Hokkaido Mathematical Journal Vol. 31 (2002) p. 283–299

Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ

Bruno de MALAFOSSE

(Received June 7, 2000; Revised May 28, 2001)

Abstract. We give some properties of the well-known operator Δ^{μ}, μ being a given integer. Then we characterize the matrix transformations that belong to $(s_r[(\Delta-\lambda I)^{\mu}], s_r)$, where $s_r[(\Delta-\lambda I)^{\mu}]=\{X\,/\,(\Delta-\lambda I)^{\mu}X\in s_r\}$.

Key words: infinite matrix, Banach algebra, sequence spaces.

1. Introduction

The Cesàro operator C_1 has been studied by many authors such as Reade [17], Maddox [7], Okutoyi [14] and de Malafosse [9]. Since there is a relation between this operator and the 1st difference operator Δ, many authors have given results on this last operator, see for instance Malkowsky [10], [11], Kizmaz [4], Çolak and Et [2]. These authors gave many characterizations of the operators A which map the space $(\Delta^{\mu})^{-1}(l^{\infty})$ into l^{∞}, that is $A\in(l^{\infty}(\Delta^{\mu}), l^{\infty})$. Let us cite Mursaleen [13] for an application of infinite matrices to Walsh functions. In this paper we establish a relation between the resolution of an infinite linear system, (see [3], [5], [8], [12], [15] and [16]) and the summability theory.

The plan of this paper is organized as follows. In Section 2 we recall the relation between an operator mapping a space of sequences into another sequence space and an infinite matrix. This lead us to the study of infinite linear systems. Then we give some spaces in which we shall solve these systems in the following. An isomorphism φ is defined permitting to do computations on infinite matrices belonging to an important class. Further in Sections 3 and 4 we give some properties of the well-known operators relatively to these new spaces. In the Section 5 we characterize the infinite matrices belonging to the sets $(s_r((\Delta-\lambda I)^{\mu}), s_r)$, $\lambda\neq 1$, $(l^{\infty}(\Delta^{\mu}), l^{\infty})$ and $(s_r((\Delta^{+})^{\mu}), s_r)$.

2000 Mathematics Subject Classification : 40C05, 46A45.

2. Definitions and well-known results

E and F are two subsets of the space s of all the sequences. We are interested in the study of some properties of an operator A mapping E into F, which can be written $A \in (E, F)$. This operator can be represented by an infinite matrix $A = (a_{nm})_{n,m \geq 1}$. For a given one column matrix $X = (x_n)_{n \geq 1}$, we define the product $Y = AX = (y_n)_{n \geq 1}$, by

$$y_n = \sum_{m=1}^{\infty} a_{nm} x_m, \quad (n = 1, 2, \ldots) \tag{1}$$

when all the series defined in the second member are convergent. Such a matrix A is often called too, a matrix transformation. For any subset E of s we shall write

$$AE = \{Y \in s \,/\, \exists X \in E \quad Y = AX\} \tag{2}$$

and for any subset F of s, we shall denote by $F(A)$ or $A^{-1}F$, the set of the sequences $X \in s$, such that $AX \in F$. In the following we shall be lead to use infinite linear systems. The resolution of such systems can be formulated in the following way: $B = (b_n)_{n \geq 1}$ being a given sequence does equation $AX = B$ admit a solution in a given space of sequences. To solve such matrix equation we need to define some particular spaces.

2.1. Spaces S_c and s_c

For a sequence $c = (c_n)_{n \geq 1}$, where $c_n > 0$, for every n, we define the space S_c, (see [5], [6], [8] and [9]) of the infinite matrices $A = (a_{nm})_{n,m \geq 1}$, such that $\sup_{n \geq 1} \left(\sum_{m \geq 1} |a_{nm}| \frac{c_m}{c_n} \right) < \infty$. S_c with respect to the norm:

$$\|A\|_{S_c} = \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \frac{c_m}{c_n} \right),$$

is a unital Banach algebra. We shall denote by $I = (\delta_{nm})_{n,m \geq 1}$, (where $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nm} = 1$ in the contrary case) the unit element. We define, too, the Banach space s_c of one-column matrices, $X = (x_n)_{n \geq 1}$, such that $\sup_{n \geq 1} \left(\frac{|x_n|}{c_n} \right) < \infty$, normed by:

$$\|X\|_{s_c} = \sup_{n} \left(\frac{|x_n|}{c_n} \right). \tag{3}$$

If $c = (c_n)_n$, and $c' = (c'_n)_n$ are two sequences, such that: $0 < c_n < c'_n \ \forall n$, then:

$$s_c \subset s_{c'}.$$

If $X \in s_c$ and $A = (a_{nm})_{n,m\geq 1} \in S_c$, the product $AX \in s_c$ and

$$\|AX\|_{s_c} \leq \|A\|_{S_c} \|X\|_{s_c}. \tag{4}$$

This permits to say that $A \in (s_c, s_c)$. A particular case, very useful, is the one where $c_n = r^n$, $r > 0$. We denote, then, by S_r and s_r, the spaces S_c, and s_c. When $r = 1$, we obtain the space of the bounded sequences $l^\infty = s_1$.

S_c being a unital algebra, we have the useful result:

if A verifies the condition $\|I - A\|_{S_c} < 1$, A is invertible in the space S_c, and for every $B \in s_c$, the equation $AX = B$ admits one and only one solution in s_c, given by:

$$X = \sum_{n=0}^{\infty} (I - A)^n B. \tag{5}$$

2.2. Infinite matrices and power series

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be the power series defined in the open disk $|z| < R$. We can associate to f the upper triangular infinite matrix $A = \varphi(f) \in \bigcup_{0<r<R} S_r$, defined by

$$\varphi(f) = \begin{pmatrix} a_0 & a_1 & a_2 & . \\ & a_0 & a_1 & . \\ O & & a_0 & . \\ & & & . \end{pmatrix}.$$

Practically we shall write $\varphi[f(z)]$ instead of $\varphi(f)$. It can be verified that

Lemma 1 i) *The map $\varphi : f \to A$ is an isomorphism from the algebra of the power series defined in $|z| < R$, into the algebra of the corresponding matrices $\bar{A}$.*

ii) *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $a_0 \neq 0$, and $\frac{1}{f(z)} = \sum_{k=0}^{\infty} a'_k z^k$ admits $R' > 0$ as radius of convergence, then*

$$\varphi\left(\frac{1}{f}\right) = [\varphi(f)]^{-1} \in \bigcup_{0<r<R'} S_r.$$

It is easy to prove that for a given $k \in N$: the matrix $\varphi\left(z^k\right) =$

$(a_{nm})_{n,m\geq 1}$ is defined by: $a_{n,n+k} = 1$ for all $n \geq 1$ the other coefficients being equal to 0. Then we can write $I = \varphi(1)$. We have of course $\varphi\left(z^k\right)\varphi\left(z^l\right) = \varphi\left(z^{k+l}\right) = [\varphi(z)]^{k+l}$ for all k, $l \in N$ and $\varphi\left(\sum_{k=0}^{\infty} a_k z^k\right) = \sum_{k=0}^{\infty} a_k \varphi\left(z^k\right)$, since the power series defined by $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is convergent in a disk $D(0,R)$. We shall use such results to calculate the product of several matrices, or express the inverse of an infinite matrix belonging to $\bar{A}$.

3. Some new properties of the operator $\Delta^{(\mu)}$

The well-known operator $\Delta^{(\mu)} : s \to s$, where μ is an integer ≥ 1, is represented by the infinite lower triangular matrix Δ^{μ}, where $\Delta = \begin{pmatrix} 1 & & O \\ -1 & 1 & \\ O & . & . \end{pmatrix}$. We have for every $X = (x_n)_{n\geq 1}$, $\Delta X = (y_n)_{n\geq 1}$, with $y_1 = x_1$ and $y_n = x_n - x_{n-1}$ if $n \geq 2$. If we suppose that $\frac{c_{n-1}}{c_n} = O(1)$ as $n \to \infty$, the product Δ^{μ} can be defined in any algebra S_c, since

$$\|\Delta\|_{S_c} = \sup_{n\geq 2}\left(1 + \frac{c_{n-1}}{c_n}\right) < \infty$$

implies that $\Delta \in S_c$. As we have seen above, we shall consider the infinite matrix Δ^{μ} instead of the operator $\Delta^{(\mu)}$. Analogously we shall denote $\Delta^{+} = {}^t\Delta$. We can generalize the definition of Δ^{μ}, when μ is a real. So, if $\mu \in R - N$, we get

$$(1-z)^{\mu} = 1 + \sum_{k=1}^{\infty} \frac{-\mu(-\mu+1)\cdots(-\mu+k-1)}{k!} z^k, \quad \text{for } |z| < 1. \tag{6}$$

If we denote

$$\begin{cases} \begin{pmatrix} -\mu+k-1 \\ k \end{pmatrix} = \dfrac{-\mu(-\mu+1)\cdots(-\mu+k-1)}{k!} & \text{if } k > 0, \\ \begin{pmatrix} -\mu+k-1 \\ k \end{pmatrix} = 1 & \text{if } k = 0, \end{cases}$$

we have for any $\mu \in R$

$$\left(\Delta^{+}\right)^{\mu} = \varphi\left[(1-z)^{\mu}\right] = \varphi\left[\sum_{k=0}^{\infty} \begin{pmatrix} -\mu+k-1 \\ k \end{pmatrix} z^k\right] \quad \text{for } |z| < 1.$$

We deduce that if $\Delta^{\mu} = (\tau_{nm})_{n,m}$,

$$\tau_{nm} = \begin{cases} \begin{pmatrix} -\mu + n - m - 1 \\ n - m \end{pmatrix} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \tag{7}$$

We need at first a lemma based on the well-known property given in Maddox [6]:

$$A \in (s_1, s_1) \Leftrightarrow A \in S_1. \tag{8}$$

Then we have for any $r > 0$,

Lemma 2 *$A \in (s_r, s_r)$ if and only if $A \in S_r$.*

Proof. Denote for all scalars $\rho > 0$ by P_ρ the matrix $(\rho^n \delta_{nm})_{n,m \geq 1}$. We have: $X \in s_r$ if and only if $P_{1/r}X \in s_1$. $A \in (s_r, s_r)$ if and only if for all $X = P_r X' \in s_r$, with $X' \in s_1$ one has $A(P_r X') \in s_r$. This last assertion is equivalent to the following one: $\forall X' \in s_1$ we have $P_{1/r}A(P_r X') \in s_1$. Thus applying (8), $A \in (s_r, s_r) \Leftrightarrow P_{1/r}AP_r \in S_1$, that is $A \in S_r$. $\square$

In the following we need to define the sequences $e_n = (0, \ldots, 1, \ldots)$, 1 being in the nth position, for $n \geq 1$ and $e = (1, 1, \ldots)$. We have:

Proposition 3 i) *The operator represented by Δ is bijective from s_r into itself, for every $r > 1$ and Δ^+ is bijective from s_r into itself, for all r, $0 < r < 1$.*

ii) *Δ^+ is surjective and not injective from s_r into itself, for all $r > 1$.*

iii) *$\forall r \neq 1$ and for every integer $\mu \geq 1$ $(\Delta^+)^{\mu} s_r = s_r$.*

iv) *We have successively*

α) *If μ is a real > 0 and $\mu \notin N$, then Δ^{μ} maps s_r into itself when $r \geq 1$ but not for $0 < r < 1$.*

If $-1 < \mu < 0$, then Δ^{μ} maps s_r into itself when $r > 1$ but not for $r = 1$.

β) *If $\mu > 0$ and $\mu \notin N$, then $\Delta^{+\mu}$ maps s_r into itself when $0 < r \leq 1$ but not if $r > 1$.*

If $-1 < \mu < 0$, then $\Delta^{+\mu}$ maps s_r into itself for $0 < r < 1$ but not for $r = 1$.

v) *For a given integer $\mu \geq 1$, we have successively*

$$\begin{cases} \forall r > 1 : A \in (s_r(\Delta^\mu), s_r) \Leftrightarrow \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n}\right) < \infty, \\ \forall r \in]0,1[: A \in \left(s_r(\Delta^+)^\mu, s_r\right) \Leftrightarrow \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n}\right) < \infty. \end{cases}$$

vi) *For every integer* $\mu \geq 1$

$$s_1 \subset s_1(\Delta^\mu) \subset s_{(n^\mu)_{n\geq 1}} \subset \bigcap_{r>1} s_r.$$

vii) *If* $\mu > 0$ *and* $\mu \notin N$ *then* q *is the greatest integer strictly less than* $(\mu + 1)$. $\forall r > 1$

$$\operatorname{Ker}\left((\Delta^+)^\mu\right) \bigcap s_r = \operatorname{span}(V_1, V_2, \ldots, V_q),$$

where:

$$\begin{cases} V_1 = {}^t e, \quad V_2 = {}^t\left(A_1^1, A_2^1, \ldots\right), \quad V_3 = {}^t\left(0, A_2^2, A_3^2, \ldots\right), \ldots \\ \qquad V_q = {}^t(0, 0, \ldots, A_{q-1}^{q-1}, A_q^{q-1}, \ldots, A_n^{q-1}, \ldots); \end{cases} \tag{9}$$

$A_i^j = \frac{i!}{(i-j)!}$, *with* $0 \leq j \leq i$, *being the number of permutations of* i *things taken* j *at a time.*

Proof. We deduce i) from the inequalities $\|I - \Delta\|_{S_r} = 1/r < 1$ and $\|I - \Delta^+\|_{S_r} = r < 1$.

Assertion ii). Consider the matrix $(\Delta^+)(-e_1)$ obtained from Δ^+, by addition of the supplementary row $-e_1$ as the initial row. We have $-(\Delta^+)(-e_1) = \Delta$. Take now $r > 1$. We see that $\|I + (\Delta^+)(-e_1)\|_{S_r} = 1/r < 1$. Then $(\Delta^+)(-e_1)$ is bijective from s_r into itself. One deduces that for all $B \in s_r$, equation $(\Delta^+)X = B$ admits in s_r infinitely many solutions

$$X = \left[(\Delta^+)(-e_1)\right]^{-1} B(u) = -u^t e - \Sigma.B(0)$$

for all scalars u (see [5]). Then $(\Delta^+) s_r = s_r$. Elsewhere Δ^+ is not injective, since $(\Delta^+)^t e = 0$ and ${}^t e \in s_r$. Using i) and ii) we see that for all $r \neq 1$, $(\Delta^+) s_r = s_r$, what implies $(\Delta^+)^\mu s_r = s_r$.

iv) α). Let now $\mu > -1$, $\mu \notin N$, and $r > 0$. Using (7), we obtain

$$\|\Delta^\mu\|_{S_r} = \sup_{n\geq 1}\left(\sum_{m=1}^{n} \left|\begin{pmatrix} -\mu + n - m - 1 \\ n - m \end{pmatrix}\right| r^{m-n}\right)$$

$$= \sum_{k=0}^{\infty} \left| \begin{pmatrix} k-\mu-1 \\ k \end{pmatrix} \right| r^{-k}.$$

If we put $u_k(r) = \left| \begin{pmatrix} k-\mu-1 \\ k \end{pmatrix} \right| r^{-k}$, we see that the series $\sum_k u_k(r)$ is convergent for $r > 1$, and divergent for $0 < r < 1$, since $\frac{u_{k+1}(r)}{u_k(r)} = \left| \frac{k-\mu}{k+1} \right| \frac{1}{r} \to \frac{1}{r}$, as $k \to \infty$. When $r = 1$, we get

$$\frac{u_{k+1}(1)}{u_k(1)} = \left| \frac{k-\mu}{k+1} \right| = 1 - \frac{\mu+1}{k} + o\left(\frac{1}{k}\right) \quad \text{as } k \to \infty$$

which proves that the series $\sum_k u_k(1)$ is convergent when $\mu > 0$ and divergent when $-1 < \mu < 0$.

By an analogous reasoning, we obtain β), since we have then

$$\begin{aligned} \|\Delta^{+\mu}\|_{S_r} &= \sup_{n \geq 1} \left(\sum_{m=n}^{\infty} \left| \begin{pmatrix} -\mu+m-n-1 \\ m-n \end{pmatrix} \right| r^{m-n} \right) \\ &= \sum_{k=0}^{\infty} \left| \begin{pmatrix} k-\mu-1 \\ k \end{pmatrix} \right| r^{k}. \end{aligned}$$

v). We have $\|I - \Delta\|_{S_r} = 1/r < 1$, for all $r > 1$, hence Δ^μ is bijective from s_r into itself. Thus $s_r(\Delta^\mu) = s_r$ and Lemma 2 gives the conclusion. The second case follows similarly by the same way, using i).

Assertion vi). Let us show at first that $s_1 \subset s_1(\Delta)$. For all $X = (x_n)_{n \geq 1} \in s_1$, we have $\Delta X \in s_1$, since $x_n - x_{n-1} = O(1)$, as $n \to \infty$. Hence $X \in s_1(\Delta)$, what proves that $s_1 \subset s_1(\Delta)$.

Let us prove that $s_1(\Delta) \subset s_{(n)_{n \geq 1}}$. Explicite, at first the matrix Δ^{-1}. From Lemma 1, we see that $\Delta^+ = \varphi(1-z)$ and since $\Sigma^{-1} = \Delta$, we deduce that $({}^t\Sigma)^{-1} = \varphi(\frac{1}{1-z})$ which implies that Σ is the lower triangular matrix all of whose entries below the main diagonal are equal to 1. Suppose now that $X = (x_n)_{n \geq 1} \in s_1(\Delta)$. We deduce that there exists a sequence $B = (b_n)_{n \geq 1} \in s_1$ such that $x_n = \sum_{i=1}^n b_i$, $\forall n \geq 1$. Hence, there is $K > 0$, such that $|x_n| = |\sum_{i=1}^n b_i| \leq Kn$, what proves that $X \in s_c$ with $c = (n)_{n \geq 1}$ and $s_1(\Delta) \subset s_{(n)}$. Further on, suppose by induction that for an integer k, with $1 \leq k \leq \mu - 1$: $s_1 \subset s_1(\Delta^k) \subset s_{(n^k)_{n \geq 1}}$. Then applying the operator Δ^{-1} to every term of the inclusions, we deduce that

$$s_1 \subset s_1(\Delta) \subset s_1(\Delta^{k+1}) \subset s_{(n^k)}(\Delta).$$

And as above, we see that $X \in s_{(n^k)}(\Delta)$ implies that there exists a sequence $B = (b_n)_{n\geq 1} \in s_{(n^k)}$ such that $X = \left(\sum_{i=1}^n b_i\right)_{n\geq 1}$. Thus, there is $K > 0$, such that $\left|\sum_{i=1}^n b_i\right| \leq \sum_{i=1}^n |b_i| \leq nKn^k$, what proves that $s_{(n^k)}(\Delta) \subset s_{(n^{k+1})}$. It remains to prove that $s_{(n^\mu)} \subset \bigcap_{r>1} s_r$. For this, consider any real $r > 1$, we have $\frac{n^\mu}{r^n} = o(1)$ as $n \to \infty$. Then $\forall X = (x_n)_{n\geq 1} \in s_{(n^\mu)}$, $x_n = O(n^\mu) = O(r^n)$, as $n \to \infty$, which proves that $X \in s_r$. Hence we deduce vi).

Assertion vii). It is well-known (see [1]) that $\operatorname{Ker}(\Delta^+)^\mu$ is the set of all the sequences $(P_{q-1}(n))_{n\geq 1}$, P_{q-1} being an arbitrary polynomial of degree less than or equal to $(q-1)$. Also it is well-known that $\dim \operatorname{Ker}(\Delta^+)^\mu = q$ and since $V_1, V_2, \ldots, V_q \in \operatorname{Ker}(\Delta^+)^\mu \bigcap s_r$ for $r > 1$ and are linearly independent, we conclude that $\operatorname{Ker}(\Delta^+)^\mu \bigcap s_r = \operatorname{span}(V_1, V_2, \ldots, V_q)$. $\square$

4. Spectra of the operators Δ, Δ^+ and Σ relatively to the space s_r.

We give here some spectral properties of several well-known operators. Recall that $C_1 = (a_{nm})_{nm\geq 1}$ is the Cesàro operator of order 1, defined by the infinite matrix

$$\begin{cases} a_{nm} = 1/n & \text{if } m \leq n, \\ a_{nm} = 0 & \text{otherwise.} \end{cases}$$

(see [3], [7], [9], [10] and [17]). We have seen in vi) of Proposition 3, that Σ denotes the operator Δ^{-1}. There exists a relation between these operators. Indeed if $D = (n\delta_{nm})_{n,m\geq 1}$ then $DC_1 = \Sigma$ and $\Delta(DC_1) = I$, which proves that $C_1^{-1} = \Delta D$. In this section A is an operator mapping s_r into itself, r being a given real > 0. We shall denote by $\sigma(A)$ its spectrum, set of complex numbers λ, such that $(A - \lambda I)$ as operator from s_r into itself, is not invertible. We can express the following results,

Theorem 4 *We have,*

$$\begin{cases} \text{i)} & \sigma(C_1) = \{0\} \bigcup \left\{ \dfrac{1}{n} \; n \geq 1 \right\}, \\ \text{ii)} & \sigma(\Delta) = \overline{D}(1, 1/r), \\ \text{iii)} & \sigma(\Delta^+) = \overline{D}(1, r). \end{cases}$$

Proof. The proof of i) is given in [9]. Let us set in the following $\Lambda_\lambda(A) = \frac{1}{\lambda-1}(\lambda I - A)$, where A is a given matrix and λ any complex number distinct from 1.

Assertion ii). Suppose that $\lambda \notin \sigma(A)$, i.e. $\lambda I - \Delta$, $\lambda \neq 1$, is bijective from s_r into itself. Then $\lambda I - \Delta$ is invertible, since it is a lower infinite triangular matrix with non zero element on the main diagonal. $(\lambda I - \Delta)^{-1}$ is also bijective from s_r into itself, which implies that $(\lambda I - \Delta)^{-1} \in S_r$, from Lemma 2. We are lead to explicitly calculate the inverse of $\lambda I - \Delta$. Indeed we have ${}^t(\lambda I - \Delta) = \varphi(\lambda - 1 + z)$ and ${}^t(\lambda I - \Delta)^{-1} = \varphi(\frac{1}{\lambda-1+z})$. Since $\frac{1}{\lambda-1+z} = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(\lambda-1)^{k+1}}$, with $|z| < |\lambda - 1|$ we deduce that $(\lambda I - \Delta)^{-1} = (\alpha_{nm})_{nm\geq 1}$, where

$$\begin{cases} \alpha_{nm} = \dfrac{(-1)^{n-m}}{(\lambda-1)^{n-m+1}} & \text{if } m \leq n, \\ \alpha_{nm} = 0 & \text{otherwise.} \end{cases}$$

The condition $(\lambda I - \Delta)^{-1} \in S_r$ is then equivalent to

$$\begin{aligned} \chi &= r \sup_{n\geq 1}\left(\sum_{m=1}^{n} \frac{1}{[|\lambda-1|r]^{n-m+1}}\right) \\ &= r \sup_{n\geq 1}\left\{\frac{\left(\frac{1}{|\lambda-1|r}\right)^{n+1} - \frac{1}{|\lambda-1|r}}{\frac{1}{|\lambda-1|r} - 1}\right\} < \infty. \end{aligned} \tag{10}$$

We see that (10) is equivalent to $\frac{1}{|\lambda-1|r} < 1$, which shows that $\lambda \notin \overline{D}(1, 1/r)$. Conversely, assume that $\lambda \notin \overline{D}(1, 1/r)$. Then

$$\|\Lambda_\lambda(\Delta) - I\|_{S_r} = \frac{1}{|\lambda - 1|r} < 1,$$

which implies that $\lambda I - \Delta$ is bijective from s_r into itself.

Assertion iii). if $\lambda I - \Delta^+$ is bijective from s_r into itself, we have $(\lambda I - \Delta^+)^{-1} = (\alpha^+_{nm})_{nm\geq 1}$, where

$$\begin{cases} \alpha^+_{nm} = \dfrac{(-1)^{m-n}}{(\lambda-1)^{m-n+1}} & \text{if } m \geq n, \\ \alpha^+_{nm} = 0 & \text{otherwise.} \end{cases}$$

The condition $(\lambda I-\Delta^{+})^{-1}\in S_r$ is then equivalent to

$$\chi' = \sup_{n\geq 1}\left(\sum_{m=n}^{\infty}\frac{1}{|\lambda-1|^{m-n+1}}r^{m-n}\right)<\infty,$$

itself equivalent to $\frac{r}{|\lambda-1|}<1$, which proves that $\lambda\notin\overline{D}(1,r)$. Conversely take $\lambda\notin\overline{D}(1,r)$. Reasoning as above, we have $\lambda I-\Delta^{+}=\varphi(\lambda-1+z)$ and

$$\left\|\Lambda_{\lambda}\left(\Delta^{+}\right)-I\right\|_{S_r}=\left\|\frac{1}{\lambda-1}\varphi\left(-z\right)\right\|_{S_r}=\frac{r}{|\lambda-1|}<1,$$

which proves that $\lambda I-\Delta^{+}$ is bijective from s_r into itself. This achieves the proof of iii). $\square$

Concerning the operator Σ we deduce the following.

Proposition 5 *Let $r>1$. We have*

i) $\frac{1}{\lambda}\in\overline{D}(1,1/r)\Leftrightarrow\lambda\in\sigma(\Sigma)$.

ii) *For all $\lambda\notin\sigma(\Sigma)$, $\lambda I-\Sigma$ is bijective from s_r into itself and if $(\lambda I-\Sigma)^{-1}=(\tau_{nm})_{n,m\geq 1}$, then*

$$\begin{cases}\tau_{nn}=\dfrac{1}{1-\lambda} & \forall n\geq 1,\\[2ex] \tau_{nm}=\dfrac{1}{(1-\lambda)^2}\left(\dfrac{-\lambda}{1-\lambda}\right)^{n-m-1} & \text{if } m\leq n,\\[2ex] \tau_{nm}=0 & \text{otherwise.}\end{cases}\tag{11}$$

Proof. i). The well-known results on the spectrum of the inverse of an operator permits us to deduce i) from ii) in Theorem 4. Indeed Σ and Δ are bijections from s_r into itself, for $r>1$. We have ${}^{t}\left(\Sigma-\lambda I\right)=\varphi\left(1-\lambda+\frac{z}{1-z}\right)$ and

$$\frac{1}{1-\lambda+\frac{z}{1-z}}=\frac{1-z}{1-\lambda+\lambda z}=\frac{1}{1-\lambda}-\frac{1}{(1-\lambda)^2}\sum_{n=1}^{\infty}\left(\frac{-\lambda}{1-\lambda}\right)^{n-1}z^n.$$

Hence we deduce the coefficients τ_{nm} given by (11). $\square$

These results lead us to the study of the set $(s_r(\Delta-\lambda I)^{\mu},s_r)$.

5. Properties of the set $(s_r((\Delta - \lambda I)^\mu), s_r)$ for $\lambda \neq 1$ and for a given integer $\mu \geq 1$.

In this part, we shall study the operators which map the space $s_r\left((\Delta - \lambda I)^\mu\right)$ into s_r. In the case where $r = 1$ and $\lambda = 0$, Malkowsky [11] introduced the sequence $\left(R_{nm}^{(\mu)}\right)_{n,m\geq 1}$, defined in the following way: $R_{nm}^{(1)} = R_{nm} = \sum_{j=m}^{\infty} a_{nj}$, $R_{nm}^{(s)} = \sum_{j=m}^{\infty} R_{nj}^{(s-1)}$ $\forall s \geq 1$. He proved that $A \in (s_1(\Delta^\mu), s_1)$ if and only if

$$\begin{cases} \text{i)} \quad \text{For all } n, \text{ the series } \displaystyle\sum_{m=1}^{\infty} m^\mu a_{nm} \text{ is convergent,} \\ \text{ii)} \quad \displaystyle\sup_n \left(\sum_{m=1}^{\infty} \left| R_{nm}^{(\mu)} \right| \right) < \infty. \end{cases}$$

In [2], it is proved that $A \in \left(s_1(\Delta^+)^\mu, s_1\right)$ if and only if

$$\begin{cases} \text{i)} \quad (a_{nm})_{n,m\geq 1} \in s_1 \quad \forall n = 1, 2, \dots \mu. \\ \text{ii)} \quad \left(\displaystyle\sum_{m=1}^{\infty} m^\mu a_{nm} \right)_{n\geq 1} \in s_1. \\ \text{iii)} \quad \displaystyle\sup_{n\geq 1} \left(\sum_{m=1}^{\infty} m^{\mu-1} \left| \sum_{j=m+1}^{\infty} a_{nj} \right| \right) < \infty. \end{cases}$$

Here we shall characterize such matrix transformations, using another method based on the resolution of infinite linear systems. We need to recall the following definitions. For any subset E of s, denote:

$$E^\alpha = \left\{ a = (a_n)_{n\geq 1} \ / \ \forall X = (x_n)_{n\geq 1} \in E \quad \sum_n |a_n x_n| < \infty \right\}.$$

E^α is the α-dual of E (see [2] and [11]). It has been proved in [11], that:

$$(s_1(\Delta^\mu))^\alpha = \left\{ a = (a_n)_{n\geq 1} \ / \ \sum_{n=1}^{\infty} n^\mu |a_n| < \infty \right\}. \tag{12}$$

In the following we shall use the well-known result,

Lemma 6 *If the double series* $\sum_n \sum_m |a_{nm}| < \infty$, *then the series* $\sum_n \sum_m a_{nm}$ *and* $\sum_m \sum_n a_{nm}$ *are convergent and* $\sum_n \sum_m a_{nm} =$

$\sum_m \sum_n a_{nm}$.

Remark 1 We see that $\sum_n \sum_m |a_{nm}| < \infty$ is equivalent to $\sum_m \sum_n |a_{nm}| < \infty$.

For a given real $r > 0$, we have the next results.

Lemma 7 *Let $A = (a_{nm})_{n,m \geq 1}$ and $P = (p_{nm})_{n,m \geq 1}$ be two infinite matrices satisfying for all $n \geq 1$:*

$$\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} |a_{nk} p_{km}| \, r^m < \infty. \tag{13}$$

Then $A(PX) = (AP)X$ for all $X \in s_r$.

Proof. If we set $A(PX) = (y_n)_{n \geq 1}$, then for every $n \geq 1$:

$$y_n = \sum_{k=1}^{\infty} a_{nk} \left(\sum_{m=1}^{\infty} p_{km} x_m \right).$$

The series intervening in the second member being convergent, since (13) holds and $X \in s_r$. Using Lemma 6, condition (13) permits us to invert the symbols $\sum_k$ and $\sum_m$ in the expression of y_n, which proves that $A(PX) = (AP)X$. □

We shall use the following condition on the matrix $A = (a_{nm})_{n,m \geq 1}$ verified for all $n \geq 1$ and $\lambda \neq 1$

$$\sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \binom{\mu + j - 1}{j} \frac{|a_{n,m+j}|}{|1 - \lambda|^{\mu + j}} r^m < \infty. \tag{14}$$

Remark 2 The previous condition can be replaced by

$$\sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \frac{|a_{n,m+j}|}{|1 - \lambda|^{\mu + j}} j^{\mu - 1} r^m < \infty \text{ and } \sum_{m=1}^{\infty} \frac{|a_{n,m}|}{|1 - \lambda|^{\mu}} r^m < \infty, \tag{15}$$

for all $n \geq 1$ and $\lambda \neq 1$. Indeed it can be proved that there are M_1 and $M_2 > 0$ such that for all $j \geq 1$:

$$M_1 j^{\mu} \leq \binom{\mu + j}{j} \leq M_2 j^{\mu}. \tag{16}$$

We shall consider the matrix $(\Delta - \lambda I)^{-\mu}$ in the case where λ does not belong necessarily to the resolvent set $\rho(\Delta)$, i.e. $|\lambda - 1| > 1/r$, (see ii) in Theorem 4). This means that $(\Delta - \lambda I)^{-\mu}$ does not belong necessarily to the space S_r. We can assert the next result:

Theorem 8 i) *If* $|\lambda - 1| > 1/r$, *then*

$$A \in (s_r((\Delta - \lambda I)^{\mu}), s_r) \Leftrightarrow \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n} \right) < \infty.$$

ii) *Take* λ, *such that* $|\lambda - 1| \leq 1/r$. *There exists a real* $R > r$, *depending on* λ, *for which*:

$$A \in (s_R((\Delta - \lambda I)^{\mu}), s_R) \Leftrightarrow \sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| R^{m-n} \right) < \infty.$$

iii) *Assume that* (14) *holds. For every* $\lambda \neq 1$, *we have successively*:

α) *For all* $Y \in s_r$ $\quad A\left[(\Delta - \lambda I)^{-\mu} Y\right] = \left[A(\Delta - \lambda I)^{-\mu}\right] Y$,

β) $A \in (s_r((\Delta - \lambda I)^{\mu}), s_r)$ *if and only if*

$$\sup_{n \geq 1} \left[\sum_{m=1}^{\infty} \left| \sum_{j=0}^{\infty} \binom{\mu + j - 1}{j} \frac{a_{n,m+j}}{(1-\lambda)^{\mu+j}} \right| r^{m-n} \right] < \infty. \tag{17}$$

Proof. i). If $|\lambda - 1| > 1/r$, using ii) in Theorem 4, we see that $\Delta - \lambda I$ is bijective from s_r into itself. Then $(\Delta - \lambda I)^{\mu}$ is bijective from s_r into itself and it is the same for $(\Delta - \lambda I)^{-\mu}$. From Lemma 2 we deduce $(\Delta - \lambda I)^{-\mu} \in S_r$, hence $s_r((\Delta - \lambda I)^{\mu}) = s_r$. Then $A \in (s_r((\Delta - \lambda I)^{\mu}), s_r)$ if and only if $A \in (s_r, s_r)$ and we obtain the conclusion using Lemma 2.

ii). Let $\lambda \neq 1$ such that $|\lambda - 1| \leq 1/r$. There exists $R > 0$ such that $1/R < |\lambda - 1|$. Doing as in Theorem 4, we have

$$\|\Lambda_{\lambda}(\Delta) - I\|_{S_R} = \frac{1}{|\lambda - 1| R} < 1.$$

This proves that $(\Delta - \lambda I)^{-\mu} \in S_R$ and the conclusion follows as above.

Assertion iii) α). Take $\lambda \neq 1$. We explicitly calculate the infinite matrix $(\Delta - \lambda I)^{-\mu}$. Here, we see that:

$${}^{t}(\Delta - \lambda I)^{\mu} = \varphi\left[(1 - \lambda - z)^{\mu}\right].$$

Thus

$$
\begin{aligned}
{}^{t}(\Delta - \lambda I)^{-\mu} &= \varphi \left[\frac{1}{(1-\lambda-z)^{\mu}} \right] \\
&= \varphi \left[\sum_{k=0}^{\infty} \frac{1}{(1-\lambda)^{\mu+k}} \binom{\mu+k-1}{k} z^{k} \right],
\end{aligned}
$$

for $|z| < |1-\lambda|$. Whence, $(\Delta - \lambda I)^{-\mu} = (\tau_{nm})_{n,m \geq 1}$, with:

$$
\tau_{nm} = \begin{cases} \dfrac{1}{(1-\lambda)^{\mu+n-m}} \dbinom{\mu+n-m-1}{n-m} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \tag{18}
$$

Set $P = (\Delta - \lambda I)^{-\mu}$ in Lemma 7. Then condition (13) means that

$$
\sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \frac{|a_{nk}|}{|1-\lambda|^{\mu+k-m}} \binom{\mu+k-m-1}{k-m} r^{m} < \infty. \tag{19}
$$

Letting $j = k - m$ in (14), we see that (14) and (19) are equivalent. It is then enough to apply Lemma 7 to conclude the proof of α).

Assertion iii) β). If $A \in (s_r((\Delta - \lambda I)^{\mu}), s_r)$, then for all $Y \in s_r$,

$$
A\left[(\Delta - \lambda I)^{-\mu} Y\right] \in s_r;
$$

and using iii) α) we deduce that $\left[A(\Delta - \lambda I)^{-\mu}\right] Y \in s_r$. From (8), we have $A(\Delta - \lambda I)^{-\mu} \in S_r$. Denote, now, by $p_{nm}(\lambda)$ the coefficients of the matrix $A(\Delta - \lambda I)^{-\mu}$, using (18) we get that:

$$
p_{nm}(\lambda) = \sum_{j=0}^{\infty} \frac{a_{n,m+j}}{(1-\lambda)^{\mu+j}} \binom{\mu+j-1}{j}. \tag{20}
$$

Hence

$$
\sup_{n \geq 1} \left(\sum_{m=1}^{\infty} |p_{nm}(\lambda)| r^{m-n} \right) < \infty.
$$

This proves the necessary condition.

Conversely, suppose that (17) is satisfied. Then $A(\Delta - \lambda I)^{-\mu} \in S_r$, i.e. for all $Y \in s_r$, $[A(\Delta - \lambda I)^{-\mu}] Y \in s_r$. Using (14) we can apply Lemma 7, as above, what gives: $A[(\Delta - \lambda I)^{-\mu} Y] = [A(\Delta - \lambda I)^{-\mu}] Y$, $\forall Y \in s_r$. The second term of the last identity belongs then to s_r, which proves that $A \in (s_r((\Delta - \lambda I)^\mu), s_r)$. This achieves the proof of iii) β. $\square$

Remark 3 We can generalize some results we have got above, considering $(\Delta^+)^\mu$ for $\mu \in C$, (see [18]). Here (6) is true when $\mu \in C - N$, for $|z| < 1$ and for all $z \in C$ whenever $\mu \in N$. Then Δ^μ is given by (7) and iii) in Theorem 8 can be extended to the case when $\mu \in C$, if the series defined in (14) is absolutely convergent.

Remark 4 We have seen in Theorem 4, that in the case where $\lambda = 0$, Δ^μ is invertible when $0 \notin \overline{D}(1, 1/r)$, that is when $r > 1$. Analogously, $(\Delta^+)^\mu$ is invertible when $r < 1$, since $0 \notin \sigma((\Delta^+)^\mu)$. These remarks lead to the study of the important particular case $r = 1$. Indeed the space $s_1 = l^\infty$ verifies the following properties: $\bigcup_{0<r<1} s_r \subsetneq s_1 \subsetneq \bigcap_{r>1} s_r$. For the first inclusion we see that $e \in s_1 - \bigcup_{0<r<1} s_r$ and if we write $c = (n)_n$, we have $c \in \bigcap_{r>1} s_r - s_1$, for the second inclusion.

Assume that (14) is satisfied for $\lambda = 0$, that is

$$\sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \binom{\mu + j - 1}{j} |a_{n,m+j}| < \infty. \tag{21}$$

We get the next result.

Corollary 9 *$A \in (s_1(\Delta^\mu), s_1)$ if and only if*

$$\sup_{n \geq 1} \left(\sum_{m=1}^{\infty} \left| \sum_{j=0}^{\infty} \binom{\mu + j - 1}{j} a_{n,m+j} \right| \right) < \infty. \tag{22}$$

Proposition 10 *Suppose that* (21) *holds. Then for every n:*

$$\sum_{m=1}^{\infty} m^\mu |a_{nm}| < \infty.$$

Proof. (21) is equivalent to

$$\sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \binom{\mu + k - m - 1}{k - m} |a_{nk}|$$
$$= \sum_{k=1}^{\infty} |a_{nk}| \sum_{m=1}^{k} \binom{\mu + k - m - 1}{k - m} = \sum_{k=1}^{\infty} |a_{nk}| \binom{\mu + k - 1}{k - 1},$$

the interchange in order of summation being justified by Lemma 6 and the result follows from (16). □

Let us replace now the condition (21) by the following one

$$\sum_{s=1}^{\infty} r^s \left[\sum_{i=0}^{s-1} \binom{\mu+i-1}{i} |a_{n,s-i}| \right] < \infty. \tag{23}$$

Then we have

Proposition 11 *Assume that condition* (23) *holds and let* r *be a real strictly less than* 1. $A \in \left(s_r\left((\Delta^+)^\mu\right), s_r\right)$ *if and only if*

$$\sup_{n \geq 1} \left(\sum_{s=1}^{\infty} \left| \sum_{i=0}^{s-1} \binom{\mu+i-1}{i} a_{n,s-i} \right| r^{s-n} \right) < \infty. \tag{24}$$

Proof. The proof is analogous to the one of Theorem 8, $p_{nm}(\lambda)$ is replaced by

$$p_{nm}^+ = \sum_{k=1}^{m} a_{nk} \binom{\mu+m-k-1}{m-k}.$$

Further we must use the fact that the product $A(\Delta^+)^{-\mu}$ belongs to S_r is equivalent to

$$\sup_{n \geq 1} \left(\sum_{s=1}^{\infty} |p_{ns}^+| \, r^{s-n} \right) < \infty,$$

itself equivalent to (24). □

Remark 5 Corollary 9 and Proposition 11 are always true if we suppose that μ is a complex number for which the series defined in (21) and (23) are absolutely convergent.

Acknowledgement The author thanks the referee for his valuable and constructive comments.

References

[1] Andersen A.F., *Summation of nonintegral order*. Mat. Tidsskr. B. (1946) 33–52.

[2] Çolak R. and Et M., *On some generalized difference spaces and related matrix transformations*. Hokkaido Mathematical Journal **26** (1997), 483–492.

[3] Cooke R.G., *Infinite matrices and sequences spaces*. Macmillan and Co., London, 1949.

[4] Kizmaz H., *On Certain Sequence Spaces*. Canad. Math. Bull. **24** (2) (1981), 169–176.

[5] Labbas R. and de Malafosse B., *On some Banach algebra of infinite matrices and applications*. Demonstratio Matematica **31** 1998.

[6] Labbas R. and de Malafosse B., *An application of the sum of linear operators in infinite matrix theory*. Faculté des Sciences de l'Université d'Ankara, Series Al Mathematics and statistics **46**, 1997.

[7] Maddox I.J., *Infinite matrices of operators*. Springer-Verlag, Berlin, Heidelberg and New York, 1980.

[8] de Malafosse B., *Systèmes linéaires infinis admettant une infinité de solutions*. Atti dell' Accademia Peloritana dei Pericolanti di Messina, Classe I di Scienze Fis. Mat. e Nat. **64** (1986).

[9] de Malafosse B., *On the spectrum of the Cesàro operator in the space s_r*. Faculté des Sciences de l'Université d'Ankara, Series Al Mathematics and statistics **48** (1999), 53–71.

[10] Malkowsky E., *The continuous duals of the spaces $c_0(\Lambda)$ and $c(\Lambda)$ for exponentially bounded sequences Λ*. Acta Sci. Math. Szeged **61** (1995), 241–250.

[11] Malkowsky E. and Parashar S.D., *Matrix transformations in spaces of bounded and convergent difference sequences of order m*. Analysis **17** (1997), 87–97.

[12] Mascart H. and de Malafosse B., *Extension de l'ensemble d'unicité d'un système linéaire infini*. Seminarberichte Fachbereich Mathematik und Informatik- Band **34**, 1989 Hagen.

[13] Mursaleen, *Application of infinite matrices to Walsh functions*. Demonstratio Mathematica, V. **XXVII**, N2, 279–282.

[14] Okutoyi J.T., *On the spectrum of C_1 as operator on bv*. Commun. Fac. Sci. Univ. Ank. Series A_1, V. **41** (1992), 197–207.

[15] Petersen G.M. and Baker Anne C., *Solvable infinite systems of linear equations*. Journal London Math. Soc. **39** (1964), 501–510.

[16] Petersen G.M., *Regular matrix transformations*. McGraw-Hill, 1966.

[17] Reade J.B., *On the spectrum of the Cesàro operator*. Bull. London Math., Soc. **17** (1985), 263–267.

[18] Zeller K. and Beckmann W., *Theory der Limitierungs verfahren*. Springer-Verlag Berlin Heidelberg New York, 1970.

Université du Havre, I.U.T.
B.P 4006, Le Havre, France
E-mail: bdemalaf@iut.univ-lehavre.fr
bdemalaf@wanadoo.fr