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On the generalized absolute convergence of Fourier series

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Abstract. Sufficient conditions are given by means of the best trigonometric approximation in L^p $(1 < p \leq 2)$ and structural properties of $f \in L^p$ for the convergence of the series

$$\sum_{n=1}^{\infty} \omega_n(\varphi(|a_n|) + \varphi(|b_n|)),$$

where a_n and b_n are the Fourier coefficients of f, $\{\omega_n\}$ is a certain sequence of positive numbers, $\varphi(u)$ $(u \geq 0)$ denotes an increasing concave function.

Key words: absolute convergence, best approximation, structural condition, Fourier coefficients.

1. Introduction

Let $f(x)$ be a 2π-periodic Lebesgue integrable to the pth power $(p \geq 1)$ function and let

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Furthermore let $E_n(p)$ denote the best approximation of f by trigonometric polynomials of order at most n in the space L^p.

In a recent paper [3], among others, we showed that

$$\sum_{n=1}^{\infty} n^{\delta} \varphi\left(n^{1/p-1} E_n(p)\right) < \infty$$

is a sufficient condition for the convergence of the series

$$\sum_{n=1}^{\infty} n^{\delta}(\varphi(|a_n|) + \varphi(|b_n|),$$

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where $\delta \geq 0$ and $\varphi(u)$ $(u \geq 0,\ \varphi(0) = 0)$ is an increasing and concave function.

In the special case $\varphi(x) = x^\beta$ $(0 < \beta \leq 1)$ in an erstwhile paper [2], instead of the factors n^δ with arbitrary nonnegative factors ω_n, that is, for the convergence of the series

$$\sum_{n=1}^{\infty} \omega_n(|a_n|^\beta + |b_n|^\beta) \tag{1.1}$$

we established such a sufficient condition which generalized a well-known result of Konjuskov [1] pertaining to the convergence of the series (1.1) in the special case $\omega_n = n^\delta$.

Already in the paper [3] we raised the problem to find a sharp sufficient condition for the convergence of the series

$$\sum_{n=1}^{\infty} \omega_n(\varphi(|a_n|) + \varphi(|b_n|)), \tag{1.2}$$

however, up to now, unfortunately, we are not able to give such a sufficient condition for an arbitrary sequence $\omega := \{\omega_n\}$.

Consequently the aim of the present work is more moderate, we shall establish sufficient conditions for the convergence of the series (1.2) setting certain additional monotonicity assumptions on the sequence ω. Naturally the sequence $\omega_n = n^\delta$ $(\delta \geq 0)$ plentifully satisfies our assumptions on ω.

In order to make easy the presentation of our results we recall some definitions and introduce certain notations.

In the sequel we shall assume that $p \geq 1$, K, K_i denote positive constants, and may vary from occurance to occurance, $K_i(\cdot)$ denotes such constant which depends only those parameters as indicated in the bracket.

We say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi β-power-monotone increasing* (*decreasing*) if there exists a constant $K := K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta\gamma_n \geq m^\beta\gamma_m \qquad (n^\beta\gamma_n \leq Km^\beta\gamma_m) \tag{1.3}$$

holds for any $n \geq m$, $m = 1, 2, \ldots$. If the terms γ_n of a sequence γ satisfy the inequalities

$$K(\gamma)\gamma_n \geq \gamma_{n+1} \qquad (\gamma_n \leq K(\gamma)\gamma_{n+1}) \tag{1.4}$$

for all $n \geq n_0(\gamma) \geq 1$, then it will be called *slowly quasi increasing* (*decreasing*).

Finally denote

$$\rho_n := (a_n^2 + b_n^2)^{1/2}, \quad \text{and} \quad p' := \frac{p}{p-1}.$$

Now we can formulate the first two theorems.

Theorem 1 *Let $1 < p \leq 2$, $f \in L^p(0, 2\pi)$, and let $\omega := \{\omega_n\}$ be a quasi η-power-monotone decreasing sequence of positive numbers with some negative η. If $\varphi(u)$ $(u \geq 0,\ \varphi(0) = 0)$ is an increasing and concave function and*

$$\sum_{n=1}^{\infty} \omega_n \varphi\left(\left\{\frac{1}{n}\sum_{k=n}^{\infty} \rho_k^{p'}\right\}^{1/p'}\right) < \infty, \tag{1.5}$$

then

$$\sum_{n=1}^{\infty} \omega_n \varphi(\rho_n) < \infty. \tag{1.6}$$

Utilizing the following known result (see [5])

$$\sum_{k=n}^{\infty} \rho_k^{p'} \leq K E_n^{p'}(p), \qquad 1 < p \leq 2,$$

Theorem 1 yields immediately the following result.

Theorem 2 *If p, f, ω and φ have the same meaning and properties as in Theorem 1 then the condition*

$$\sum_{n=1}^{\infty} \omega_n \varphi(n^{-1/p'} E_n(p)) < \infty$$

implies (1.6).

Since the sequence $\omega := \{n^\delta\}$ is clearly quasi $(-\delta)$-power-monotone decreasing, thus Theorem 2 is an extension of Theorem 2 given in [3] from positive δ to arbitrary δ, but the enlargement visibly has sense only if $\delta \geq -1$.

We also mention that Theorem 2 in the special case $\varphi(x) = x^\beta$ and

$\omega_n = n^\delta$ was proved by Konjuskov [1], that is, that

$$\sum_{n=1}^{\infty} n^{\delta-\beta/p'} E_n^\beta(p) < \infty$$

implies

$$\sum_{n=1}^{\infty} n^\delta \rho_n^\beta < \infty.$$

In our old-time paper [2] we can also find a result (see Hilfssatz III) which gives a structural condition. Namely it is proved that

$$\int_0^1 t^{-2-\delta} \left(\int_0^{2\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p dx \right)^{\beta/p} dt < \infty \tag{1.7}$$

implies

$$\sum_{n=1}^{\infty} n^\delta \left\{ \sum_{k=n}^{\infty} \rho_k^{p'} \right\}^{\beta/p'} < \infty. \tag{1.8}$$

Now we raise the following problem: Can we replace in (1.7) the function x^β with an arbitrary increasing and concave function $\varphi(x)$ such that the new condition should imply (1.8) also with $\varphi(x)$ in place of x^β? The answer is yes.

The next problem: Can we also substitute the function x^δ in (1.7) and (1.8) by a suitable function $\omega(x)$ such that the new structural condition with $\varphi(x)$ and $\omega(x)$ should be sufficient for (1.8) naturally with $\varphi(x)$ and $\omega(x)$ in place of x^β and x^δ? The answer is incompletely yes, namely we have to restrict the assumption presented in Theorem 1 on the sequence $\omega := \{\omega_n\}$.

The above statements follow from the following result, where we shall use the function defined as follows:

$$\omega(x) := \begin{cases} \omega_n, & \text{if } x = n, \ n \geq 1, \\ \text{linear between } n \text{ and } n+1. \end{cases}$$

Theorem 3 *Let $1 < p \leq 2$, $f \in L^p(0, 2\pi)$, and let $\omega := \{\omega_n\}$ be a quasi η-power-monotone decreasing sequence of positive numbers with some negative η, and simultaneously quasi ρ-power-monotone increasing with some*

$\rho < 1$. *If* $\varphi(u)$ $(u \geq 0,\ \varphi(0) = 0)$ *is an increasing and concave function, furthermore*

$$\int_0^1 t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left\{\int_0^{2\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p dx\right\}^{1/p}\right)dt < \infty, \tag{1.9}$$

then

$$\sum_{n=1}^{\infty} \omega_n \varphi\left(\left\{\sum_{k=n}^{\infty} \rho_k^{p'}\right\}^{1/p'}\right) < \infty. \tag{1.10}$$

2. Lemmas

We require the following lemmas.

Lemma 1 ([4], or see [3]) *Let k and m be natural numbers. Then the following inequalities*

$$m2^{1-m} \leq \sum_{j=k^m}^{(k+1)^m-1} j^{\frac{1}{m}-1} \leq m2^{m-1} \tag{2.1}$$

hold.

In the sequel $[\alpha]$ will denote the integer part of α.

Lemma 2 *Let $1 < p \leq 2$, $\omega := \{\omega_n\}$ be a sequence of positive numbers, and let m be an arbitrary natural number. Furthermore let $\{\alpha_n\}$ be a monotone nonincreasing sequence of nonnegative numbers and let $\varphi(u)$ $(u \geq 0$, $\varphi(0) = 0)$ be an increasing concave function. Then the conditions*

$$\sigma(\omega, m) := \sum_{k=1}^{\infty} k^{\frac{1}{m}-1}\omega_{[k^{1/m}]}\varphi\left(k^{\frac{1-p}{pm}}\alpha_{[k^{1/m}]}\right) < \infty \tag{2.2}$$

and

$$\sigma(\omega) := \sum_{k=1}^{\infty} \omega_k \varphi\left(k^{\frac{1-p}{p}}\alpha_k\right) < \infty \tag{2.3}$$

are equivalent.

Proof. First we show that (2.2) implies (2.3). Considering the first inequality in (2.1), the monotonicity of $\varphi(x)/x$ and that $p > 1$, an easy calculation gives that

$$\begin{aligned}
\sigma(\omega, m) &= \sum_{k=1}^{\infty} \sum_{j=k^m}^{(k+1)^m-1} j^{\frac{1}{m}-1} \omega_{[j^{1/m}]} \varphi\left(j^{\frac{1-p}{pm}} \alpha_{[j^{1/m}]}\right) \\
&\geq \sum_{k=1}^{\infty} \omega_k \varphi\left((k+1)^{\frac{1-p}{p}} \alpha_k\right) \sum_{j=k^m}^{(k+1)^m-1} j^{\frac{1}{m}-1} \\
&\geq K(m,p) \sum_{k=1}^{\infty} \omega_k \varphi\left(k^{\frac{1-p}{p}} \alpha_k\right) = K(m,p)\sigma(\omega). \qquad (2.4)
\end{aligned}$$

This verifies the implication (2.2) $\Rightarrow$ (2.3).

The proof of (2.3) $\Rightarrow$ (2.2) runs likewise. Using the first equality in (2.4), and the second inequality in (2.1), we obtain immediately that

$$\sigma(\omega, m) \leq K(m) \sum_{k=1}^{\infty} \omega_k \varphi\left(k^{\frac{1-p}{p}} \alpha_k\right) = K(m)\sigma(\omega).$$

The proof is complete. □

Lemma 3 (Jensen's inequality) *Let $\varphi(u)$ ($u \geq 0$, $\varphi(0) = 0$) be an increasing concave function. Then, for any infinite sequence of nonnegative numbers $x_1, x_2, \ldots, x_n, \ldots$ and any infinite sequence of positive numbers $p_1, p_2, \ldots, p_n, \ldots$, the following inequality*

$$\frac{\sum_{k=1}^{\infty} p_k \varphi(x_k)}{\sum_{k=1}^{\infty} p_k} \leq \varphi\left(\frac{\sum_{k=1}^{\infty} p_k x_k}{\sum_{k=1}^{\infty} p_k}\right) \qquad (2.5)$$

holds, assuming that each series in (2.5) converges.

3. Proofs of the theorems

Proof of Theorem 1. In order to simplify writing we shall write only $k^{1/m}$ instead of $[k^{1/m}]$.

Let $m > -\eta + 1$. An elementary calculation, using an Abel rearrangement and the Jensen inequality, gives that

$$\sum_{n=1}^{\infty} \omega_n \varphi(\rho_n) = \sum_{n=1}^{\infty} \sum_{k=1}^{n^m} \omega_n n^{-m} \varphi(\rho_n)$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{\infty} \sum_{n=k^{1/m}}^{\infty} \omega_n n^{-m} \varphi(\rho_n) \\
&\leq \sum_{k=1}^{\infty} \Bigg(\sum_{n=k^{1/m}}^{\infty} \omega_n n^{-m} \Bigg) \varphi\Bigg(\Bigg\{ \sum_{n=k^{1/m}}^{\infty} \omega_n n^{-m} \Bigg\}^{-1} \sum_{n=k^{1/m}}^{\infty} \omega_n n^{-m} \rho_n \Bigg) \\
&=: S_1. \qquad (3.1)
\end{aligned}
$$

Since $-m-\eta < -1$ and the sequence ω is quasi η-power-monotone decreasing, we get that

$$
\begin{aligned}
\sum_{n=\mu}^{\infty} \omega_n n^{-m} &= \sum_{n=\mu}^{\infty} \omega_n n^{\eta} n^{-m-\eta} \leq K \omega_\mu \mu^{\eta} \sum_{n=\mu}^{\infty} n^{-m-\eta} \\
&\leq K_1 \omega_\mu \mu^{1-m}. \qquad (3.2)
\end{aligned}
$$

Thus

$$
S_1 \leq K_1 \sum_{k=1}^{\infty} \omega_{k^{1/m}} k^{\frac{1}{m}-1} \varphi\Bigg(\omega_{k^{1/m}}^{-1} k^{1-\frac{1'}{m}} \sum_{n=k^{1/m}}^{\infty} \omega_n n^{-m} \rho_n \Bigg).
$$

Now we use the Hölder inequality and an analogous estimate as in (3.2) and then we obtain that

$$
\begin{aligned}
S_1 &\leq K_1 \sum_{k=1}^{\infty} \omega_{k^{1/m}} k^{\frac{1}{m}-1} \\
&\qquad \varphi\Bigg(\omega_{k^{1/m}}^{-1} k^{1-\frac{1}{m}} \Bigg(\sum_{n=k^{1/m}}^{\infty} \rho_n^{p'} \Bigg)^{1/p'} \Bigg(\sum_{n=k^{1/m}}^{\infty} \omega_n^{p} n^{-pm} \Bigg)^{1/p} \Bigg) \\
&\leq K_2 \sum_{k=1}^{\infty} \omega_{k^{1/m}} k^{\frac{1}{m}-1} \varphi\Bigg(k^{\frac{1-p}{pm}} \Bigg(\sum_{n=k^{1/m}}^{\infty} \rho_n^{p'} \Bigg)^{1/p'} \Bigg) =: S_2. \qquad (3.3)
\end{aligned}
$$

To estimate the sum S_2 we use Lemma 2 with $\alpha_k := \left(\sum_{n=k}^{\infty} \rho_n^{p'} \right)^{1/p'}$, whence we get that $S_2 < \infty$ if and only if the inequality (1.5) holds, namely $\frac{1-p}{p} = -\frac{1}{p'}$.

Since the inequality (1.5) is assumed to be true, thus, by (3.1) and (3.3), the statement (1.6) is proved.

The proof is complete. □

Proof of Theorem 3. The monotonicity assumptions on the sequence ω imply that it is slowly quasi monotone increasing, thus there exists a constant

$K := K(\omega) \geq 1$ such that

$$K\omega_n \geq \omega_{n+1}, \quad \text{for all } \ n \geq 1, \tag{3.4}$$

$$n^\eta \omega_n \leq K m^\eta \omega_m, \ \eta < 0, \quad \text{for all } \ n \geq m \geq 1, \tag{3.5}$$

and

$$K n^\rho \omega_n \geq m^\rho \omega_m, \ \rho < 1, \quad \text{for all } \ n \geq m \geq 1 \tag{3.6}$$

hold.

Now let $A := \max(K + 1, \omega_1)$. Taking into account (3.4) an easy consideration shows that we can define an increasing sequence $\{p_m\}$ of integers such that $p_0 = 0$ and for all $m \geq 0$ the inequalities

$$A^m \leq \sum_{n=p_m+1}^{p_{m+1}} \omega_n \leq A^{m+1} \tag{3.7}$$

hold.

Taking into account this estimations and using the monotonicity properties of the sequence ω we shall show that the terms of the sequence $\{p_m\}$ satisfy the inequality

$$p_{m+1} \leq K(\omega) p_m \quad \text{for } \ m \geq 1. \tag{3.8}$$

Let

$$\Omega_n := \sum_{k=1}^{n} \omega_k.$$

Using the property (3.5) of ω we get that

$$\Omega_n \geq K_1(\omega) n \omega_n, \tag{3.9}$$

namely

$$\Omega_n = \sum_{k=1}^{n} \omega_k k^\eta k^{-\eta} \geq \frac{1}{K} n^\eta \omega_n \sum_{k=1}^{n} k^{-\eta},$$

and (3.6) implies similarly that

$$\Omega_n \leq K_2(\omega) n \omega_n. \tag{3.10}$$

By (3.7) we know that

$$\Omega_{p_m} \geq A^{m-1}$$

and

$$\Omega_{p_{m+1}} - \Omega_{p_m} \leq A^{m+1}.$$

Hence it follows that

$$\frac{\Omega_{p_{m+1}}}{\Omega_{p_m}} \leq 1 + \frac{A^{m+1}}{\Omega_{p_m}} \leq 1 + A^2, \tag{3.11}$$

furthermore, by (3.9) and (3.10),

$$\begin{aligned}
\frac{\Omega_{p_{m+1}}}{\Omega_{p_m}} &\geq K_3(\omega)\frac{p_{m+1}\omega_{p_{m+1}}}{p_m\omega_{p_m}} \\
&= K_3\frac{(p_{m+1})^{\rho}\omega_{p_{m+1}}(p_{m+1})^{1-\rho}}{(p_m)^{\rho}\omega_{p_m}(p_m)^{1-\rho}} \\
&\geq K_3K^{-1}\left(\frac{p_{m+1}}{p_m}\right)^{1-\rho}.
\end{aligned}$$

Since $\rho < 1$ the last estimation and (3.11) imply (3.8).

Since the functions $\varphi(u)$ and $u^{1/p'}$ are concave, and (3.7) holds, we obtain that

$$\begin{aligned}
\sum_{n=1}^{\infty}\omega_n\varphi\left(\left\{\sum_{k=n}^{\infty}\rho_k^{p'}\right\}^{1/p'}\right) &\leq \sum_{m=0}^{\infty}\sum_{n=p_m+1}^{p_{m+1}}\omega_n\varphi\left(\sum_{\nu=m}^{\infty}\left\{\sum_{k=p_\nu+1}^{p_{\nu+1}}\rho_k^{p'}\right\}^{1/p'}\right) \\
&\leq A\sum_{m=0}^{\infty}A^m\sum_{\nu=m}^{\infty}\varphi\left(\left\{\sum_{k=p_\nu+1}^{p_{\nu+1}}\rho_k^{p'}\right\}^{1/p'}\right) \\
&\leq A^2\sum_{\nu=0}^{\infty}A^{\nu}\varphi\left(\left\{\sum_{k=p_\nu+1}^{p_{\nu+1}}\rho_k^{p'}\right\}^{1/p'}\right).
\end{aligned} \tag{3.12}$$

Next we set

$$F(t) := \left\{\int_0^{2\pi}|f(x+2t)+2f(x-t)-2f(x)|^p dx\right\}^{1/p}.$$

Then the Hausdorff-Young theorem (see [6], p. 101) gives that

$$F(t) \geq \left(\sum_{k=1}^{\infty} \rho_k^{p'} |\sin kt|^{2p'} \right)^{1/p'}. \tag{3.13}$$

Hence, (1.9) and (3.13) imply that

$$I := \int_0^1 t^{-2} \omega\left(\frac{1}{t}\right) \varphi\left(\left\{ \sum_{k=1}^{\infty} \rho_k^{p'} |\sin kt|^{2p'} \right\}^{1/p'} \right) dt < \infty. \tag{3.14}$$

On the other hand, it is obvious that

$$I \geq \sum_{m=1}^{\infty} \int_{1/p_{m+1}}^{1/p_m} t^{-2} \omega\left(\frac{1}{t}\right) \varphi\left(\left\{ \sum_{k=p_{m-1}+1}^{p_m} \rho_k^{p'} |\sin kt|^{2p'} \right\}^{1/p'} \right) dt. \tag{3.15}$$

Since by (3.8)

$$0 < c \leq \frac{p_{m-1}}{p_{m+1}} \leq kt \leq \frac{p_m}{p_m} = 1$$

holds, thus (3.14) and (3.15) yield that

$$\sum_{m=1}^{\infty} \varphi\left(\left\{ \sum_{k=p_{m-1}+1}^{p_m} \rho_k^{p'} \right\}^{1/p'} \right) \int_{1/p_{m+1}}^{1/p_m} t^{-2} \omega\left(\frac{1}{t}\right) dt < \infty. \tag{3.16}$$

Because, by (3.7),

$$\int_{1/p_{m+1}}^{1/p_m} t^{-2} \omega\left(\frac{1}{t}\right) dt \geq A^m$$

maintains, thus (3.12) and (3.16) verify the implication (1.9)⇒(1.10), and this ends the proof. □

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