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On geometric spectral radius of commuting n-tuples of operators

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Dedicated to Professor George Maltese on his 60th birthday

Abstract. The aim of this paper is to prove that for most classical joint spectra as e.g. the Taylor spectrum, the Harte spectrum, Słodkowski spectra, also left and right spectrum, the joint approximate point spectrum, and some other spectroids, the geometric spectral radius is the same and depends only upon a commuting n-tuple of operators. We generalize also this result by showing that for all these spectra or subspectra their convex envelopes coincide.

1. Definitions and notation

Let X be a complex Banach space. Denote by $B(X)$ the algebra of all continuous linear operators on X. Put $B_{\mathrm{com}}^{n}(X)$ for the set of all n-tuples of commuting operators in $B(X)$ and put $B_{\mathrm{com}}(X)=\bigcup_{n=1}^{\infty}B_{\mathrm{com}}^{n}(X)$, in particular $B(X)$ identified with $B_{\mathrm{com}}^{1}(X)$ is a subset of $B_{\mathrm{com}}(X)$. Suppose that to each n-tuple $(T_1, \ldots, T_n)$ in $B_{\mathrm{com}}(X)$ there corresponds a subset $\sigma_s(T_1, \ldots, T_n)\subset \boldsymbol{C}^n$, and consider the following axioms (it is a small modification of axioms given by the second author in [11]).

(i) $\sigma_s(T_1, \ldots, T_n)$ is a non-void compact subset of $\boldsymbol{C}^n$, for all $(T_1, \ldots, T_n)\in B_{\mathrm{com}}(X)$,

(ii) $\sigma_s(T_1, \ldots, T_n)\subset\prod_{i=1}^{n}\sigma(T_i)$, where $\sigma(T)$ denotes the usual spectrum of an operator T in $B(X)$, $(T_1, \ldots, T_n)\in B_{\mathrm{com}}(X)$. In particular, for a single operator T we have

$$\sigma_s(T)\subset\sigma(T).$$

The next axiom is the equality in the above formula

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(iii) $\sigma_s(T)=\sigma(T)$

for all T in $B(X)$.

Let $p_{n,k}$ be a polynomial map from $\boldsymbol{C}^n$ to $\boldsymbol{C}^k$, i.e. a map given by the formula

$$p_{n,k}(z_1, \dots, z_n)=(p_1(z_1, \dots, z_n), \dots, p_k(z_1, \dots, z_n)),$$

wher p_i are polynomials in n complex variables. Such a polynomial map induces a map (denoted also by $p_{n,k}$) from $B^n_{\text{com}}(X)$ to $B^k_{\text{com}}(X)$ given by $(T_1, \dots, T_n) \longrightarrow (p_1(T_1, \dots, T_n), \dots, p_k(T_1, \dots, T_n))$.
The fourth axiom is given as the equality

(iv) $\sigma_s(p_{n,k}(T_1, \dots, T_n))=p_{n,k}(\sigma_s(T_1, \dots, T_n))\subset \boldsymbol{C}^k$,

for all polynomial maps $p_{n,k}$ and all n-tuples $(T_1, \dots, T_n)$ in $B^n_{\text{com}}(X)$ $n=1, 2, \dots$. It is the most essential axioms concerning the map σ_s and it is called the spectral mapping property.

The last axiom gives so called translation property of σ_s, it is a particular case of the axiom (iv).

(v) $\sigma_s(T_1+\alpha_1 I, \dots, T_n+\alpha_n I)=\sigma_s(T_1, \dots, T_n)+(\alpha_1, \dots, \alpha_n)$,

for all n-tuples in $B_{\text{com}}(X)$ and all points in $\boldsymbol{C}^n$, $n=1, 2, \dots$

DEFINITION. A (joint) spectrum on X is a map σ_s from $B_{\text{com}}(X)$ to subsets of $\bigcup_{n=1}^{\infty} \boldsymbol{C}^n$ satisfying axioms (i)-(iv), and consequently also the axiom (v). A subspectrum is a map satisfying axioms (i), (ii) and (iv). A spectroid is a map satisfying axioms (i), (ii) and (v).

Thus every spectrum is a subspectrum and every subspectrum is a spectroid.

EXAMPLES OF SPECTRA: the Taylor spectrum σ_T ([3], [4], [5], [6], [8], [9], [11]), the Harte spectrum σ_H ([4], [5], [7], [11]), the double sequence or Słodkowski spectra $\sigma_{s,j,k}$, j, $k=0, 1, 2, \dots$ ([6], Słodkowski denotes these spectra by $\sigma_{\pi,j}\cup\sigma_{\delta,k}$). By the way, Słodkowski in [6] proved that all these spectra $\sigma_{s,j,k}$ are different only in the Hilbert space situation (dim $(\mathfrak{H})=\infty$). It is not known, whether every infinite dimensional Banach space has an infinite family of different spectra. Examples of subspectra which is not spectra: the left spactrum σ_l and the right spectrum σ_r ([5], [7], [11]), the joint approximate point spectrum σ_π and the joint defect spectrum σ_δ ([5], [7], [11]), many kinds of essential spectra ([5]). Examples of spectroids which are not subspectra: the commutant spectrum σ' and the bicommutant spectrum σ'' ([5], [7], [11]). All spectroids in above examples are difined on all Banach spaces. There are some relations among the above spectroids. For instance we have (by the definition of the Harte spectrum)

(1) $\sigma_H = \sigma_l \cup \sigma_r$

in the sense that for each n-tuple $(T_1, \dots, T_n)$ in $B_{\mathrm{com}}(X)$ it is

$$\sigma_H(T_1, \dots, T_n) = \sigma_l(T_1, \dots, T_n) \cup \sigma_r(T_1, \dots, T_n).$$

Also

(2) $\sigma_{s,0,0} = \sigma_n \cup \sigma_\delta,$

and

(3) $\sigma_\pi \subset \sigma_l.$

For a spectroid σ_s and an n-tuple $(T_1, \dots, T_n)$ in $B_{\mathrm{com}}(X)$ we define the geometric spectral radius of $(T_1, \dots, T_n)$ relative to σ_s by the formula

(4) $r_{\sigma_s}(T_1, \dots, T_n) = \max \{|z| : z \in \sigma_s(T_1, \dots, T_n)\},$

where $|z| = |(z_1, \dots, z_n)| = (\sum_{i=1}^{n} |z_i|^2)^{1/2}$. We have choosen the term "geometric spectral radius" in order to distinguish it from the "algebraic spactral radius" considered in the paper [1] in the case when X is a Hilbert space.

The main result of this paper states that the geometric spectral radius relative to a spectrum is in fact independent of this spectrum and equals to the geometric spectral radius of the joint approximate point spectrum as well as to the spectral radii of some other spectroids (we omit in the sequel the word "geometric"). Using this result we will show that the convex envelopes of all spectra as well as of some subspectra coincide. This is a more general result. Our result says that for all spectra and some other spectroids the set conv $\sigma_s(T_1, \dots, T_n)$ does not depend upon σ_s for a given n-tuple $(T_1, \dots, T_n) \in B_{\mathrm{com}}(X)$.

2. Theorems on spectral radii and convex envelopes

Our main result reads as follows.

THEOREM 1. *Let X be a complex Banach space and let σ_0 be a spectrum on X. Then for any n-tuple $(T_1, \dots, T_n)$ of commuting operators in $B(X)$ we have*

(5) $r_{\sigma_0}(T_1, \dots, T_n) = r_{\sigma_\pi}(T_1, \dots, T_n),$

in particular, $r_{\sigma_0}(T_1, \dots, T_n)$ does not depend upon σ_0.
The proof will follow Lemma 2 and Proposition 3.

LEMMA 2. *Let X be a complex Banach space and σ_0 be a subspectrum on X. Let $(T_1, \dots, T_n)$ be in $B_{\mathrm{com}}(X)$. Then there exists an operator T_0 such that $r_{\sigma_0}(T_1, \dots, T_n)=r_{\sigma_0}(T_0)$ and $r_{\sigma_s}(T_0)\leq r_{\sigma_s}(T_1, \dots, T_n)$ for any subspectrum σ_s.*

PROOF. If $r_{\sigma_0}(T_1, \dots, T_n)=0$, we can put $T_0=0$. We assume that $r_{\sigma_0}(T_1, \dots, T_n) > 0$. choose $z^{(0)}\in\sigma_0(T_1, \dots, T_n)$ such that $|z^{(0)}|=r_{\sigma_0}(T_1, \dots, T_n)$. Consider the orthogonal projection of $\boldsymbol{C}^n$ onto $\boldsymbol{C}$, given by the formula $P(z)=\frac{1}{|z^{(0)}|}\sum_{i=1}^{n} z_i\cdot \bar{z}_i^{(0)}$ for $z=(z_1, \dots, z_n)$. It projects a ball centered at origin, with any radius r onto a disk centered at origin with the same radius and $|P(z^{(0)})|=r_{\sigma_0}(T_1, \dots, T_n)$. Put $T_0=P(T_1, \dots, T_n)$. By the spectral mapping property (iv) and the above property of P, it follows that T_0 has the desired properties.

PROPOSITION 3. *With the same notation as in Lemma 2, if for any single operator T, $\partial\sigma(T)\subset\sigma_0(T)$, then $r_{\sigma_\pi}(T_1, \dots, T_n)=r_{\sigma_0}(T_1, \dots, T_n)$.*

PROOF. By Lemma 2, we have that there exists an operator T_0 and $r_{\sigma_0}(T_1, \dots, T_n)=r_{\sigma_0}(T_0)\leq r(T_0)=r_{\sigma_\pi}(T_0)\leq r_{\sigma_\pi}(T_1, \dots, T_n)$. Applying Lemma 2 to σ_π, there exists an operator T_π such that $r_{\sigma_\pi}(T_1, \dots, T_n)=r_{\sigma_\pi}(T_\pi)$ and $r_{\sigma_s}(T_\pi)\leq r_{\sigma_s}(T_1, \dots, T_n)$ for any subspectrum σ_s. Then we have

$$r_{\sigma_\pi}(T_1, \dots, T_n)=r_{\sigma_\pi}(T_\pi)=r(T_\pi)=r_{\sigma_0}(T_\pi)\leq r_{\sigma_0}(T_1, \dots, T_n)$$

and hence $r_{\sigma_\pi}(T_1, \dots, T_n)=r_{\sigma_0}(T_1, \dots, T_n)$.

The common value of the left hand side of (5) we denote from now on by $r(T_1, \dots, T_n)$. From Proposition 7 in [1], it follows that for every commuting n-tuple of operators the algebraic spectral radius is not smaller than the geometric one (in the case when X is a Hilbert space). It is not known whether both radii coincide, for all commuting n-tuples.

For $(T_1, \dots, T_n)\in B_{\mathrm{com}}(X)$, let A be a commutative closed subalgebra of $B(X)$ containing the operators I, $T_1, \dots, T_n$. Then the point $z=(z_1, \dots, z_n)\in\boldsymbol{C}^n$ is in $\sigma_A(T_1, \dots, T_n)$ if and only if for all $S_1, \dots, S_n$ in A

$$\sum_{i=1}^{n} S_i(T_i-z_i) \neq I.$$

PROPOSITION 4. *For the commutant and bicommutant spectra we have*

$$r_{\sigma'}(T_1, \dots, T_n)=r_{\sigma''}(T_1, \dots, T_n)=r(T_1, \dots, T_n),$$

where $(T_1, \dots, T_n) \in B_{\mathrm{com}}(X)$.

PROOF. We use the following well known relations

(6) $\sigma_\pi(T_1, \dots, T_n) \subset \sigma'(T_1, \dots, T_n) \subset \sigma''(T_1, \dots, T_n) \subset \sigma_A(T_1, \dots, T_n)$,

where $A = A(T_1, \dots, T_n)$ is the smallest closed subalgebra of $B(X)$ containing the operators $T_1, \dots, T_n$. Similarly as Lemma 2 we find a T_0 in A with $r(T_0) = r_{\sigma_A}(T_0) = r_{\sigma_A}(T_1, \dots, T_n)$. We cannot use Theorem 1, since σ_A is not always a spectroid; it is defined only on k-tuples of operators belonging to A. Thus there is $z_0 \in \sigma(T_0)$ with $r(T_0) = |z_0|$. Clearly $z_0 \in \partial\sigma(T_0)$ and so $z_0 \in \sigma_\pi(T_0)$. Since $T_0 = P(T_1, \dots, T_n)$ for some projection P, we infer by the spectral mapping property of σ_π that $z_0 = P(z_1, \dots, z_n)$ for some $(z_1, \dots, z_n) \in \sigma_\pi(T_1, \dots, T_n)$. We have $|z_0| \le (\sum_{i=1}^{n} |z_i|^2)^{1/2}$ and so $|z_0| \le r_{\sigma_\pi}(T_1, \dots, T_n)$. Together with relation (6) we obtain the conclusion.

For any commutative closed subalgebra A of $B(X)$ with identity we denote the set of all non-zero multiplicative linear functionals on A by $\mathfrak{M}(A)$. Then by Theorem 1 and relation (6) we obtain the following

COROLLARY 5. *For any commuting n-tuple of operators* $(T_1, \dots, T_n)$ *in* $B_{\mathrm{com}}(X)$ *the spectral radius* $r(T_1, \dots, T_n)$ *is given by the following formula*

$$r(T_1, \dots, T_n) = max\ \{(\sum_{i=1}^{n} |f(T_i)|^2)^{1/2} : f \in \mathfrak{M}(A)\},$$

where A *is any closed commutative subalgebra of* $B(X)$ *containing the operators* $I, T_1, \dots, T_n$. *In particular it can be the algebra* $A(T_1, \dots, T_n)$ *used in the proof of Proposition* 4.

PROPOSITION 6. *For any n-tuple* $(T_1, \dots, T_n)$ *in* $B_{\mathrm{com}}(X)$ *we have*

$$r_{\sigma_\delta}(T_1, \dots, T_n) = r_{\sigma_r}(T_1, \dots, T_n) = r(T_1, \dots, T_n).$$

PROOF. Using a well known relation $\sigma_\delta(T_1, \dots, T_n) \subset \sigma_r(T_1, \dots, T_n)$ we obtain

(7) $r_{\sigma_\delta}(T_1, \dots, T_n) \le r_{\sigma_r}(T_1, \dots, T_n)$

for all n-tuples $(T_1, \dots, T_n)$ in $B_{\mathrm{com}}(X)$. The conclusion follows now from Proposition 3 and the well known fact that for a single operator T we have $\partial\sigma(T) \subset \sigma_\delta(T)$.

PROPOSITION 7. *For any n-tuple* $(T_1, \dots, T_n)$ *in* $B_{\mathrm{com}}(X)$ *we have*

$$r_{\sigma_1}(T_1, \dots, T_n) = r(T_1, \dots, T_n).$$

The proof follows immediately from the formulas (1) and (3).

It can be easily seen that for many so called essential spectra their geometrical spectral radius is smaller for some commuting n-tuples of operators than the radius r.

We shall consider now the convex envelopes for some spectroids. We say that a spectroid σ_s is in the class $\sum_0$ if for every $(T_1, \ldots, T_n)$ in $B_{\mathrm{com}}(X)$ we have

$$(8)\qquad r_{\sigma_s}(T_1, \ldots, T_n)=r(T_1, \ldots, T_n).$$

By the above results the class $\sum_0$ contains all spectra as well as certain subspectra (σ_π, σ_δ, σ_r, σ_l) and certain spectroids (σ', σ'').

Our result reads as follows.

THEOREM 8. *Let σ_1 and σ_2 be spectroids of the class $\sum_0$. Then for every commuting n-tuple of operators $(T_1, \ldots, T_n)$ we have*

$$conv\ \sigma_1(T_1, \ldots, T_n)=conv\ \sigma_2(T_1, \ldots, T_n).$$

PROOF. Fix an n-tuple $(T_1, \ldots, T_n)\in B_{\mathrm{com}}(X)$ and take any closed ball $B(z^{(0)}, r)\subset \boldsymbol{C}^n$, with center $z^{(0)}=(z_1^{(0)}, \ldots, z_n^{(0)})$ and radius r, which contains $\sigma_1(T_1, \ldots, T_n)$. Applying the translation property (v) to the spectroid σ_1 we obtain

$$\sigma_1(T_1\text{-}z_1^{(0)}\cdot I, \ldots, T_n\text{-}z_n^{(0)}\cdot I)\subset B(0, r).$$

Thus $r\ (T_1\text{-}z_1^{(0)}\cdot I, \ldots, T_n\text{-}z_n^{(0)}\cdot I)\leq r$ and so

$$(9)\qquad \sigma_2(T_1\text{-}z_1^{(0)}\cdot I, \ldots, T_n\text{-}z_n^{(0)}\cdot I)\subset B(0, r).$$

Applying again the translation property (v) to the spectrum σ_2 we obtain by (9)

$$\sigma_2(T_1, \ldots, T_n)\subset B(z^{(0)}, r).$$

We have shown that any ball $B(z^{(0)}, r)$ containing $\sigma_1(T_1, \ldots, T_n)$ must also contain $\sigma_2(T_1, \ldots, T_n)$. Using an obvious fact, that the convex envelope of a compact set in $\boldsymbol{C}^n$ equals to the intersection of all closed balls containing this set, we obtain

$$\sigma_2(T_1, \ldots, T_n)\subset \mathrm{conv}\ \sigma_1(T_1, \ldots, T_n).$$

Interchanging the role of σ_1 and σ_2 we obtain an opposite inclusion. The conclusion follows.

Let S be a compact set in $\boldsymbol{C}^n$, its convex envelope conv S is also compact and equals to the convex envelope of all its extreme points.

Moreover these extreme points must belong to S. For a spectroid σ_s on a Banach space X and for $(T_1, \dots, T_n) \in B_{\mathrm{com}}(X)$ denote by $E_{\sigma_s}(T_1, \dots, T_n)$ the set of all extreme points of the set $\sigma_s(T_1, \dots, T_n)$ (=extreme points of conv $\sigma_s(T_1, \dots, T_n)$). Under this notation we have

COROLLARY 9. *Let σ_1 and σ_2 be spectroids of the class Σ_0. Then for every commuting n-tuple of operators $(T_1, \dots, T_n)$ we have*

$$E_{\sigma_1}(T_1, \dots, T_n) = E_{\sigma_2}(T_1, \dots, T_n),$$

in particular the set $E_{\sigma_s}(T_1, \dots, T_n)$ depends only upon $T_1, \dots, T_n$ and is independent of σ_s whenever it belongs to Σ_0.

In defining our geometrical spectral radius we were using the Euclidean norm in $\boldsymbol{C}^n$. However, our theorem 8 implies that we can use any other norm, for instance an l_p-norm. Thus we have

COROLLARY 10. *Suppose we have on each $\boldsymbol{C}^n$ a norm $\|\cdot\|_n$ (it is automatically equivalent to the Euclidean norm) and we define spectral radius $\rho_\sigma(T_1, \dots, T_n)$ of a commuting n-tuple of operators as the minimal radius r, such that the ball $\{z \in \boldsymbol{C}^n : \|z\|_n \leq r\}$ contains $\sigma(T_1, \dots, T_n)$. Then for each spectroid σ in Σ_0 the spectral radius $\rho_\sigma(T_1, \dots, T_n)$ does not depend upon σ.*

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Added in proof. The problem of Bunce mentioned before the proposition 7 was racently solved in positive by V. Miiller and A. Sołtysiak " Spectral radius formula for commuting Hilbert space operators."

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