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Biharmonic green domains in $\boldsymbol{R}^n$

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Abstract. The properties of biharmonic functions with a singularity at a finite or infinite point in $\boldsymbol{R}^n$, $n \geq 2$, are investigated, leading to a generalization of the classical Bôcher theorem for harmonic functions with positive singularity, when $2 \leq n \leq 4$. This latter result is useful in identifying some biharmonic Green domains in $\boldsymbol{R}^n$.

Key words: biharmonic point singularities in $\boldsymbol{R}^n$.

1. Introduction

The behaviour of a biharmonic function $u(x)$ in $0 < |x| < 1$ in $\boldsymbol{R}^n$, $n \geq 2$, is considered, leading to a necessary and sufficient condition for u to extend as a distribution in $|x| < 1$; a case of particular interest is when u is bounded.

The corresponding results when the biharmonic function is defined outside a compact set K in $\boldsymbol{R}^n$ lead to an analogue of Bôcher's theorem (after a Kelvin transformation) for positive harmonic functions in $\boldsymbol{R}^n \backslash K$; but this is valid only when $2 \leq n \leq 4$. A corollary to this is: let Ω be a domain in $\boldsymbol{R}^n$, $2 \leq n \leq 4$ such that $\boldsymbol{R}^n \backslash \Omega$ is compact. Then Ω is not a biharmonic Green domain; that is, a biharmonic Green function cannot be defined on Ω.

2. Preliminaries

For $n \geq 2$, let E_n and S_n denote the fundamental solutions of the Laplacian Δ and Δ^2 in $\boldsymbol{R}^n$; that is, $\Delta E_n = \delta$ and $\Delta^2 S_n = \delta$ in the sense of distributions.

Given a locally integrable function f on $\boldsymbol{R}^n$, let $M(r, f)$ denote the mean value of $f(x)$ on $|x| = r$.

Proposition 2.1 *Let $u(x)$ be a harmonic function in $0 < |x| < 1$ in $\boldsymbol{R}^n$. Then the following are equivalent*:

1) *u extends as a distribution in $|x| < 1$ (in which case, it is of order*

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m for an integer $m \geq 0$).

2) *$u(x) = v(x) + \sum_{|k| \leq m} a_k \partial^k E_n(x)$ where $v(x)$ is harmonic in $|x| < 1$.*

3) *$u(x) \geq \varphi(x)$ in $0 < |x| < 1$ where $M(r, |\varphi|) = o(r^{1-m-n})$ when $|x| = r \to 0$.*

Proof.

1) $\Rightarrow$ 2): If u extends as a distribution in $|x| < 1$, Δu is a distribution in $|x| < 1$ with point support $\{0\}$ and hence $\Delta u = \sum_{|k| \leq m} a_k \partial^k \delta$ for some integer $m \geq 0$.

Consequently, the distribution $T = u - \sum_{|k| \leq m} a_k \partial^k E_n$ in $|x| < 1$ satisfies the equation $\Delta T = 0$ and hence T is equal a.e. to a harmonic function v in $|x| < 1$.

Since $u - \sum_{|k| \leq m} a_k \partial^k E_n$ is continuous in $0 < |x| < 1$, we have $u(x) = v(x) + \sum_{|k| \leq m} a_k \partial^k E_n(x)$ in $0 < |x| < 1$.

2) $\Rightarrow$ 3): This follows from the observation $\partial^k E_n(x) = O(|x|^{2-k-n})$ when $|x| \to 0$ (Mizohata [9], p. 145).

3) $\Rightarrow$ 1): Setting $h(x) = -u(x)$, we note that $h^+(x) \leq |\varphi(x)|$ and hence by hypothesis, $r^{m+n-1} M(r, h^+) \to 0$ when $|x| = r \to 0$.

Now the series expansion of $u(x)$ in $0 < |x| < 1$ (M. Brelot [7], p. 201) gives $h(x) = -u(x) = -v(x) + \sum_k a_k \partial^k E_n(x)$, where $v(x)$ is harmonic in $|x| < 1$. However here the series reduces to a finite number of terms, since the assumption $r^{m+n-1} M(r, h^+) \to 0$ when $r \to 0$ implies that $a_k = 0$ if $|k| \geq m + 1$.

Thus, $u(x) = v(x) - \sum_{|k| \leq m} a_k \partial^k E_n(x)$ in $0 < |x| < 1$ extends as a distribution of order $\leq m$ in $|x| < 1$. □

Corollary 2.2 *Let $u(x)$ be harmonic in $0 < |x| < 1$ in $\mathbf{R}^n$. Then the following are equivalent:*

1) *$u(x)$ is bounded on one-side.*
2) *$u(x) = v(x) + \alpha E_n(x)$ where $v(x)$ is harmonic in $|x| < 1$.*
3) *$|x|^{n-1} u(x)$ tends to 0 when $|x| \to 0$.*
4) *$r^{n-1} M(r, |u|) \to 0$ when $r \to 0$.*
5) *$u(x) \geq \varphi(x)$ in $0 < |x| < 1$ where $M(r, |\varphi|) = o(r^{1-n})$ when $r \to 0$.*

Proof. This is an immediate consequence of the above Proposition 2.1 when we remark that if u is lower bounded, it extends as a superharmonic function in $|x| < 1$ (M. Brelot [7], p. 39). Consequently it is a locally integrable function and defines a distribution of order 0. □

An application of the Kelvin transformation gives the following equivalent version of the above Corollary 2.2, stated here for $m = 0$ in which form it is used later.

Corollary 2.3 *Let $u(x)$ be a harmonic function in $|x| > R$ in $\boldsymbol{R}^n$, $n \geq 2$. Then the following are equivalent:*

1) $u(x) = o(|x|)$ *when* $|x| \to \infty$.

2) $\liminf_{|x|\to\infty} \dfrac{u(x)}{|x|} \geq 0$

3) $u(x) = \begin{cases} \alpha \log|x| + b(x) & \text{if } n = 2 \\ \alpha + b_1(x) & \text{if } n \geq 3 \end{cases}$

where α is a constant; $b(x)$ and $b_1(x)$ are harmonic functions in $|x| > R$ such that $\lim_{|x|\to\infty} b(x)$ is finite and $\lim_{|x|\to\infty} b_1(x) = 0$.

4) *There exists a locally integrable function $\varphi(x)$ such that $u(x) \geq \varphi(x)$ outside a compact set and $M(R, |\varphi|) = o(R)$ when $R \to \infty$.*

Remark. The above two corollaries are variously known as the Bôcher theorem or the Picard principle for harmonic functions with point singularity ([1], [8] and [5]). In the next section we prove similar results for biharmonic functions.

3. Removable biharmonic point singularities

Let u be a biharmonic function in $0 < |x| < 1$ in $\boldsymbol{R}^n$. Since Δu is harmonic in $0 < |x| < 1$, using its series expansion we can obtain the series expansion for $u(x)$ as $u(x) = b(x) + \sum_\alpha a_\alpha \partial^\alpha S_n(x) + \sum_\alpha b_\alpha \partial^\alpha E_n(x)$, $0 < |x| < 1$, where $b(x)$ is biharmonic in $|x| < 1$ (Aronszajn et al. [4] p. 82).

Lemma 3.1 *Let $u(x)$ be biharmonic in $0 < |x| < 1$ in $\boldsymbol{R}^n$. Then u extends as a distribution in $|x| < 1$ if and only if $u(x) = b(x) + \sum_f a_\alpha \partial^\alpha S_n(x)$ where $\sum_f$ stands for a finite sum and $b(x)$ is biharmonic in $|x| < 1$.*

Proof. Suppose u extends as a distribution in $|x| < 1$. Then $\Delta^2 u$ is a distribution with point support $\{0\}$. Hence $\Delta^2 u = \sum_f a_\alpha \partial^\alpha \delta = \sum_f a_\alpha \partial^\alpha (\Delta^2 S_n)$.

Consequently, $T = u - \sum_f a_\alpha \partial^\alpha S_n$ is a distribution in $|x| < 1$ such that $\Delta^2 T = 0$; this implies that there exists a biharmonic function $b(x)$ in

$|x| < 1$ such that $T = b$ a.e.

That is, $u = b + \sum_f a_\alpha \partial^\alpha S_n$ in $0 < |x| < 1$ because of continuity. The converse is obvious. □

Theorem 3.2 *Let $u(x)$ be biharmonic in $0 < |x| < 1$ in $\boldsymbol{R}^n$. Then the following are equivalent:*

1) *u extends as a distribution in $|x| < 1$ and $\Delta u \geq \varphi$ in $0 < |x| < 1$ where $M(r, |\varphi|) = o(r^{1-n})$ when $r \to 0$.*

2) *$u(x) = b(x) + \alpha S_n(x)$ where $b(x)$ is biharmonic in $|x| < 1$.*

Proof.

1) $\Rightarrow$ 2): Since u extends as a distribution in $|x| < 1$, by Lemma 3.1, $u = b + \sum_f a_\alpha \partial^\alpha S_n$ in $0 < |x| < 1$ and hence $\Delta u =$ (a harmonic fucntion in $|x| < 1) + \sum_f a_\alpha \partial^\alpha E_n$.

But Δu being harmonic in $0 < |x| < 1$ and $\Delta u \geq \varphi$ where $M(r, |\varphi|) = o(r^{1-n})$ when $r \to 0$, $\Delta u =$ (a harmonic function in $|x| < 1) + \beta E_n$ by Corollary 2.2.

This implies that $a_\alpha = 0$ if $|\alpha| > 0$ and consequently $u(x) = b(x) + a_0 S_n(x)$ in $0 < |x| < 1$.

2) $\Rightarrow$ 1): Obvious. □

Corollary 3.3 *Let $u(x)$ be biharmonic in $0 < |x| < 1$ in $\boldsymbol{R}^n$. Then u extends as a biharmonic function in $|x| < 1$ if both $M(r, |u|)$ and $M(r, |\Delta u|)$ are $o(E_n(r))$ when $r \to 0$.*

Proof. Since Δu is harmonic in $0 < |x| < 1$ and by the assumption on $M(r, |\Delta u|)$, there exists a harmonic function h in $|x| < 1$ such that $\Delta u = h$ in $0 < |x| < 1$ (Corollary 2.2).

If b is a biharmonic function in $|x| < 1$ such that $\Delta b = h$, there exists a harmonic function $H(x)$ in $0 < |x| < 1$ such that $u(x) = b(x) + H(x)$ in $0 < |x| < 1$; and by the assumption on u, $M(r, |H|) = o(r^{1-n})$ when $r \to 0$. Hence, by Corollary 2.2, H extends as a harmonic function in $|x| < 1$.

This proves the corollary. □

Bounded biharmonic functions with point singularity. The above corollary in particular implies that a bounded biharmonic function $u(x)$ in $0 < |x| < 1$ in $\boldsymbol{R}^n$, $n \geq 2$, extends as a biharmonic function in $|x| < 1$ if and only if $M(r, |\Delta u|) = o(E_n(r))$ when $r \to 0$.

However, when $n \geq 4$ we have a better result relating to the removability of the point singularity.

Theorem 3.4 *Let u be a bounded biharmonic function in $0 < |x| < 1$ in $\boldsymbol{R}^n$, $n \geq 4$. Then u extends as a biharmonic function in $|x| < 1$.*

Proof. Define $u(0) = \liminf_{x \to 0} u(x)$. Thus defined, u is a l.s.c. function in $|x| < 1$, bounded and a distribution.

Hence by Lemma 3.1, it is of the form $u(x) = b(x) + \sum_f a_\alpha \partial^\alpha S_n(x)$ in $0 < |x| < 1$, where $b(x)$ is biharmonic in $|x| < 1$.

Since $n \geq 4$, the form of $S_n(x)$ together with the fact that u is bounded near 0 implies that $a_\alpha = 0$ for every α.

Hence $u(x)$ extends as a biharmonic function in $|x| < 1$. □

Corollary 3.5 (Sario et al. [11] p. 152) *There exist no nonconstant bounded biharmonic functions on the punctured Euclidean n-space $|x| > 0$ if $n \geq 4$.*

Proof. Let $u(x)$ be a bounded biharmonic function in $|x| > 0$. By the above Theorem 3.4, u extends as a bounded biharmonic function in $\boldsymbol{R}^n$ and hence is a constnat. □

Proposition 3.6 *Let u be a bounded biharmonic function in $0 < |x| < 1$ in $\boldsymbol{R}^3$. Then $u(x) = b(x) + \alpha|x| + \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{|x|}$ where $x = (x_1, x_2, x_3)$ and $b(x)$ is biharmonic in $|x| < 1$.*

Proof. Since u extends as a distribution in $|x| < 1$, by Lemma 3.1, $u(x) = b(x) + \sum_f a_\alpha \partial^\alpha S_3(x)$ in $0 < |x| < 1$, where $b(x)$ is biharmonic in $|x| < 1$.

Now $S_3(x) = |x|$ and $\frac{\partial S_3}{\partial x_i}(x) = \frac{x_i}{|x|}$ $(i = 1, 2, 3)$. Consequently, the assumption that u is bounded near 0 implies that $a_\alpha = 0$ if $|\alpha| \geq 2$.

Hence $u(x)$ is of the form $u(x) = b(x) + \alpha|x| + \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{|x|}$. □

Corollary 3.7 (Sario et al. [11] p. 151) *Let $u(x)$ be a bounded biharmonic fnctions in $\boldsymbol{R}^3 \backslash \{0\}$. Then u is a linear combination of 1, $\frac{x_1}{|x|}$, $\frac{x_2}{|x|}$ and $\frac{x_3}{|x|}$.*

Proof. Since $u(x) = b(x) + \alpha|x| + \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{|x|}$ in $0 < |x|$, $b(x)$ is a biharmonic function in $\boldsymbol{R}^3$ such that $b(x) = -\alpha|x| +$ (a bounded function near ∞).

Now $b(x)$ is of the form (Almansi), $b(x) = |x|^2 h_1(x) + h_2(x)$ where h_1 and h_2 are harmonic in $\boldsymbol{R}^3$.

Using these two expressions for $b(x)$, when we calculate the mean-value $M(r, b)$, we obtain

$$r^2 h_1(0) + h_2(0) = -\alpha r + (\text{a bounded function of } r \text{ near infinity}).$$

When r becomes large, we note that we should have $h_1(0) = 0$ and then $\alpha = 0$. Consequently, the first expression for $b(x)$ says that the biharmonic functions $b(x)$ is a bounded function near ∞, and hence a constant.

Consequently, $u(x) =$ (a constant) $+ \frac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{|x|}$ in $|x| > 0$. This completes the proof of the corollary. □

4. Biharmonic functions with singularity at infinity

In this section we consider the superharmonic functions u defined outside a compact set in $\boldsymbol{R}^n$, satisfying the condition $\Delta^2 u \geq 0$. For such functions we show that a Bôcher-type representation (Corollary 2.3) is valid if and only if $2 \leq n \leq 4$.

Theorem 4.1 *Let $u(x)$ be a continuous function defined in $|x| > 1$ in $\boldsymbol{R}^n$, $2 \leq n \leq 4$. Suppose $u \geq 0$, $\Delta u \leq 0$ and $\Delta^2 u \geq 0$. Then u is harmonic.*

Proof. This is an immediate consequence of the following lemma. □

Lemma 4.2 *Let $u(x)$ be a superharmonic function in $|x| > 1$ in $\boldsymbol{R}^n$, $2 \leq n \leq 4$, for which $\Delta^2 u \geq 0$. Suppose u satisfies an additional condition:*

1) *When $n = 2$, there exists a superharmonic function $v > 0$ outside a disc such that $\liminf_{|x|\to\infty} \frac{u(x)}{v(x)} > -\infty$.*

2) *When $n = 3$, $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \geq 0$.*

3) *When $n = 4$, $\liminf_{|x|\to\infty} \frac{u(x)}{\log|x|} \geq 0$.*

Then $u(x)$ is harmonic in $|x| > 1$.

Proof.

1) When $n = 2$, since u is superharmonic in $|x| > 1$, there exists a superharmonic function $s(x)$ in $\boldsymbol{R}^2$ such that $s(x) = u(x) - \alpha \log |x|$ outside a disc [2].

Let $\liminf_{|x|\to\infty} \frac{u(x)}{v(x)} = \beta > -\infty$.

Then, for ϵ small, $u(x) \geq (\beta - \epsilon)v(x)$ outside a disc. Hence, whatever be the sign of $(\beta - \epsilon)$, $u(x)$ majorizes a subharmonic function outside a disc.

Consequently, s has a harmonic minorant outside a compact set in $\boldsymbol{R}^2$. That is, the flux at infinity of s is finite.

Since $t(x) = \Delta u(x) \leq 0$ is subharmonic in $|x| > 1$, $t < 0$ or $t \equiv 0$ in $|x| > 1$.

Suppose at some point x, $|x| > 1$, $t(x) < 0$; let $\max_{|x|=r>1} t(x) = M$. Then $M < 0$ and $\Delta u(x) = t(x) \leq M$ in $|x| > r$ (since the harmonic measure of the point at infinity of $\boldsymbol{R}^2$ is 0). This would imply that the flux at infinity of u and hence that of s is infinite, a contradiction.

Hence $t(x) \equiv 0$ in $|x| > 1$; that is $u(x)$ is harmonic in $|x| > 1$.

2) When $n = 3$, since $t = \Delta u \leq 0$ is subharmonic in $|x| > 1$, $t < 0$ or $t \equiv 0$ in $|x| > 1$.

Suppose $t < 0$ for $|x| > 1$. Then if $r > 1$, for some $c > 0$, $t(x) \leq -\frac{c}{|x|}$ on $|x| = r$.

Let $s(x) = t(x) + \frac{c}{|x|}$ in $|x| > r$. Then $s(x)$ is subharmonic and $\limsup s(x) \leq 0$ when x tends to a finite or infinite boundary point. Hence $s(x) \leq 0$ in $|x| > r$.

Now $s(x) = \Delta(u(x) + \frac{c}{2}|x|)$ in $|x| > r$ and since $s(x) \leq 0$, we conclude that $u(x) + \frac{c}{2}|x| = v(x)$ a.e. where $v(x)$ is a superharmonic function in $|x| > r$.

By the assumption on u, given $\epsilon < \frac{c}{2}$, $u(x) \geq -\epsilon|x|$ outside a compact set. Thus $u(x) + \frac{c}{2}|x|$ is a function in $|x| > 1$ tending to ∞ at the point at infinity, consequently, the superharmonic function $v(x)$ in $|x| > r$ tends to ∞ at the point at infinity. This would mean that the harmonic measure of the point at infinity of $\boldsymbol{R}^3$ is 0, a contradiction.

Hence $\Delta u = t \equiv 0$ in $|x| > 1$; that is, u is harmonic in $|x| > 1$.

3) When $n = 4$, we repeat the arguments for the case $n = 3$.

With $t = \Delta u \leq 0$, if $t < 0$ for $|x| > 1$, then for some $c > 0$, $t(x) \leq -\frac{c}{|x|^2}$ on $|x| = r > 1$.

Then $s(x) = t(x) + \frac{c}{|x|^2} \leq 0$ is subharmonic in $|x| > r$.

But $s(x) = \Delta(u(x) + \frac{c}{2}\log|x|)$. Consequently, using the assumption on u we conclude that $u(x) + \frac{c}{2}\log|x| = v(x)$ a.e. where $v(x)$ is a superharmonic function in $|x| > r$ tending to ∞ at the point at infinity. This is a contradiction since the harmonic measure of the point at infinity of $\boldsymbol{R}^4$ is nonzero.

We conclude therefore that u is harmonic in $|x| > 1$.

This completes the proof of the lemma which leads to a representation analogous to the one in Corollary 2.3 when $2 \leq n \leq 4$. $\square$

Theorem 4.3 *Let $u(x)$ be a superharmonic function defined in $|x| > 1$ in $\boldsymbol{R}^n$, $2 \le n \le 4$. Suppose $\Delta^2 u \ge 0$ and $\liminf_{|x|\to\infty} \frac{u(x)}{\log|x|} \ge 0$. Then $u(x)$ is of the form*

$$u(x) = \begin{cases} \alpha \log|x| + b(x) & \text{if } n = 2 \\ \alpha + b_1(x) & \text{if } n = 3 \text{ or } 4 \end{cases}$$

where α is a constant; $b(x)$ and $b_1(x)$ are harmonic functions in $|x| > 1$ such that $\lim_{|x|\to\infty} b(x)$ is finite and $\lim_{|x|\to\infty} b_1(x) = 0$.

Proof. The assumed conditions on u imply that the conditions stated in Lemma 4.2 are satisfied. Hence u is harmonic.

Again the assumed conditions on u imply that $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \ge 0$. A condition like this for the harmonic function u gives the representation stated in the theorem. □

Remark. A similar extension of the Bôcher Theorem 4.3 is not possible in $\boldsymbol{R}^n$ when $n \ge 5$. For example, if $u(x) = |x|^{4-n}$ in $|x| > 1$ in $\boldsymbol{R}^n, n \ge 5$, $u > 0$, $\Delta u < 0$ and $\Delta^2 u = 0$.

In analogy with Lemma 4.2, we have the following theorem. Recall that if u is a superharmonic function outside a compact set in $\boldsymbol{R}^n$, $n \ge 2$, then

$$\lambda(u) = \begin{cases} \lim\limits_{r\to\infty} \dfrac{M(r,u)}{\log r} & \text{if } n = 2, \\ \lim\limits_{r\to\infty} M(r,u) & \text{if } n \ge 3 \end{cases}$$

is well-defined and $\lambda(u) < \infty$.

Theorem 4.4 *Let Ω be a domain with compact complement in $\boldsymbol{R}^n$, $2 \le n \le 4$. Let $u(x)$ be a superharmonic function in Ω for which $\Delta^2 u \ge 0$ and $\lambda(u) > -\infty$. Then $u(x)$ is harmonic in Ω.*

Proof. Let $\boldsymbol{R}^n \backslash \Omega \subset \{x : |x| < R\}$. By hypothesis, $t(x) = \Delta u(x) \le 0$ is subharmonic in Ω and hence $t \equiv 0$ or $t < 0$ in Ω. Suppose $t < 0$ in Ω.

1) When $n = 2$,
let $\max_{|x|=R} t(x) = M < 0$; then by the maximum principle, $t(x) \le M$ in $|x| \ge R$.

Let $s(x) = u(x) - \frac{M}{4}|x|^2$; then in $|x| > R$, $\Delta s(x) = t(x) - M \le 0$ and hence $s(x)$ is superharmonic.

Hence $M(r,u) = M(r,s) + \frac{M}{4} r^2$ if $r > R$. Since $\lambda(s) < \infty$ and $M < 0$, this would imply $\lambda(u) = -\infty$, a contradiction.

2) When $n = 3$,
for some $c > 0$, $t(x) \leq -\frac{c}{|x|}$ on $|x| = R$ which by the maximum principle, as in Lemma 4.2, implies $t(x) \leq -\frac{c}{|x|}$ in $|x| > R$. Then as before we conclude that $s(x) = u(x) + \frac{c}{2}|x|$ is superharmonic in $|x| > R$.

Hence $M(r, u) = M(r, s) - \frac{c}{2}r$ if $r > R$. Since $c > 0$ and $\lambda(s) < \infty$, this would imply $\lambda(u) = -\infty$, a contradiction.

3) When $n = 4$,
as in Lemma 4.2, we find that $s(x) = u(x) + \frac{c}{2}\log|x|$ is superharmonic in $|x| > R$ for some $c > 0$. This would again imply that $\lambda(u) = -\infty$, a contradiction.

To conclude, in all the three cases the hypothesis $t < 0$ in Ω leads to a contradiction. Hence $t \equiv 0$ in Ω; that is, $u(x)$ is harmonic in Ω. This completes the proof of the theorem. □

Liouville's theorem states that a superharmonic function > 0 in $\boldsymbol{R}^2$ is a constant. An analogous result using the operator Δ^2 is given in the following

Corollary 4.5 *Let $u(x)$ be a superharmonic function in $\boldsymbol{R}^n$, $2 \leq n \leq 4$, for which $\Delta^2 u \geq 0$. Let $u^* = \inf(u, 0)$ and suppose $\lambda(u^*) > -\infty$. Then u is a constant.*

Proof. Since $\lambda(u^*) > -\infty$, so is $\lambda(u)$ and hence by the above Theorem 4.4, u is harmonic in $\boldsymbol{R}^n$, $2 \leq n \leq 4$.

Since $M(r, |u|) = M(r, u) + 2M(r, u^-) = u(0) - 2M(r, u^*)$, we conclude that $\lambda(-|u|)$ is finite. This means in particular that the harmonic function $u(x)$ in $\boldsymbol{R}^n$ satisfies the condition $\lim_{r\to\infty} \frac{M(r,|u|)}{r} = 0$.

Hence u is a constant (M. Brelot [7] p. 202). □

5. Biharmonic green domains in $\boldsymbol{R}^n$

A domain Ω in $\boldsymbol{R}^n$, $n \geq 2$, is said to be a (harmonic) Green domain if the Green function $G(x, y)$ exists in Ω.

Definition 5.1 A domain Ω in $\boldsymbol{R}^n$, $n \geq 2$, is said to be a biharmonic Green domain if it is a (harmonic) Green domain and if for any $y \in \Omega$ there exists a potential $q_y(x)$ in Ω such that $\Delta q_y(x) = -G_y(x)$ in Ω.

Note This definition is a variant of the one given by L. Sario [10] which is as follows: Let Ω be a domain in $\boldsymbol{R}^n$, $n \geq 2$. Let Ω_n be a regular exhaustion of Ω such that $y \in \Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}$ and $\Omega = \cup \Omega_n$. Let $p_n(x)$ be the Green potential in Ω_n with pole $\{y\}$. Let $q_n(x)$ be the function in $\bar{\Omega}_n$ such that $\Delta q_n = -p_n$ in Ω_n, $q_n = 0 = \Delta q_n$ on $\partial \Omega_n$. Let $q = \sup q_n$. If $q \not\equiv \infty$, Ω is said to possess biharmonic Green functions, that is Ω is a biharmonic Green domain.

The following Lemma is proved in [3].

Lemma 5.2 *A domain Ω in $\boldsymbol{R}^n$, $n \geq 2$, is a biharmonic Green domain if and only if there exist potentials p and q in Ω such that $\Delta q = -p$.*

Proposition 5.3 *Every relatively compact domain Ω in $\boldsymbol{R}^n$, $n \geq 2$, is a biharmonic Green domain.*

Proof. Let Ω_0 be a relatively compact domain $\supset \bar{\Omega}$. Choose a superharmonic function q_0 in Ω_0 such that $\Delta q_0 = -1$. (M. Brelot [6] has proved that given any measure μ on an open set w in $\boldsymbol{R}^n$, there exists a subharmonic functions s in w such that $\Delta s = \mu$).

Since $\bar{\Omega} \subset \Omega_0$, q_0 has a harmonic minorant in Ω; if h_0 is the greatest harmonic minorant of q_0 in Ω, $q_1 = q_0 - h_0$ is a potential in Ω such that $\Delta q_1 = -1$ in Ω.

Choose a potential p in Ω such that $p \leq 1$ in Ω. Let s be a superharmonic function in Ω such that $\Delta s = -p$ and let t be a superharmonic function in Ω such that $\Delta t = -(1-p)$.

Then $s + t = q_1 +$ a harmonic function in Ω, which implies that s has a subharmonic minorant in Ω; hence we can construct the greatest harmonic minorant H of s in Ω.

Then $q = s - H$ is a potential in Ω such that $\Delta q = -p$.

That is, Ω is a biharmonic Green domain. □

Theorem 5.4 *Let Ω be a domain in $\boldsymbol{R}^n$, $2 \leq n \leq 4$, such that $K = \boldsymbol{R}^n \backslash \Omega$ is compact (K can be empty). Then Ω is not a biharmonic Green domain.*

Proof. Suppose Ω is a biharmonic Green domain; that is, there exist potentials p and q in Ω such that $\Delta q = -p$.

Let $r > 0$ be large so that $K \subset \{x : |x| < r\}$.

Then q is a positive superharmonic function in $|x| > r$ for which $\Delta^2 q \geq 0$. Then by Lemma 4.2, q is harmonic in $|x| > r$. This implies that $p \equiv 0$ in Ω, a contradiction.

This completes the proof of the theorem. □

However, when $n \geq 5$ there is no place for such exceptions. For we have the following theorem.

Theorem 5.5 *Any domain* Ω *in* $\boldsymbol{R}^n$, $n \geq 5$, *is a biharmonic Green domain.*

Proof. Without loss of generality, we'll assume that $0 \in \Omega$. Write $r = |x|$. Since r^{2-n} is a positive superharmonic function, write $r^{2-n} = p + h$ in Ω where p is a potential in Ω and h is harmonic.

Corresponding to the potential p in Ω, there exists a superharmonic function s in Ω such that $\Delta s = -p \geq -p - h = -r^{2-n} = \Delta u$ where $u = \frac{r^{4-n}}{2(n-4)}$ is a positive superharmonic function in Ω.

Since $\Delta s \geq \Delta u$, there exists a subharmonic function v in Ω such that $s = u + v$ in Ω; this means, since $u > 0$, that s has a subharmonic minorant in Ω. Hence we can write $s = q + H$ in Ω where q is a potential in Ω and H is harmonic (not necessarily positive).

Then $\Delta q = \Delta s = -p$ in Ω. Since p and q are potentials in Ω, this means that Ω is a biharmonic Green domain. □

Question In $\boldsymbol{R}^2$, a domain Ω is a (harmonic) Green domain if and only if $\boldsymbol{R}^2 \backslash \Omega$ is not locally polar. In view of Theorem 5.4, can we prove that a harmonic Green domain Ω in $\boldsymbol{R}^n$, $2 \leq n \leq 4$, is a biharmonic Green domain if and only if $\boldsymbol{R}^n \backslash \Omega$ is not compact?

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