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A note on duality of first order partial differential equations

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0. Introduction

The dual relationships among the equations are given by the Legendre transformation in the classical theory of differential equations and it is useful to solve some type of differential equations [**5**]. However situations are not so clear in the classical theory as usual. So Izumiya establishes the principle of duality among first order ordinary differential equations with complete solutions [**6**]. Our purpose in this note is to generalize the result for systems of first order partial differential equations with complete solutions.

The geometrical theory of first order partial differential equations is described natually in the context of contact geometry, which can be considered as a generalization of projective geometry ([**1**], [**2**]). A particular aspect of projective geometry is the principle of duality. So we may expect that some type of duality holds also among first order partial differential equations.

In § 1 we shall prepare some basic notions and construct the framework. In § 2 we shall establish the principle of duality among pairs of completely integrable system of first order partial differential equations and its complete solution. In the special case of holonomic systems of first order partial differential equations, we can assert a more strong result (i.e. the principle of duality among completely integrable holonomic systems of first order partial differential equations themselves), which is discussed in § 3.

All arguments should be understood locally and all maps considered here are differentiable of class C^{∞}.

1. Basic notions

In this section we shall state our basic notions. *A system of partial differential equations of first order* (or briefly, *an equation*) is a submersion germ $F:(J^1(\boldsymbol{R}^n,\boldsymbol{R}), z_0)\rightarrow(\boldsymbol{R}^d, 0)$, $1\leq d\leq n$, on the 1-jet space of functions of n-variables. If $d=n$, we call it *an holonomic system of par-*

tial differential equations of first order (or briefly, *an holonomic equation*). Let θ be the canonical contact form on $J^1(\boldsymbol{R}^n, \boldsymbol{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$, where (x, y, p) are canonical coordinates of $J^1(\boldsymbol{R}^n, \boldsymbol{R})$. We define *a geometric solution of* $F=0$ to be a Legendrian immersion germ $i: (L, q_0) \to (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$ such that $i(L) \subset F^{-1}(0)$, where *the Legendrian immersion germ* is an immersion $i: (L, q_0) \to J^1(\boldsymbol{R}^n, \boldsymbol{R})$ of an n-dimensional manifold such that $i^*\theta = 0$. For a Legendrian immersion germ $i: (L, q_0) \to (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$, $q_0 \in L$ is said to be *a Legendrian singular point* if $\pi \circ i$ is not an immersion at q_0, where $\pi(x, y, p) = (x, y)$. We remark that q_0 is a Legendrian non-singular point if and only if $\tilde{\pi} \circ i$ is a local diffeomorphism at q_0, where $\tilde{\pi}(x, y, p) = x$.

An equation $F=0$ is *completely integrable at* z_0 if there exists an immersion germ $f: (\boldsymbol{R}^{n-d+1} \times \boldsymbol{R}^n, 0) \to (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$ such that *Image* $f \subset F^{-1}(0)$ and $f|\{c\} \times \boldsymbol{R}^n$ is a Legendrian immersion germ for any $c \in (\boldsymbol{R}^{n-d+1}, 0)$. In this case f is calld *a complete solution of* $F=0$ *at* z_0. An equation $F=0$ is *completely integrable at* z_0 *in the classical sense* if there exists a function germ $f: (\boldsymbol{R}^{n-d+1} \times \boldsymbol{R}^n, (c_0, x_0)) \to (\boldsymbol{R}, y_0)$ such that $F(x, f(c, x), \partial f/\partial x(c, x)) = 0$ and $rank(\partial f/\partial c_i, \partial^2 f/\partial c_i \partial x_j) = n-d+1$.

Then we have the following lemma.

Lemma 1.1 ([7]). *For an equation* $F: (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0) \to (\boldsymbol{R}^d, 0)$, *the followings are equivalent.*

(1) $F=0$ *is completely integrable at* z_0 *in the classical sense.*
(2) $F=0$ *has a complete solution* f *at* z_0 *such that* $f|\{0\} \times \boldsymbol{R}^n$ *is Legendrian non-singular.*

2. The principle of duality

We use the following generalized Legendre transformation [4] to express the principle of duality.

Let $(x, y, p) = (x_1, \cdots, x_n, y, p_1, \cdots, p_n)$ be a coordinate system of $J^1(\boldsymbol{R}^n, \boldsymbol{R})$ with the contact structure given by $\theta = dy - \sum_{i=1}^n p_i dx_i$. We adopt another coordinate system $(X, Y, P) = (X_1, \cdots, X_n, Y, P_1, \cdots, P_n)$ of $J^1(\boldsymbol{R}^n, \boldsymbol{R})$ whose contact structure is given by $\Theta = dY - \sum_{i=1}^n P_i dX_i$. For any subset I of $\{1, \cdots, n\}$, we now define a diffeomorphism $L_I: J^1(\boldsymbol{R}^n, \boldsymbol{R}) \to J^1(\boldsymbol{R}^n, \boldsymbol{R})$ by

$$X_{I^c} = x_{I^c},\ X_I = p_I,\ Y = p_I x_I - y,\ P_{I^c} = -p_{I^c},\ P_I = x_I,$$

where $I^c = \{1, \cdots, n\} \backslash I$, $x_I = \{x_i \mid i \in I\}$, $X_I = x_I$ means that $X_i = x_i$ for any $i \in I$, $p_I \cdot x_I = \sum_{i \in I} p_i x_i$ and $(x_1, \cdots, x_n) = (x_I, x_{I^c}) = (x_{I^c}, x_I)$. We call L_I *the*

Legendre I-transformation. It is easy to see that L_I is a contact diffeomorphism $L_I:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),\theta)\to(J^1(\boldsymbol{R}^n,\boldsymbol{R}),\Theta)$ and $L_{\{1,\cdots,n\}}$ is the classical Legendre transformation.

Let $F:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),z_0)\to(\boldsymbol{R}^d,0)$ be an equation. If we apply the Legendre I-transformation to our equation, we obtain a new equation

$$F_I=F\circ(L_I)^{-1}:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),Z_0)\to(\boldsymbol{R}^d,0),\ Z_0=L_I(z_0),$$

in the new coordinate system (X,Y,P).

Then we have the following simple lemma.

Lemma 2.1 *Let $i:(L,q_0)\to(J^1(\boldsymbol{R}^n,\boldsymbol{R}),z_0)$ be a geometric solution of $F=0$. Then $L_I\circ i:(L,q_0)\to(J^1(\boldsymbol{R}^n,\boldsymbol{R}),Z_0)$, $Z_0=L_I(z_0)$, is a geometric solution of $F_I=0$.*

Proof. Since L_I is a contact diffeomorphism and $F_I^{-1}(0)=L_I(F^{-1}(0))$, it is easy to see that $L_I\circ i$ is a geometric solution of $F_I=0$. Q. E. D.

Let d be an integer such that $1\leq d\leq n$. We denote $CI^{(n,d)}(z_0)$ the set of completely integrable equations $F:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),z_0)\to(\boldsymbol{R}^d,0)$. We also denote $CS^{(n,d)}(z_0)$ the set of pairs (F,Γ_F), where $F\in CI^{(n,d)}(z_0)$ and $\Gamma_F:(\boldsymbol{R}^{n-d+1}\times\boldsymbol{R}^n,0)\to(J^1(\boldsymbol{R}^n,\boldsymbol{R}),z_0)$ is a complete solution of $F=0$ at z_0. We set $\Pi:CS^{(n,d)}(z_0)\to CI^{(n,d)}(z_0)$ by $\Pi((F,\Gamma_F))=F$. We denote $\Gamma_F(c,t)=(x^F(c,t),y^F(c,t),p^F(c,t))$ for $(c,t)\in(\boldsymbol{R}^{n-d+1}\times\boldsymbol{R}^n,0)$.

Then we define $2^{2^n}-1$ subsets of $CS^{(n,d)}(z_0)$ as follows:

$$CS_0^{(n,d)}(z_0)=\{(F,\Gamma_F)\in CS^{(n,d)}(z_0)|\\ \partial(x^F_{K^c},p^F_K)/\partial(t)|_0\neq0 \text{ for any } K\subset\{1,\cdots,n\}\}$$

and

$$CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)=\{(F,\Gamma_F)\in CS^{(n,d)}(z_0)|\\ \partial(x^F_{J_i^c},p^F_{J_i})/\partial(t)|_0=0 \quad\text{for}\quad i=1,\cdots,l \quad\text{and}\\ \partial(x^F_{K^c},p^F_K)/\partial(t)|_0\neq0 \text{ for any } K\subset\{1,\cdots,n\} \text{ such that } K\notin\{J_1,\cdots,J_l\}\},$$

where $l=1,\cdots,2^n-1$, $J_i\subset\{1,\cdots,n\}$, $J_i\neq J_j(i\neq j)$ and $\partial(x^F_{K^c},p^F_K)/\partial(t)|_0$ is the Jacobian of the map $t=(t_1,\cdots,t_n)\mapsto(x^F_{K^c}(0,t),p^F_K(0,t))$ at $t=0$.

Let $L_I^*:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),\Theta)\to(J^1(\boldsymbol{R}^n,\boldsymbol{R}),\theta)$ be the Legendre I-transformation from the coordinates (X,Y,P) to the coodinates (x,y,p). Then we can check that $L_I^*\circ L_I=id$ and $L_I\circ L_I^*=id$. Let $F:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),Z_0)\to(\boldsymbol{R}^d,0)$ be an equation in the coordinate system (X,Y,P). Then we obtain a new equation $F_I^*=F\circ(L_I^*)^{-1}:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),z_0)\to(\boldsymbol{R}^d,0)$, $z_0=L_I^*(Z_0)$, in the coordinate system (x,y,p).

We denote $*CS^{(n,d)}(Z_0)$ the set of pairs (F,Γ_F) of a completely integrable equation $F:(J^1(\boldsymbol{R}^n,\boldsymbol{R}),Z_0)\rightarrow(\boldsymbol{R}^d,0)$ and its complete solution Γ_F at Z_0 in the coordinate system (X,Y,P). We also define $2^{2^n}-1$ subsets $*CS_0^{(n,d)}(Z_0)$ and $*CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(Z_0)$ of $*CS^{(n,d)}(Z_0)$ in exactly the same definition as those of the above.

Then we have the following lemma.

Lemma 2.2 (1) *$CS^{(n,d)}(z_0)$ is a disjoint union of $CS_0^{(n,d)}(z_0)$ and $CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)$, where $l=1,\cdots,2^n-1$, $J_i\subset\{1,\cdots,n\}$ and $J_i\neq J_j(i\neq j)$.*

(2) *$*CS^{(n,d)}(Z_0)$ is a disjoint union of $*CS_0^{(n,d)}(Z_0)$ and $*CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(Z_0)$, where $l=1,\cdots,2^n-1$, $J_i\subset\{1,\cdots,n\}$ and $J_j\neq J_j(i\neq j)$.*

Proof. We only consider (1). By the definitions, $CS_0^{(n,d)}(z_0)$ and $CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)$ are disjoint subsets of $CS^{(n,d)}(z_0)$. So it is enough to show that any element of $CS^{(n,d)}(z_0)$ belongs to one of $CS_0^{(n,d)}(z_0)$ and $CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)$.

For $(F,\Gamma_F)\in CS^{(n,d)}(z_0)$, we denote $\Gamma_{F,c}(t)=\Gamma_F(c,t)$. Because $L_0=$ *Image* $\Gamma_{F,0}(t)$ is a Legendrian submanifold of $J^1(\boldsymbol{R}^n,\boldsymbol{R})$, by Arnold-Zakalyukin theory ([**3**], Corollary 20.2) there exist a subset $K\subset\{1,\cdots,n\}$ and a function germ $S_0(x_{K^c},p_K)$ such that

$$L_0=\{(x_{K^c},-\partial S_0/\partial p_K,S_0-\langle\partial S_0/\partial p_K,p_K\rangle,\partial S_0/\partial x_{K^c},p_K)|(x_{K^c},p_K)\in\boldsymbol{R}^n\}.$$

Then there exists a function germ $S(c,x_{K^c},p_K)$ such that we can regard the complete solution as follows:

$$\Gamma_F(c,x_{K^c},p_K)=(x_{K^c},-\partial S/\partial p_K,S-\langle\partial S/\partial p_K,p_K\rangle,\ \partial S/\partial x_{K^c},p_K).$$

Therefore we get

$$\begin{aligned}(F,\Gamma_F)&\in\{(F,\Gamma_F)\,|\,\partial(x_{K^c}^F,p_K^F)/\partial(t)\neq0\}\\&=CS_0^{(n,d)}(z_0)\cup(\cup_{\substack{l=1,\cdots,2^n-1\\ K\notin\{J_1,\cdots,J_l\}}}CS_{l,\{J_1\cdots J_l\}}^{(n,d)}(z_0)).\end{aligned}$$

Q. E. D.

As for the set of completely integrable equations in the classical sense we can easily get the following as a corollary of Lemmas 1.1 and 2.2.

Lemma 2.3 (1) *$\{F\in CI^{(n,d)}(z_0)|F=0$ is completely integrable at z_0 in the classical sense$\}=\Pi(CS_0^{(n,d)}(z_0)\cup(\cup_{\substack{l=1,\cdots,2^n-1\\ \phi\notin\{J_1,\cdots,J_l\}}}CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)))$.*

(2) *$\{F\in CI^{(n,d)}(z_0)|F=0$ is not completely integrable at z_0 in the classical sense$\}=CI^{(n,d)}(z_0)\backslash\Pi(CS_0^{(n,d)}(z_0)\cup(\cup_{\substack{l=1,\cdots,2^n-1\\ \phi\notin\{J_1,\cdots,J_l\}}}CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)))$.*

Now we have the following duality theorem.

Theorem 2.4 *Let n, d be integers such that $1 \leq d \leq n$. For any subset $I \subset \{1, \cdots, n\}$, we have one-to-one correspondences*

$$D_I^{(n,d)}: CS^{(n,d)}(z_0) \to *CS^{(n,d)}(Z_0) \quad \text{and}$$
$$*D_I^{(n,d)}: *CS^{(n,d)}(Z_0) \to CS^{(n,d)}(z_0)$$

defined by

$$D_I^{(n,d)}(F, \Gamma_F) = (F_I, L_I \circ \Gamma_F), \quad *D_I^{(n,d)}(F, \Gamma_F) = (F_I^*, L_I^* \circ \Gamma_F) \quad \text{and}$$
$$Z_0 = L_I(z_0),$$

which satisfy

$$*D_I^{(n,d)} \circ D_I^{(n,d)} = id \quad \text{and} \quad D_I^{(n,d)} \circ *D_I^{(n,d)} = id.$$

Furthermore there exist disjoint unions of $2^{2^n}-1$ subsets

$$CS^{(n,d)}(z_0) = CS_0^{(n,d)}(z_0) \cup \left(\cup_{\substack{l=1,\cdots,2^n-1 \\ J_i \subset \{1,\cdots,n\}, J_i \neq J_j (i \neq j)}} CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0) \right)$$

and

$$*CS^{(n,d)}(Z_0) = *CS_0^{(n,d)}(Z_0) \cup \left(\cup_{\substack{l=1,\cdots,2^n-1 \\ J_i \subset \{1,\cdots,n\}, J_i \neq J_j (i \neq j)}} *CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(Z_0) \right)$$

such that

$$D_I^{(n,d)}(CS_0^{(n,d)}(z_0)) = *CS_0^{(n,d)}(Z_0),$$
$$D_I^{(n,d)}(CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)) = *CS_{l,\{[I,J_1],\cdots,[I,J_l]\}}^{(n,d)}(Z_0)$$

and

$$*D_I^{(n,d)}(*CS_0^{(n,d)}(Z_0)) = CS_0^{(n,d)}(z_0),$$
$$*D_I^{(n,d)}(*CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(Z_0)) = CS_{l,\{[I,J_1],\cdots,[I,J_l]\}}^{(n,d)}(z_0),$$

where $[I, J] = (I \cup J) \backslash (I \cap J)$.

Proof. For any $(F, \Gamma_F) \in CS^{(n,d)}(z_0)$, $L_I \circ \Gamma_F$ is a complete solution of $F_I = 0$ at Z_0 by Lemma 2.1. Then $(F_I, L_I \circ \Gamma_F) \in *CS^{(n,d)}(Z_0)$. Then $D_I^{(n,d)}$ is a well defined and one-to-one correspondence. $*D_I^{(n,d)}$ is also a well defined and one-to-one correspondence in exactly the same reason.

Since $*D_I^{(n,d)} \circ D_I^{(n,d)}(F, \Gamma_F) = *D_I^{(n,d)}(F_I, L_I \circ \Gamma_F) = (F \circ (L_I^* \circ L_I)^{-1}, (L_I^* \circ L_I) \circ \Gamma_F) = (F, \Gamma_F)$ and $D_I^{(n,d)} \circ *D_I^{(n,d)}(F, \Gamma_F) = D_I^{(n,d)}(F_I^*, L_I^* \circ \Gamma_F) = (F \circ (L_I \circ L_I^*)^{-1}, (L_I \circ L_I^*) \circ \Gamma_F) = (F, \Gamma_F)$, then we have $*D_I^{(n,d)} \circ D_I^{(n,d)} = id$ and $D_I^{(n,d)} \circ *D_I^{(n,d)} = id$.

Now we only show the relations for the map $D_I^{(n,d)}$. The relations for the map $*D_I^{(n,d)}$ are shown in the similar way. From the definitions of $D_I^{(n.d)}$, $CS_0^{(n,d)}(z_0)$ and $*CS_0^{(n,d)}(Z_0)$, it is clear that $D_I^{(n,d)}(CS_0^{(n,d)}(z_0)) =$

$*CS_0^{(n,d)}(Z_0)$.

If $D_I^{(n,d)}(CS_{1,\{J_i\}}^{(n,d)}(z_0)) = *CS_{1,\{K_i\}}^{(n,d)}(Z_0)$ for each $i=1,\cdots,l$, then we have a relation $D_I^{(n,d)}(CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)) = *CS_{l,\{K_1,\cdots,K_l\}}^{(n,d)}(Z_0)$ from the definitions of $CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)$ and $*CS_{l,\{K_1,\cdots K_l\}}^{(n,d)}(Z_0)$. Therefore it is enough to show that $D_I^{(n,d)}(CS_{1,\{J\}}^{(n,d)}(z_0)) = *CS_{1,\{[I,J]\}}^{(n,d)}(Z_0)$.

Let

$$D_I^{(n,d)}(CS_{1,\{J\}}^{(n,d)}(z_0)) = *CS_{1,\{K\}}^{(n,d)}(Z_0),$$

where

$$D_I^{(n,d)}(F, \Gamma_F) = (F_I, L_I \circ \Gamma_F),$$
$$L_I(x, y, p) = (x_{I^c}, p_I, p_I \cdot x_I - y, -p_{I^c}, x_I) = (X_{I^c}, X_I, Y, P_{I^c}, P_I),$$
$$CS_{1,\{J\}}^{(n,d)}(z_0) = \{(F, \Gamma_F) \in CS^{(n,d)}(z_0) \mid \partial(x_{J^c}^F, p_J^F)/\partial(t)|_0 = 0 \text{ and } \partial(x_{L^c}^F, p_L^F)/\partial(t)|_0 \neq 0 \text{ for any } L \subset \{1, \cdots, n\} \text{ such that } L \neq J\}$$

and

$$*CS_{1,\{K\}}^{(n,d)}(Z_0) = \{(F, \Gamma_F) \in *CS^{(n,d)}(Z_0) \mid \partial(X_{K^c}^F, P_K^F)/\partial(t)|_0 = 0 \text{ and } \partial(X_{L^c}^F, P_L^F)/\partial(t)|_0 \neq 0 \text{ for any } L \subset \{1, \ldots, n\} \text{ such that } L \neq K\}.$$

Therefore $x_{J^c}^F = (x_{J^c \cap I^c}^F, x_{J^c \cap I}^F) = (X_{I^c \cap J^c}^F, P_{I \cap J^c}^F)$ and $p_J^F = (p_{J \cap I}^F, p_{J \cap I^c}^F) = (X_{I \cap J}^F, -P_{I^c \cap J}^F)$. Then $(x_{J^c}^F, p_J^F) = (X_{I^c \cap J^c}^F, X_{I \cap J}^F, P_{I \cap J^c}^F, -P_{I^c \cap J}^F)$. Since $\partial(x_{J^c}^F, p_J^F)/\partial(t)|_0 = 0$, we have

$$\partial(X_{(I^c \cap J^c) \cup (I \cap J)}^F, P_{(I \cap J^c) \cup (I^c \cap J)}^F)/\partial(t)|_0 = \partial(X_{K^c}^F, P_K^F)/\partial(t)|_0 = 0.$$

Hence we get $K = (I \cap J^c) \cup (I^c \cap J) = [I, J]$. Q. E. D.

We show an example.

Example 2. 5 Consider the following equation:

$$F = x_1 + p_1^2 : (J^1(\boldsymbol{R}^2, \boldsymbol{R}), 0) \to (\boldsymbol{R}, 0).$$

Two of the complete solutions of $F=0$ are given by

$$\Gamma_1(c, t) = (-t_1^2, c_2, -(2/3)t_1^3 + c_1, t_1, t_2)$$

and

$$\Gamma_2(c, t) = (-t_1^2, t_2 + c_2, -(2/3)t_1^3 + (1/2)t_2^2 + c_1, t_1, t_2),$$

where $c = (c_1, c_2)$ is the parameter. Then we see that

$$(F, \Gamma_1) \in CS_{3,\{\phi,\{1\},\{2\}\}}^{(2,1)}(0) \quad \text{and} \quad (F, \Gamma_2) \in CS_{2,\{\phi,\{2\}\}}^{(2,1)}(0).$$

We can calculate that

$$F_{\{1,2\}} = P_1 + X_1^2,$$
$$L_{\{1,2\}} \circ \Gamma_1(c, t) = (t_1, t_2, -(1/3)t_1^3 + c_2 t_2 - c_1, -t_1^2, c_2)$$

and

$$L_{\{1,2\}} \circ \Gamma_2(c, t) = (t_1, t_2, -(1/3)t_1^3 + (1/2)t_2^2 + c_2 t_2 - c_1, -t_1^2, t_2 + c_2).$$

Then we see that

$$(F_{\{1,2\}}, L_{\{1,2\}} \circ \Gamma_1) \in *CS^{(2,1)}_{3,\{\{1\},\{2\},\{1,2\}\}}(0) \quad \text{and}$$
$$(F_{\{1,2\}}, L_{\{1,2\}} \circ \Gamma_2) \in *CS^{(2,1)}_{2,\{\{1\},\{1,2\}\}}(0).$$

By Theorem 2. 4 we can also see that

$$\begin{aligned}(F_{\{1,2\}}, L_{\{1,2\}} \circ \Gamma_1) = D^{(2,1)}_{\{1,2\}}(F, \Gamma_1) &\in D^{(2,1)}_{\{1,2\}}(CS^{(2,1)}_{3,\{\phi,\{1\},\{2\}\}}(0)) \\ &= *CS^{(2,1)}_{3,\{\{1,2\},\{2\},\{1\}\}}(0)\end{aligned}$$

and

$$\begin{aligned}(F_{\{1,2\}}, L_{\{1,2\}} \circ \Gamma_2) = D^{(2,1)}_{\{1,2\}}(F, \Gamma_2) &\in D^{(2,1)}_{\{1,2\}}(CS^{(2,1)}_{2,\{\phi,\{2\}\}}(0)) \\ &= *CS^{(2,1)}_{2,\{\{1,2\},\{1\}\}}(0).\end{aligned}$$

As a corollary of the above theorem, we have a characterization of completely integrable equations.

Corollary 2. 6 *Let F be an equation at z_0. Then $F=0$ is completely integrable at z_0 if and only if there exists a subset $I \subset \{1, \cdots, n\}$ such that $F_I = 0$ is completely integrable at $Z_0 = L_I(z_0)$ in the classical sense.*

Proof. Since $F=0$ is completely integrable at z_0, there exists a complete solution Γ_F of $F=0$ such that $(F, \Gamma_F) \in CS^{(n,d)}(z_0)$. By Lemma 2. 2 there exist an integer l and subsets J_i $(i=1, \cdots, l)$ such that $0 \leq l \leq 2^n - 1$, $J_i \subset \{1, \cdots, n\}$, $J_i \neq J_j$ $(i \neq j)$ and $(F, \Gamma_F) \in CS^{(n,d)}_{l,\{J_1,\cdots,J_l\}}(z_0)$. Since $0 \leq l \leq 2^n - 1$, there exists a subset $I \subset \{1, \cdots, n\}$ such that $I \neq J_i$ for any $i=1, \cdots, l$. By Theorem 2. 4 we have

$$\begin{aligned}D^{(n,d)}_I(F, \Gamma_F) = (F_I, L_I \circ \Gamma_F) &\in D^{(n,d)}_I(CS^{(n,d)}_{l,\{J_1,\cdots,J_l\}}(z_0)) \\ &= *CS^{(n,d)}_{l,\{[I,J_1],\cdots,[I,J_l]\}}(Z_0).\end{aligned}$$

Since $[I, J] = \phi$ if and only if $I = J$, we get $[I, J_i] \neq \phi$ for any $i=1, \cdots, l$. Therefore by Lemma 2. 3 $F_I = 0$ is completely integrable at Z_0 in the classical sense.

Conversely suppose that there exists a subset $I \subset \{1, \cdots, n\}$ such that $F_I = 0$ has a complete solution Γ_{F_I} of $F_I = 0$ at Z_0 such that $(F_I, \Gamma_{F_I}) \in *CS^{(n,d)}(Z_0)$. By Theorem 2. 4 we get $*D^{(n,d)}_I(F_I, \Gamma_{F_I}) \in CS^{(n,d)}(z_0)$. Since $*D^{(n,d)}_I(F_I, \Gamma_{F_I}) = (F_I \circ (L_I^*)^{-1}, L_I^* \circ \Gamma_{F_I}) = (F \circ (L_I^* \circ L_I)^{-1}, L_I^* \circ \Gamma_{F_I}) = (F, L_I^* \circ \Gamma_{F_I})$, the equation $F=0$ is completely integrable at z_0. Q. E. D.

A characterization of completely integrable equations in the classical sense is given by Izumiya [7] as follows:

An equation $F=0$ is said to be *Clairaut type at* z_0 if there exist smooth function germs B_{ji}, $A_{ik}^l: (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)\rightarrow \boldsymbol{R}$ for $i, j=1, \cdots, n$; $k, l=1, \cdots, d$ such that

$$\partial F_l/\partial x_i+p_i\partial F_l/\partial y=\sum_{j=1}^n B_{ji}\partial F_l/\partial p_j+\sum_{k=1}^d A_{ik}^l F_k \qquad (i=1, \cdots, n \text{ and } l=1, \cdots, d)$$

and satisfy that

(1) $B_{ji}=B_{ij}$

(2) $\partial B_{jk}/\partial x_i+p_i\partial B_{jk}/\partial y+\sum_{l=1}^n B_{li}\partial B_{jk}/\partial p_l=\partial B_{ji}/\partial x_k+p_k\partial B_{ji}/\partial y+\sum_{l=1}^n B_{lk}\partial B_{ji}/\partial p_l$ at any $z\in(F^{-1}(0), z_0)$ for $i, j, k=1, \cdots, n$.

Then we have the following

Theorem 2.7 ([7]). *For an equation germ $F=0$, the followings are equivalent.*

(1) *$F=0$ is Clairaut type at z_0.*

(2) *$F=0$ is completely integrable at z_0 in the classical sense.*

Therefore by Corollary 2.6 and Theorem 2.7 we have the following.

Corollary 2.8 *Let F be an equation at z_0. Then $F=0$ is completely integrable at z_0 if and only if there exists a subset $I\subset\{1, \cdots, n\}$ such that $F_I=0$ is Clairaut type at $Z_0=L_I(z_0)$.*

3. The principle of duality for holonomic equations

In the special case of holonomic equations(i.e. $d=n$) we can show the following local uniqueness theorem by using the uniqueness of codimension one foliations, so that we have the principle of duality among the completely integrable equations themselves.

Proposition 3.1 *Let $F: (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)\rightarrow(\boldsymbol{R}^n, 0)$ be a completely integrable holonomic equation. Let $\Gamma_i: (\boldsymbol{R}\times\boldsymbol{R}^n, 0)\rightarrow(J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$, $i=1, 2$, be complete solutions of $F=0$. Then there exists a diffeomorphism germ Φ: $(\boldsymbol{R}\times\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}\times\boldsymbol{R}^n, 0)$ of the form $\Phi(c, t)=(\phi_1(c), \phi_2(c, t))$ such that $\Gamma_1\circ\Phi=\Gamma_2$.*

So we naturally introduce the following equivalence relation among complete solutions. Let $F: (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0) \to (\boldsymbol{R}^n, 0)$ be a completely integrable holonomic equation. Let $\Gamma_i: (\boldsymbol{R} \times \boldsymbol{R}^n, 0) \to (J^1(\boldsymbol{R}^n, \boldsymbol{R}), z_0)$, $i=1, 2$, be complete solutions of $F=0$. We say that Γ_1 and Γ_2 are *equivalent* if there exists a diffeomorphism germ $\Phi: (\boldsymbol{R} \times \boldsymbol{R}^n, 0) \to (\boldsymbol{R} \times \boldsymbol{R}^n, 0)$ of the form $\Phi(c, t)=(\phi_1(c), \phi_2(c, t))$ such that $\Gamma_1 \circ \Phi = \Gamma_2$.

Let $CIH^{(n)}(z_0)=CI^{(n,n)}(z_0)$, which is the set of complete integrable holonomic equations at z_0. By using the equivalence relation $CIH^{(n)}(z_0)$ can be identified with $CS^{(n,n)}(z_0)$ from the local uniqueness theorem. Then we can define $2^{2^n}-1$ subsets $CIH_0^{(n)}(z_0)$ and $CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(z_0)$ of $CIH^{(n)}(z_0)$ in the similar way as the definition of the subsets $CS_0^{(n,d)}(z_0)$ and $CS_{l,\{J_1,\cdots,J_l\}}^{(n,d)}(z_0)$ of $CS^{(n,d)}(z_0)$.

We denote $*CIH^{(n)}(Z_0)$ the set of complete integrable holonomic equations $(J^1(\boldsymbol{R}^n, \boldsymbol{R}), Z_0) \to (\boldsymbol{R}^n, 0)$ in the coordinate system (X, Y, P). We also define $2^{2^n}-1$ subsets $*CIH_0^{(n)}(Z_0)$ and $*CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(Z_0)$ of $*CIH^{(n)}(Z_0)$ in exactly the same definition as those of the above.

Then by Lemma 2.2 we have the following

Lemma 3.2 (1) *$CIH^{(n)}(z_0)$ is a disjoint union of $CIH_0^{(n)}(z_0)$ and $CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(z_0)$, where $l=1, \cdots, 2^n-1$, $J_i \subset \{1, \cdots, n\}$ and $J_i \neq J_j (i \neq j)$.*

(2) *$*CIH^{(n)}(Z_0)$ is a disjoint union of $*CIH_0^{(n)}(Z_0)$ and $*CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(Z_0)$, where $l=1, \cdots, 2^n-1$, $J_i \subset \{1, \cdots, n\}$ and $J_i \neq J_j (i \neq j)$.*

As for the set of completely integrable holonomic equations in the classical sense we can get the following by Lemmas 2.3. and 3.2.

Lemma 3.3 (1) *$\{F \in CIH^{(n)}(z_0) | F=0$ is completely integrable at z_0 in the classical sense$\} = CIH_0^{(n)}(z_0) \cup (\bigcup_{\substack{l=1,\cdots,2^n-1 \\ \phi \notin \{J_1,\cdots,J_l\}}} CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(z_0))$.*

(2) *$\{F \in CIH^{(n)}(z_0) | F=0$ is not completely integrable at z_0 in the classical sense$\} = \bigcup_{\substack{l=1,\cdots,2^n-1 \\ \phi \in \{J_1,\cdots,J_l\}}} CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(z_0)$.*

Now we have the following duality theorem by Theorem 2.4.

Theorem 3.4 *Let n be an integer. For any subset $I \subset \{1, \cdots, n\}$, we have one-to-one correspondences*

$$DH_I^{(n)}: CIH^{(n)}(z_0) \to *CIH^{(n)}(Z_0) \quad \text{and}$$
$$*DH_I^{(n)}: *CIH^{(n)}(Z_0) \to CIH^{(n)}(z_0)$$

defined by

$$DH_I^{(n)}(F)=F_I, \quad *DH_I^{(n)}(F)=F_I^* \text{ and } Z_0=L_I(z_0),$$

which satisfy $*DH_I^{(n)} \circ DH_I^{(n)}=id$ and $DH_I^{(n)} \circ *DH_I^{(n)}=id$.

Furthermore there exist disjoint unions of $2^{2^n}-1$ *subsets*

$$CIH^{(n)}(z_0)=CIH_0^{(n)}(z_0)\cup(\cup_{\substack{l=1,\cdots,2^n-1\\ J_i\subset\{1,\cdots,n\},J_i\neq J_j(i\neq j)}}CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(z_0))$$

and

$$*CIH^{(n)}(Z_0)=*CIH_0^{(n)}(Z_0)\cup(\cup_{\substack{l=1,\cdots,2^n-1\\ J_i\subset\{1,\cdots,n\},J_i\neq J_j(i\neq j)}}*CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(Z_0))$$

such that

$$DH_I^{(n)}(CIH_0^{(n)}(z_0))=*CIH_0^{(n)}(Z_0),$$
$$DH_I^{(n)}(CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(z_0))=*\ CIH_{l,\{[I,J_1],\cdots,[I,J_l]\}}^{(n)}(Z_0)$$

and

$$*DH_I^{(n)}(*CIH_0^{(n)}(Z_0))=CIH_0^{(n)}(z_0),$$
$$*DH_I^{(n)}(*CIH_{l,\{J_1,\cdots,J_l\}}^{(n)}(Z_0))=CIH_{l,\{[I,J_1],\cdots,[I,J_l]\}}^{(n)}(z_0),$$

where $[I,J]=(I\cup J)\backslash(I\cap J)$.

Proof. We only have to check that $DH_I^{(n)}$ and $*DH_I^{(n)}$ are well-defined maps. By the difinition $DH_I^{(n)}(F)=F_I=F\circ L_I^{-1}$. For any $F\in CIH^{(n)}(z_0)$, $L_I\circ\Gamma_F$ is the unique complete solution of $F_I=0$ by Lemma 2.1 and Proposition 3.1. Then $F_I\in *CIH^{(n)}(Z_0)$. Then $DH_I^{(n)}$ is well-defined. $*DH_I^{(n)}$ is also a well-defined in exactly the same reason. Q. E. D.

Finally we show an example.

Example 3.5 Consider the following holonomic equation:

$$F=(x_1-x_2,\ p_1+p_2-2x_1):(J^1(\boldsymbol{R}^2,\boldsymbol{R}),0)\to(\boldsymbol{R}^2,0).$$

The complete solution is given by

$$\Gamma_F(c,t_1,t_2)=((t_1+t_2)/2,\ (t_1+t_2)/2,\ ((t_1+t_2)/2)^2+c,\ t_1,\ t_2),$$

where c is parameter. Then $(F,\Gamma_F)\in CS_{1,\{\phi\}}^{(2,2)}(0)$ and hence $F\in CIH_{1,\{\phi\}}^{(2)}(0)$.

Therefore $F=0$ is not completely integrable at 0 in the classical sense and $F_{\{1\}}=0$, $F_{\{2\}}=0$ and $F_{\{1,2\}}=0$ are completely integrable at 0 in the classical sense by Lemma 3.3 and Theorem 3.4.

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