g}$ a $\boldsymbol{Z}$-graded Lie algebra when $\mathfrak{g}$ has such a $\boldsymbol{Z}$-gradation. A subalgebra (resp. an ideal) $\mathfrak{s}\subset\mathfrak{g}$ is called a $\boldsymbol{Z}$-graded subalgebra (resp. ideal) if $\mathfrak{s}=\oplus_{i\in\boldsymbol{Z}}\mathfrak{s}\cap\mathfrak{g}_i$. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two $\boldsymbol{Z}$-graded Lie algebras. Then, by definition, a homomorphism $\phi:\mathfrak{g}\to\mathfrak{g}'$ of $\boldsymbol{Z}$-graded Lie algebras preserves the $\boldsymbol{Z}$-gradation in the sense that $\phi(\mathfrak{g}_i)\subset\mathfrak{g}'_i$. Similarly isomorphisms and epimorphisms of $\boldsymbol{Z}$-graded Lie algebras are defined.

Let $\mathfrak{g}=\oplus_{i\in\boldsymbol{Z}}\mathfrak{g}_i$ be a $\boldsymbol{Z}$-graded Lie algebra. We will denote by $\mathfrak{g}_-$ the subalgebra $\oplus_{i\leqq -1}\mathfrak{g}_i$. Then a $\boldsymbol{Z}$-graded Lie algebra $\mathfrak{g}$ is called *transitive* if it satisfies the following conditions:

(1.1.1) $\mathfrak{g}_{-1}\neq\{0\}$ and $\mathfrak{g}_-$ is an FGLA.
(1.1.2) For $x\in\mathfrak{g}_i (i\geqq 0)$, $[x, \mathfrak{g}_{-1}]=\{0\}$ implies $x=0$.

1.2. Correspondence between local Lie algebra and graded Lie algebras (see V. G. Kac. [1]. page 1276-1277)

A direct sum of vector spaces $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ is called a *local Lie algebra* if one has bilinear maps : $\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ for $|i|$, $|j|$, $|i+j| \leqq 1$, such that anticommutativity and the Jacobi identity hold whenever they make sense. Homomorphisms and isomorphisms of local Lie algebras are defined as in the case of graded Lie algebras. Given a $\boldsymbol{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \boldsymbol{Z}} \mathfrak{g}_i$, the subspace $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a local Lie algebra, which is called the local part of $\mathfrak{g}$.

Now, let $\mathfrak{g} = \bigoplus_{i \in \boldsymbol{Z}} \mathfrak{g}_i$ be a $\boldsymbol{Z}$-graded Lie algebra generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then the $\boldsymbol{Z}$-graded Lie algebra $\mathfrak{g}$ is called *minimal* if, for any other $\boldsymbol{Z}$-graded Lie algebra $\mathfrak{g}'$, each isomorphism of the local parts $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ extends to an epimorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$. Indeed, for any local Lie algebra $\hat{\mathfrak{g}}$, there is a minimal $\boldsymbol{Z}$-graded Lie algebra $\mathfrak{g}$ whose local part is isomorphic to $\hat{\mathfrak{g}}$ (see V. G. Kac [1], page 1276). We will utilize this fact in the proof of Lemma 2. 2.

§ 2. Finite dimensional transitive graded Lie algebras

In this section, we state a necessary and sufficient condition under which a finite dimensional transitive $\boldsymbol{Z}$-graded Lie algebra over $\boldsymbol{C}$ be semisimple. Also, throughout this section, we assume that the ground field is the field of complex numbers $\boldsymbol{C}$.

2. 1. Throughout this section, $\mathfrak{g} = \bigoplus_{i \in \boldsymbol{Z}} \mathfrak{g}_i$ will denote a finite dimensional transitive $\boldsymbol{Z}$-graded Lie algebra for which the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is completely reducible. We denote by ϕ_i the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_i$ induced by restriction of the adjoint representation of $\mathfrak{g}$. By the assumption, we can decompose $\mathfrak{g}_{-1}$ into a direct sum of $\mathfrak{g}_0$-submodules

(2. 1. 1) $\mathfrak{g}_{-1} = \tilde{\mathfrak{g}}_{-1} \oplus \mathfrak{g}'_{-1}$, $\tilde{\mathfrak{g}}_{-1} = \bigoplus_{j=1}^{t} \mathfrak{g}_{-1}^{(j)}$, $\mathfrak{g}'_{-1} = \bigoplus_{j=t+1}^{n(-1)} \mathfrak{g}_{-1}^{(j)}$,

where each $\mathfrak{g}_{-1}^{(j)}$ is an irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$ such that

$$[\mathfrak{g}_{-1}^{(j)}, \mathfrak{g}_1] \neq \{0\} \text{ for } 1 \leqq j \leqq t$$
$$\text{and } [\mathfrak{g}_{-1}^{(j)}, \mathfrak{g}_1] = \{0\} \text{ for } t < j \leqq n(-1).$$

We denote by $\phi_{-1}^{(j)}$ the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}^{(j)}$ given by $[\mathfrak{g}_0, \mathfrak{g}_{-1}^{(j)}] \subset \mathfrak{g}_{-1}^{(j)}$. Since ϕ_{-1} is faithful and completely reducible, $\mathfrak{g}_0$ is a reductive Lie algebra, i. e, $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathfrak{c}(\mathfrak{g}_0)$, where $\mathfrak{g}_0'$ denotes the semisimple part of $\mathfrak{g}_0$ and $\mathfrak{c}(\mathfrak{g}_0)$ the conter of $\mathfrak{g}_0$.

From the assumption, we first deduce

LEMMA 2. 1 *The representation of* $\mathfrak{g}_0$ *on* $\mathfrak{g}$ *is completely reducible.*

PROOF. We first prove that $\mathfrak{g}_0$-module $\mathfrak{g}_-$ is completely reducible. By transitivity, we can consider $\mathfrak{g}_0$ as a subalgebra of the Lie algebra Dergr($\mathfrak{g}_-$) of all the derivations of $\mathfrak{g}_-$ preserving the gradation of $\mathfrak{g}_-$. On the other hand, $\mathrm{Dergr}(\mathfrak{g}_-)$ contains the semisimple and nilpotent components of its elements (see N. Bourbaki [2], Ch. VII, § 5, n^{0}1). Thus we can decompose the element x of $\mathfrak{c}(\mathfrak{g}_0)$ as follows:

$$x=x_s+x_n,\ x_s,\ x_n\in\mathrm{Dergr}(\mathfrak{g}_-),$$

where x_s (resp. x_n) is the semisimple (resp. nilpotent) component of x. Since $x|\mathfrak{g}_{-1}$ is semisimple and $x_n|\mathfrak{g}_{-1}$ is the nilpotent component of $x|\mathfrak{g}_{-1}$, we have $x_n|\mathfrak{g}_{-1}=0$. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, we have $x_n=0$, so $x=x_s$. Thus $\mathfrak{g}_0$-module $\mathfrak{g}_-$ is completely reducible. Next we prove that $\mathfrak{g}_p(p\geqq 0)$ is a completely reducible $\mathfrak{g}_0$-module. We will use induction on p. Since $\mathfrak{g}_0$ is reductive, the statement holds for $p=0$. We assume now that the statement holds for k. We consider the mapping

$$\iota:\mathfrak{g}_{k+1}\longrightarrow\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_k),$$

where for $x\in\mathfrak{g}_{k+1}$, $\iota(x)=\mathrm{ad}(x)|\mathfrak{g}_{-1}$. Then, by transitivity, it is easy to prove that ι is a monomorphism of $\mathfrak{g}_0$-modules, so we may regard $\mathfrak{g}_{k+1}$ as a $\mathfrak{g}_0$-submodule of $\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_k)$. Owing to the induction hypothesis, $\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_k)$ is a completely reducible $\mathfrak{g}_0$-module, so $\mathfrak{g}_{k+1}$ is a completely reducible $\mathfrak{g}_0$-module. This proves the Lemma. Q. E. D

2.2. Now we decompose $\mathfrak{g}_1$ into a direct sum of irreducible $\mathfrak{g}_0$-submodules:

$$\mathfrak{g}_1=\oplus_{j=1}^{n(1)}\mathfrak{g}_1^{(j)},$$

and we denote by $\phi_1^{(j)}$ the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1^{(j)}$ given by $[\mathfrak{g}_0,\mathfrak{g}_1^{(j)}]\subset\mathfrak{g}_1^{(j)}$. In this paragraph, we will investigate the relation between $\phi_1^{(k)}$ and $\phi_{-1}^{(j)}$.

Here we note that the elements of $\mathfrak{c}(\mathfrak{g}_0)$ act on $\mathfrak{g}_1^{(i)}(i=1,\dots,n(1))$ by scalar multiplications.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$. Then, associated to this choice is the system of weights of the representations ϕ_p, $\phi_{-1}^{(i)}$ and $\phi_1^{(i)}$.

We now fix a Cartan subalgebra $\mathfrak{h}$ and a Weyl chamber, and by Λ_i(resp. M_i) we will denote the highest (resp. lowest) weight of $\phi_{-1}^{(i)}$(resp. $\phi_1^{(i)}$). For each Λ_i(resp. M_i), $F_{\Lambda_i}\in\mathfrak{g}_{-1}^{(i)}$(resp. $E_{\mathrm{M}_i}\in\mathfrak{g}_1^{(i)}$) denotes a non-zero weight vector for Λ_i(resp. M_i). Also, for a root α of $\mathfrak{g}_0'$, we denote by e_α a root vector for α, and let h_α be the unique element of $\boldsymbol{C}[e_\alpha,e_{-\alpha}]$ for which $\alpha(h_\alpha)=2$. Fix $1\leqq i\leqq n(1)$. Then there is an integer i_0 such that $[\mathfrak{g}_{-1}^{(i_0)},\mathfrak{g}_1^{(i)}]\neq\{0\}$, since $\mathfrak{g}$ is transitive. Here we remark that $[\mathfrak{g}_{-1}^{(i_0)},\mathfrak{g}_1^{(i)}]\neq\{0\}$ if and only if $[E_{\mathrm{M}_i},F_{\Lambda_{i_0}}]\neq\{0\}$.

Then we have

LEMMA 2.2. *The representations of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_{-1}^{(i_0)}$ *and* $\mathfrak{g}_1^{(i)}$ *are contra-*

gredient (*i. e.*, $\Lambda_{i_0}+\mathrm{M}_i=0$). *Consequently* $h:=[E_{\mathrm{M}_1}, F_{\Lambda_{i_0}}]\in\mathfrak{h}$.

PROOF. For covenience, we suppose that $i_0=i=1$. We first suppose that $\Lambda_1+\mathrm{M}_1=\alpha$ is a root of $\mathfrak{g}_0'$(i. e., $[E_{\mathrm{M}_1}, F_{\Lambda_1}]=e_\alpha$). Then we have

$$\mathfrak{g}_0'=\mathfrak{a}_1\oplus\mathfrak{a}_2\oplus\mathfrak{a}_3\oplus\mathfrak{a}_4,$$

where each $\mathfrak{a}_i$ is a semisimple ideal in $\mathfrak{g}_0'$ such that

$$\mathrm{Ker}\ \phi_{-1}^{(1)}=\mathfrak{a}_1\oplus\mathfrak{a}_2 \text{ and } \mathrm{Ker}\ \phi_1^{(1')}=\mathfrak{a}_2\oplus\mathfrak{a}_3.$$

Here we consider four cases. If α is a root of $\mathfrak{a}_2$, then we have $[e_\gamma[E_{\mathrm{M}_1}, F_{\Lambda_1}]]=[e_{-\gamma}[E_{\mathrm{M}1}, F_{\Lambda_1}]]=0$ for any root γ of $\mathfrak{a}_2$, which is a contradiction because of the semisimplicity of $\mathfrak{a}_2$. Next suppose that α is a root of $\mathfrak{a}_3$. Since $\Lambda_1+\mathrm{M}_1=\alpha$ and $\mathrm{M}_1(h_\alpha)=0$, we have $\Lambda_1(h_\alpha)\neq0$. Let $\mathfrak{b}$ be the three dimensional subalgebra of $\mathfrak{g}$ with a basis $\{[F_{\Lambda_1}, e_{-\alpha}], h_\alpha, E_{\mathrm{M}_1}\}$. We consider $\mathfrak{b}$-submodule N of $\mathfrak{b}$-module $\mathfrak{g}$ generated by F_{Λ_1}(i. e., $N=\mathrm{Ad}(U(\mathfrak{b}))F_{\Lambda_1}$, where $U(\mathfrak{b})$ is the universal enveloping algebra of $\mathfrak{b}$). Then we have $0=\mathrm{tr}(\mathrm{ad}\ h_\alpha|N)=(\dim N)\Lambda_1(h_\alpha)$, which is a contradiction. Similarly, when we suppose that α is a root of $\mathfrak{a}_1$, we reach a contradiction by applying the above arguments to $\mathfrak{b}=\boldsymbol{C}F_{\Lambda_1}\oplus\boldsymbol{C}h_\alpha\oplus\boldsymbol{C}[E_{\mathrm{M}_1}, e_\alpha]$ and $N=\mathrm{Ad}(U(\mathfrak{b}))E_{\mathrm{M}_1}$. Finally, we suppose that α is a root of $\mathfrak{a}_4$. Let $\mathfrak{a}$ be a simple component of $\mathfrak{a}_4$ such that α is a root of $\mathfrak{a}$, and $\hat{\mathfrak{g}}_{-1}^{(1)}$ (resp. $\hat{\mathfrak{g}}_1^{(1')}$) be an irreducible $\mathfrak{a}$-submodule of $\mathfrak{g}_{-1}^{(1)}$ (resp. $\mathfrak{g}_1^{(1')}$) containing F_{Λ_1}(resp. E_{M_1}). Then the representations of $\mathfrak{a}$ on $\hat{\mathfrak{g}}_{-1}^{(1)}$ and $\hat{\mathfrak{g}}_1^{(1')}$ are faithful and irreducible. Since $\Lambda_1+\mathrm{M}_1=\alpha$, by V. G. Kac ([1], page 1299 Theorem 2), we know that $\hat{\mathfrak{g}}_{-1}^{(1)}\oplus\mathfrak{a}\oplus\hat{\mathfrak{g}}_1^{(1')}$ is isomorphic to the local part of the special algebra S_n or the Hamiltonian algebra H_n as a local Lie algebra. Since S_n and H_n is minimal, it follows that $\mathfrak{g}$ contains a subalgebra whose factor algebra is isomorphic to S_n or H_n. But since S_n and H_n is infinite dimensional, we obtain that $\mathfrak{g}$ is infinite dimensional, which is a contradiction due to the assumption. Q. E. D

For the behavior of $h:=[E_{\mathrm{M}_i}, F_{\Lambda_{i_0}}]$, we have

LEMMA 2. 3. $\mathrm{M}_i(h)=-\Lambda_{i_0}(h)\neq0$. *Consequently* $[h, E_{\mathrm{M}_i}]\neq0$, $[h, F_{\Lambda_{i_0}}]\neq0$.

PROOF. For convenience, we suppose $i=i_0=1$. We now suppose that $[h, E_{\mathrm{M}_1}]=0$. By transitivity, there is a weight vector v_λ of $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ with a weight λ such that $[h, v_\lambda]\neq0$. We put $\mathfrak{b}=\boldsymbol{C}E_{\mathrm{M}_1}\oplus\boldsymbol{C}h\oplus\boldsymbol{C}F_{\Lambda_1}$ and $N=\mathrm{Ad}(U(\mathfrak{b}))v_\lambda$. Then we have $0=\mathrm{tr}(\mathrm{ad}\ h|N)=(\dim N)\lambda(h)$, which is a contradiction. Q. E. D

For the pair $(\mathfrak{g}_{-1}^{(k)}, \mathfrak{g}_1^{(i')})$ of $\mathfrak{g}_0$-modules such that $[\mathfrak{g}_{-1}^{(k)}, \mathfrak{g}_1^{(i')}]\neq\{0\}$, we have

LEMMA 2. 4. *For each* i ($1\leqq i\leqq n(1)$), *there is a unique integer* k *such* *that* $[\mathfrak{g}_{-1}^{(k)}, \mathfrak{g}_1^{(i')}]\neq\{0\}$. *Furthermore,* $\Lambda_{k'}+\mathrm{M}_i\neq 0$ *for any* k' *such that* $k\neq k'$.

PROOF. We first suppose that there are two integers k_1, k_2 such that $[E_{\mathrm{M}_i}, F_{\Lambda_{i_1}}]\neq\{0\}$ and $[E_{\mathrm{M}_i}, F_{\Lambda_{i_2}}]\neq\{0\}$. We put $\alpha_1^\vee=[E_{\mathrm{M}_i}, F_{\Lambda_{i_1}}]$ and $\alpha_2^\vee=[E_{\mathrm{M}_i}, F_{\Lambda_{i_2}}]$. Then, by Lemma 2.3, we have

$$[\alpha_1^\vee, E_{\mathrm{M}_i}]=c_1E_{\mathrm{M}_i},\ [\alpha_2^\vee, E_{\mathrm{M}_i}]=c_2E_{\mathrm{M}_i},\ c_1, c_2\in \boldsymbol{C}^\times.$$

First suppose that $\{\alpha_1^\vee, \alpha_2^\vee\}$ is linearly independent. Replace $\phi_{-1}^{(k_1)}$ by the irreducible representation $\tilde{\phi}_{-1}^{(k_1)}$ with the highest weight Λ_{k_1} and corresponding weight vector $F_{\Lambda_{k_1}}-c_2^{-1}c_1F_{\Lambda_{k_2}}$. Then we have

$$\begin{aligned}&\tilde{h}:=[E_{\mathrm{M}_i}, F_{\Lambda_{k_1}}-c_2^{-1}c_1F_{\Lambda_{k_2}}]=\alpha_1^\vee-c_2^{-1}h_1\alpha_2^\vee\neq 0\\&[\tilde{h}, E_{\mathrm{M}_i}]=0,\end{aligned}$$

which is a contradiction by Lemma 2.3. Thus $\{\alpha_1^\vee, \alpha_2^\vee\}$ is linearly dependent. Moreover, multiplying E_{M_i} and $F_{\Lambda_{k_2}}$ by some non-zero scalars, we may assume that $\alpha_1^\vee=\alpha_2^\vee$ and $c_1=c_2=1$. Also we put $F_\Lambda=F_{\Lambda_{k_2}}-F_{\Lambda_{k_1}}$. Then, for $s\geqq 0$, we obtain by induction:

$$(\mathrm{ad}\ E_{\mathrm{M}_i})(\mathrm{ad}\ F_{\Lambda_{k_1}})^{s+1}F_\Lambda=-(s+1)(s+2)/2(\mathrm{ad}\ F_{\Lambda_{k_1}})^sF_\Lambda.$$

If s_0 is the last integer such that $(\mathrm{ad}\ F_{\Lambda_{k_1}})^{s_0}F_\Lambda\neq 0$, then we have $s_0=-1$ or $s_0=-2$, which is a contradiction.

Next we suppose that there are two integers k_1, k_2 such that $[E_{\mathrm{M}_i}, F_{\Lambda_{k_1}}]\neq\{0\}$, $\Lambda_{k_2}+\mathrm{M}_i=0$ and $[E_{\mathrm{M}_i}, F_{\Lambda_{k_2}}]=0$. Using the notation above, for $s\geqq 0$, we obtain by induction

$$(\mathrm{ad}\ E_{\mathrm{M}_i})(\mathrm{ad}\ F_{\Lambda_{k_1}})^{s+1}F_{\Lambda_{k_2}}=-(s+1)(s+2)/2(\mathrm{ad}\ F_{\Lambda_{k_1}})^sF_{\Lambda_{k_2}}$$

Similarly we can reach a contradiction as above. Q. E. D

2.3. Using our previous results, we prove the following proposition which will play a crucial role in the investigation of the pseudo-product graded Lie algebra.

PROPOSITION 2.5. *Let* $\mathfrak{g}=\bigoplus_{i\in\boldsymbol{Z}}\mathfrak{g}_i$ *be a finite dimensional transitive* $\boldsymbol{Z}$*-graded Lie algebra over* $\boldsymbol{C}$ *for which the representation of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_{-1}$ *is completely reducible. Then we have the following*

(i) *Let* $\tilde{\mathfrak{g}}_{-1}$ *and* $\mathfrak{g}'_{-1}$ *be as in* (2.1.1) *so that* $\mathfrak{g}_{-1}=\tilde{\mathfrak{g}}_{-1}\oplus\mathfrak{g}'_{-1}$. *Then the* $\boldsymbol{Z}$*-graded subalgebra* $\tilde{\mathfrak{g}}=\bigoplus_{i\in\boldsymbol{Z}}\tilde{\mathfrak{g}}_i$ *of* $\mathfrak{g}$ *generated by* $\tilde{\mathfrak{g}}_{-1}\oplus[\tilde{\mathfrak{g}}_{-1}, \mathfrak{g}_1]\oplus\mathfrak{g}_1$ *is a semisimple Lie algebra. Furthermore the subalgebra* $\bigoplus_{i\geqq 1}\mathfrak{g}_i$ *of* $\mathfrak{g}$ *is generated by* $\mathfrak{g}_1$.

(ii) *The radical* $\mathfrak{r}$ *of* $\mathfrak{g}$ *is a* $\boldsymbol{Z}$*-graded ideal in* $\mathfrak{g}$(*i. e.,* $\mathfrak{r}=\bigoplus_{i\in\boldsymbol{Z}}\mathfrak{r}_i$, *where* $\mathfrak{r}_i=\mathfrak{r}\cap\mathfrak{g}_i$) *and* $\mathfrak{r}_i=\mathfrak{d}_i(i\leqq 0)$ *and* $\mathfrak{r}_i=\{0\}(i\geqq 1)$, *where* $\mathfrak{d}_{-k}=\{x\in\mathfrak{g}_{-k}:(\mathrm{ad}\ \mathfrak{g}_1)^kx$

$=\{0\}\}(k\geqq 1)$ *and* $\mathfrak{d}_0=\{x\in\mathfrak{c}(\mathfrak{g}_0):(\mathrm{ad}\ \mathfrak{g}_1)x=\{0\}\}$. *Moreover we have* $\mathfrak{r}_{-1}=\mathfrak{g}'_{-1}$ *and* $\mathfrak{g}_0=\mathfrak{r}_0\oplus[\tilde{\mathfrak{g}}_{-1},\mathfrak{g}_1]\oplus\mathfrak{g}''_0$, *where* $\mathfrak{g}''_0$ *is the centralizer of* $\tilde{\mathfrak{g}}_{-1}$ *in* $\mathfrak{g}'_0$.

PROOF. (i) Let E be the element of $\mathrm{Dergr}(\mathfrak{g})$ such that

$$E(x)=px \text{ for } x\in\mathfrak{g}_p.$$

Regarding $\mathfrak{c}(\mathfrak{g}_0)$ as the subalgebra of $\mathrm{Dergr}(\mathfrak{g})$, $(\mathfrak{c}(\mathfrak{g}_0)+\boldsymbol{C}E)$-module $\mathfrak{g}$ is completely reducible by Lemma 2.1. By O. Mathieu ([7], page 402, Lemma 34), there is a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $(\mathfrak{c}(\mathfrak{g}_0)+\boldsymbol{C}E)(\mathfrak{s})\subset\mathfrak{s}$. Then $\mathfrak{s}$ is graded, which we write $\mathfrak{s}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{s}_p$. Also, the radical $\mathfrak{r}$ of $\mathfrak{g}$ is graded, which we write $\mathfrak{r}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{r}_p$. Then, since $\mathfrak{c}(\mathfrak{g}_0)\supset\mathfrak{r}_0$, $\mathfrak{s}$ is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}$, so $\mathfrak{s}$ is a completely reducible $\mathfrak{g}_0$-submodule by Lmma 2.1. Hence we can decompose $\mathfrak{s}_p$ into a direct sum of irreducible $\mathfrak{g}_0$-submodules of $\mathfrak{s}_p$:

$$\mathfrak{s}_p=\bigoplus_{k=1}^{s(p)}\mathfrak{s}_p^{(k)}.$$

Let W be an irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{r}_1$. Then, by transitivity, there is an integer k such that $[\mathfrak{g}_{-1}^{(k)},W]\neq\{0\}$. Let E_{M} be the highest weight vector of W with the highest weight M. By Lemma 2.3, the subspace $\boldsymbol{C}F_{\Lambda_k}\oplus\boldsymbol{C}[F_{\Lambda_k},E_{\mathrm{M}}]\oplus\boldsymbol{C}E_{\mathrm{M}}$ is a simple three dimensional Lie subalgebra of $\mathfrak{r}$, which is a contradiction. Thus we have $\mathfrak{r}_1=\{0\}$, so, by transitivity, $\mathfrak{r}_p=\{0\}(p\geqq 1)$. Hence we have $\mathfrak{s}_p=\mathfrak{g}_p(p\geqq 1)$. For each $i(1\leqq i\leqq t)$, $\mathfrak{g}_{-1}^{(i)}$ is contragredient to $\mathfrak{s}_1^{(j)}$ as a $\mathfrak{g}_0$-module for some $j(1\leqq j\leqq s(1))$. Also we remark that $\mathfrak{s}_{-1}$ is contragredient to $\mathfrak{s}_1$ as a $\mathfrak{g}_0$-module. Indeed, let $(\ |\ )$ be the Killing form of $\mathfrak{s}$. Since the restriction of $(\ |\ )$ on $\mathfrak{s}_1\times\mathfrak{s}_{-1}$ is non-degenerate, we have an isomorphism $\nu:\mathfrak{s}_1\longrightarrow\mathfrak{s}_{-1}^*$ defined by

$$<\nu(x),y>=(x|y),\ x\in\mathfrak{s}_1,\ y\in\mathfrak{s}_{-1}.$$

Since $(\ |\ )$ is completely invariant (i.e, $(x|Dy)+(Dx|y)=0$ for $x,y\in\mathfrak{s}$ and $D\in\mathrm{Der}(\mathfrak{s})$), we can prove easily that ν is an isomorphism of $\mathfrak{g}_0$-modules, so $\mathfrak{s}_1$ is contragredient to $\mathfrak{s}_{-1}$ as a $\mathfrak{g}_0$-module. Thus $\mathfrak{g}_{-1}^{(i)}$ is isomorphic to $\mathfrak{s}_{-1}^{(k)}$ as a $\mathfrak{g}_0$-module for some $k(1\leqq k\leqq s(-1))$. However, since $\mathfrak{g}_{-1}^{(i)}$ is not isomorphic to $\mathfrak{g}_{-1}^{(j)}$ as a $\mathfrak{g}_0$-module for all j such that $i\neq j$ by Lemma 2.4, we have $\mathfrak{g}_{-1}^{(i)}=\mathfrak{s}_{-1}^{(k)}$. In particular, we have $l(\mathfrak{s}_{-1})\leqq l(\tilde{\mathfrak{g}}_{-1})$, where we denote by $l(N)$ the number of the irreducible components of $\mathfrak{g}_0$-module N. On the other hand, by Lemma 2.4, we have $l(\mathfrak{s}_{-1})=l(\mathfrak{s}_1)\geqq l(\tilde{\mathfrak{g}}_{-1})$. Thus we have $l(\tilde{\mathfrak{g}}_{-1})=l(\mathfrak{s}_{-1})$, so $\tilde{\mathfrak{g}}_{-1}=\mathfrak{s}_{-1}$. Since the subalgebra $(\mathfrak{g}/\mathfrak{r})_-$ of $\mathfrak{g}/\mathfrak{r}$ is generated by $\mathfrak{g}_{-1}/\mathfrak{r}_{-1}$, the subalgebra $\mathfrak{s}_-$ of $\mathfrak{s}$ is generated by $\mathfrak{s}_{-1}$. Moreover, since an ideal in a semisimple $\boldsymbol{Z}$-graded Lie algebra is graded, we can decompose $\mathfrak{s}$ into a direct sum of two semisimple ideals $\mathfrak{t}$ and $\mathfrak{u}$ (i.e., $\mathfrak{s}=\mathfrak{t}\oplus\mathfrak{u}$, $\mathfrak{t}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{t}_p$ and $\mathfrak{u}=\bigoplus_{p\in\boldsymbol{Z}}\mathfrak{u}_p$) such that $\mathfrak{t}$ is a semisimple transitive $\boldsymbol{Z}$-graded Lie algebra and $\mathfrak{u}=$

$\mathfrak{u}_0$. Then the subalgebra $\oplus_{i\geqq 1}\mathfrak{t}_i$ of $\mathfrak{t}$ is generated by $\mathfrak{t}_1$, and we have $\mathfrak{t}_0=[\mathfrak{t}_{-1}, \mathfrak{t}_1]=[\tilde{\mathfrak{g}}_{-1}, \mathfrak{g}_1]$ (see N. Tanaka [5], page 28). Also, since $\mathfrak{s}_p=\mathfrak{t}_p(p\geqq 1)$, the subalgebra $\oplus_{i\geqq 1}\mathfrak{g}_i$ of $\mathfrak{g}$ is generated by $\mathfrak{g}_1$. Thus we have $\tilde{\mathfrak{g}}=\mathfrak{t}$, so $\tilde{\mathfrak{g}}$ is semisimple.

(ii) We put $\mathfrak{d}=\oplus_{i\leqq 0}\mathfrak{d}_i$. Then $\mathfrak{d}$ is a solvable ideal in $\mathfrak{g}$, so $\mathfrak{d}\subset\mathfrak{r}$. In particular, we have $\mathfrak{g}'_{-1}\subset\mathfrak{d}_{-1}\subset\mathfrak{r}_{-1}$. Since $\tilde{\mathfrak{g}}_{-1}=\mathfrak{s}_{-1}$, we have $\dim\mathfrak{g}'_{-1}=\dim\mathfrak{r}_{-1}$, so $\mathfrak{r}_{-1}=\mathfrak{g}'_{-1}$. Moreover, since $(\operatorname{ad}\ \mathfrak{g}_1)^{i-1}\mathfrak{r}_{-i}\subset\mathfrak{r}_{-1}(i\geqq 2)$ and $[\mathfrak{r}_0, \mathfrak{g}_1]\subset\mathfrak{r}_1=\{0\}$, we have $\mathfrak{r}_{-i}\subset\mathfrak{d}_{-i}(i\geqq 0)$. Hence we have $\mathfrak{d}=\mathfrak{r}_{-i}(i\geqq 0)$. Finally, from the proof of (i), we have $\mathfrak{u}_0=\mathfrak{g}''_0$ and $\mathfrak{t}_0=[\tilde{\mathfrak{g}}_{-1}, \mathfrak{g}_1]$, so $\mathfrak{g}_0=\mathfrak{r}_0\oplus[\tilde{\mathfrak{g}}_{-1}, \mathfrak{g}_1]\oplus\mathfrak{g}''_0$.

Q. E. D

As a corollary of Proposition 2.5, we have

COROLLARY 2. 5. *Let $\mathfrak{g}=\oplus_{i\in\boldsymbol{Z}}\mathfrak{g}_i$ be a finite dimensional transitive $\boldsymbol{Z}$-graded Lie algebra over $\boldsymbol{C}$. Then following statements are equivalent.*

(1) $\mathfrak{g}$ *is semisimple.*

(2) (i) *The representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is completely reducible, and* (ii) *there is no non-trivial $\mathfrak{g}_0$-invariant subspace of $\mathfrak{g}_{-1}$ contained in the centralizer of $\mathfrak{g}_1$ in $\mathfrak{g}$.*

(3) (i) *The representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is completely reducible, and* (ii) $\mathfrak{g}'_{-1}=\{0\}$.

PROOF. Let W be a $\mathfrak{g}_0$-invariant subspace of $\mathfrak{g}_{-1}$ such that $[W, \mathfrak{g}_1]=\{0\}$. Then an ideal in $\mathfrak{g}$ generated by W is a solvable ideal in $\mathfrak{g}$; thus (1)$\longrightarrow$(2) (ii). It follows from I. L. Kantor ([3], page 44, Proposition 12) that (1) $\longrightarrow$(2) (i). If (3) holds, then $\tilde{\mathfrak{g}}_{-1}=\mathfrak{g}_{-1}$. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, we have $\tilde{\mathfrak{g}}_-=\mathfrak{g}_-$. Moreover, since $\tilde{\mathfrak{g}}$ is semisimple by Proposition 2. 5., we have $\mathfrak{r}_i=\{0\}(i\leqq -1)$, so, by transitivity, $\mathfrak{r}_p=\{0\}(p\geqq 0)$. Hence $\mathfrak{g}$ is semisimple, which proves (3)$\rightarrow$(1). (2)$\rightarrow$(3) is clear. Q. E. D

§ 3. Pseudo-product graded Lie algebras

3. 1. Let $\mathfrak{g}=\oplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ be a transitive $\boldsymbol{Z}$-graded Lie algebra over K, where K is $\boldsymbol{R}$ or $\boldsymbol{C}$. Let $\mathfrak{g}^{(1)}_{-1}$ and $\mathfrak{g}^{(2)}_{-1}$ be subspaces of $\mathfrak{g}_{-1}$. Then the $\boldsymbol{Z}$-graded Lie algebra $\mathfrak{g}=\oplus_{p\in\boldsymbol{Z}}\mathfrak{g}_p$ is called a *pseudo-product graded Lie algebra*, if it satisfies the following conditions

(3.1.1) $\mathfrak{g}_{-1}=\mathfrak{g}^{(1)}_{-1}\oplus\mathfrak{g}^{(2)}_{-1}$

(3.1.2) $[\mathfrak{g}^{(1)}_{-1}, \mathfrak{g}^{(1)}_{-1}]=[\mathfrak{g}^{(2)}_{-1}, \mathfrak{g}^{(2)}_{-1}]=\{0\}$

(3.1.3) $[\mathfrak{g}_0, \mathfrak{g}^{(1)}_{-1}]\subset\mathfrak{g}^{(1)}_{-1}$, $[\mathfrak{g}_0, \mathfrak{g}^{(2)}_{-1}]\subset\mathfrak{g}^{(2)}_{-1}$

Then we have the following

LEMMA 3. 1. (see N. Tanaka [6], page 292) *Let $\mathfrak{g}=\oplus_{p\in\boldsymbol{Z}}\ \mathfrak{g}_p$ be a* *pseudo-product graded Lie algebra over K. If* $\mathfrak{g}_-$ *is non-degenerate (that is, for any* $x\in\mathfrak{g}_{-1}$, $[x, \mathfrak{g}_{-1}]=\{0\}$ *implies* $x=0$), *then the Lie algebra* $\mathfrak{g}$ *is finite dimensional.*

3. 2. Now we give our main theorem.

THEOREM 3. 2. *Let* $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}} \mathfrak{g}_p$ *be a pseudo-product graded Lie algebra over* $\boldsymbol{C}$, *and suppose that the subalgebra* $\mathfrak{g}_-$ *of* $\mathfrak{g}$ *is non-degenerate and the representations of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_{-1}^{(1)}$ *and* $\mathfrak{g}_{-1}^{(2)}$ *are irreducible. Then we have*:

(i) *If the representation of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_1$ *is reducible, then* $\mathfrak{g}$ *is a simple Lie algebra.*

(ii) *If the representation of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_1$ *is irreducible, then we have* $\mathfrak{g}_2=\{0\}$.

As a consequence of (i) *and* (ii), *if* $\mathfrak{g}_2\neq\{0\}$, *then* $\mathfrak{g}$ *is a simple Lie algebra.*

PROOF. By Lemma 3. 1, we can apply all arguments in § 2.

(i) Using the notation of (2. 1. 1), we have $\mathfrak{g}'_{-1}=\{0\}$ by Lemma 2. 4. Hence, by Corollary 2. 5, $\mathfrak{g}$ is semisimple. If $\mathfrak{g}$ is not simple, then we have $\mathfrak{g}=\mathfrak{a}^{(1)}\oplus\mathfrak{a}^{(2)}$, where $\mathfrak{a}^{(i)}(i=1, 2)$ is a non-trivial semisimple ideal in $\mathfrak{g}$. Since $\mathfrak{a}^{(i)}$ is graded, we can write $\mathfrak{a}^{(i)}=\bigoplus_{p\in \boldsymbol{Z}} \mathfrak{a}_p^{(i)}$. Here we remark that $\mathfrak{a}_{-1}^{(i)}\neq\{0\}$ because of transitivity of $\mathfrak{g}$. Since $\mathfrak{g}_{-1}^{(1)}$ is not isomorphic to $\mathfrak{g}_{-1}^{(2)}$ as a $\mathfrak{g}_0$-module by Lemma 2. 4, we have $\mathfrak{g}_{-1}^{(1)}=\mathfrak{a}_{-1}^{(1)}$, $\mathfrak{g}_{-1}^{(2)}=\mathfrak{a}_{-1}^{(2)}$ or $\mathfrak{g}_{-1}^{(1)}=\mathfrak{a}_{-1}^{(2)}$, $\mathfrak{g}_{-1}^{(2)}=\mathfrak{a}_{-1}^{(1)}$. Thus $[\mathfrak{g}_{-1}^{(1)}, \mathfrak{g}_{-1}^{(2)}]=\{0\}$, which is a contradiction to the fact that $\mathfrak{g}_-$ is non-degenerate. Hence $\mathfrak{g}$ is a simple Lie algebra.

(ii) Now we can assume that $[\mathfrak{g}_{-1}^{(1)}, \mathfrak{g}_1]=\{0\}$. Then the subalgebra $\mathfrak{g}_{-1}^{(2)}\oplus[\mathfrak{g}_{-1}^{(2)}, \mathfrak{g}_1]\oplus\mathfrak{g}_1$ is a simple Lie algebra by Proposition 2. 5, so $[\mathfrak{g}_1, \mathfrak{g}_1]=\{0\}$. Therefore it follows from Proposition 2. 5 that $\mathfrak{g}_2=\{0\}$. Q. E. D

3. 3. We now prove the real version of Theorem 3. 2.

THEOREM 3. 3. *Let* $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}} \mathfrak{g}_k$ *be a pseudo-product graded Lie algebra over* $\boldsymbol{R}$, *and suppose that its subalgebra* $\mathfrak{g}_-$ *of* $\mathfrak{g}$ *is non-degenerate and the representation of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_{-1}^{(1)}$ *and* $\mathfrak{g}_{-1}^{(2)}$ *is irreducible. Then we have*:

(i) If the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is reducible, then $\mathfrak{g}$ is a simple Lie algebra.

(ii) If the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is irreducible, then we have $\mathfrak{g}_2=\{0\}$.

As a consequence of (i) and (ii), if $\mathfrak{g}_2\neq\{0\}$, then $\mathfrak{g}$ is a simple Lie algebra over $\boldsymbol{R}$.

PROOF. First of all, we note that $\mathfrak{g}$ is finite dimentional by Lemma 3. 1. Let $\mathfrak{g}^{\boldsymbol{C}}=\bigoplus_{p\in \boldsymbol{Z}} \mathfrak{g}_p^{\boldsymbol{C}}$ denote the complexification of $\mathfrak{g}=\bigoplus_{p\in \boldsymbol{Z}} \mathfrak{g}_p$. Then $\mathfrak{g}^{\boldsymbol{C}}$ is a transitive $\boldsymbol{Z}$-graded Lie algebra over $\boldsymbol{C}$. Then, since $\mathfrak{g}_{-1}^{(1)\boldsymbol{C}}$ is not isomorphic to $\mathfrak{g}_{-1}^{(2)\boldsymbol{C}}$ as a $\mathfrak{g}_0^{\boldsymbol{C}}$ module by Lemma 2. 4, $\mathfrak{g}_{-1}^{(1)}$ is not isomorphic to $\mathfrak{g}_{-1}^{(2)}$ as a

$\mathfrak{g}_0$-module. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Then $\mathfrak{r}$ is a $\boldsymbol{Z}$-graded ideal in $\mathfrak{g}$, which we write $\mathfrak{r}=\bigoplus_{p\in Z}\mathfrak{r}_p$. Moreover its complexification $\mathfrak{r}^C$ is the radical of $\mathfrak{g}^C$. Then $\mathfrak{r}_{-1}$ is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$, so we have $\mathfrak{r}_{-1}=\mathfrak{g}_{-1}^{(1)}$ or $\mathfrak{r}_{-1}=\mathfrak{g}_{-1}^{(2)}$ or $\mathfrak{r}_{-1}=\{0\}$. Here we remark that $\mathfrak{g}_{-1}^C/\mathfrak{r}_{-1}^C$ is contragredient to $\mathfrak{g}_1^C$ as a $\mathfrak{g}_0^C$ module by Proposition 2. 5, so $\mathfrak{g}_{-1}/\mathfrak{r}_{-1}$ is contragredient to $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module. If $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is reducible, then $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}/\mathfrak{r}_{-1}$ is reducible. Hence we have $\mathfrak{r}_{-1}=\{0\}$. By Corollary 2. 5, $\mathfrak{g}^C$ is semisimple, so $\mathfrak{g}$ is semisimple. Then, by the same method of proof of theorem 3. 2, we can prove the fact that $\mathfrak{g}$ is simple. This proves (i). If $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is irreducible, then we have $\mathfrak{g}_{-1}^{(1)}=\mathfrak{r}_{-1}$ or $\mathfrak{g}_{-1}^{(2)}=\mathfrak{r}_{-1}$. Now we suppose that $\mathfrak{g}_{-1}^{(2)}=\mathfrak{r}_{-1}$. Then, by Proposition 2. 5, the subalgebra generated by $\mathfrak{g}_{-1}^{(1)C}\oplus[\mathfrak{g}_{-1}^{(1)C}, \mathfrak{g}_1^C]\oplus\mathfrak{g}_1^C$ is semisimple. However, since $[\mathfrak{g}_{-1}^{(1)C}, \mathfrak{g}_{-1}^{(1)C}]=\{0\}$, we have $[\mathfrak{g}_1^C, \mathfrak{g}_1^C]=\{0\}$. Hence, by Proposition 2. 5, we have $\mathfrak{g}_2^C=\{0\}$, so $\mathfrak{g}_2=\{0\}$. Similarly we can prove that $\mathfrak{g}_2=\{0\}$ when $\mathfrak{g}_{-1}^{(1)}=\mathfrak{r}_{-1}$. This proves (ii). Q. E. D

Reference

[1] V. G. KAC: Simple irreducible graded Lie algebras of finite growth, Math USSR-Izvestija (1968) 1271-1311.

[2] N. BOURBAKI: Groupes et algèbres de Lie, Chaps. 7, 8, Diffusion C. C. L. S Paris (1975).

[3] I. L. KANTOR: Some generalizations of Jordan algebra, Trudy. sem. tenzor. anal. 17 (1974) 250-313 (in Russian)

[4] N. TANAKA: On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. 10 (1970) 1-82.

[5] N. TANAKA: On the equivalence problems associated with simple graded Lie algebras, Hokkaido Math. J. 8 (1979)23-84.

[6] N. TANAKA: On affine symmetric spaces and the automorphism groups of product manifolds, Hokkaido Math. J. 14 (1985) 277-351.

[7] O. MATHIEU: Classification des algèbras de Lie graduéss simples de croissance $\leqq 1$, Invent. math. 86 (1986) 371-426.

Department of Mathematics
Hokkaido University