Hokkaido Mathematical Journal Vol. 31 (2002) p. 61–77

On the class of univalent functions starlike with respect to N-symmetric points

I. R. NEZHMETDINOV and S. PONNUSAMY

(Received March 21, 2000)

Abstract. In the present paper we study certain generalizations of the class $\mathcal{SSP}_N$ of functions starlike with respect to N-symmetric points. We obtain a structural formula for functions in $\mathcal{SSP}_N$, and deduce a sharp lower bound for $|f'(z)|$ when N is even (this case completes the distortion theorem for $\mathcal{SSP}_N$). Improved estimates for Koebe constants are also given. Further, it is proved that for any $N \geq 2$ the class $\mathcal{SSP}_N$ contains non-starlike functions. Finally, we characterize the class $\mathcal{SSP}_N$ in terms of Hadamard convolution.

Key words: univalent, starlike, close-to-convex and convex functions.

1. Introduction and main results

Denote by $\mathcal{A}$ the class of all functions f, analytic in the unit disc Δ and normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the class of functions in $\mathcal{A}$ that are univalent in Δ. A function $f \in \mathcal{A}$ is said to be starlike with respect to symmetric points [8] if for any r close to 1, $r < 1$, and any z_0 on the circle $|z| = r$, the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at z_0 as z traverses the circle $|z| = r$ in the positive direction, i.e.,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z_0)}\right) > 0, \quad \text{for } z = z_0, \ |z| = r.$$

Denote by $\mathcal{SSP}$ the class of all functions in $\mathcal{S}$ which are starlike with respect to symmetric points and, functions f in this class is characterized by

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in \Delta.$$

We also have the following generalization of the class $\mathcal{SSP}$ introduced by K. Sakaguchi [8]. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, set

$$\mathcal{SSP}_N = \left\{ f \in S : \mathrm{Re}\left(\frac{zf'(z)}{f_N(z)}\right) > 0, \ z \in \Delta \right\},$$

2000 Mathematics Subject Classification : 30C45.

where

$$f_N(z) = z + \sum_{m=1}^{\infty} a_{mN+1} z^{mN+1}.$$

The elements of the class $\mathcal{SSP}_N$ are said to be starlike with respect to N-symmetric points.

Set $\varepsilon := \exp(2\pi i/N)$. For $f \in \mathcal{A}$ we consider its weighted mean defined by

$$M_{f,N}(z) = \frac{1}{\sum_{j=1}^{N-1} \varepsilon^{-j}} \sum_{j=1}^{N-1} \varepsilon^{-j} f(\varepsilon^j z).$$

It can be easily seen that

$$\frac{f(z) - M_{f,N}(z)}{N} = \frac{1}{N} \sum_{j=0}^{N-1} \varepsilon^{-j} f(\varepsilon^j z) = f_N(z).$$

The geometric characterization of this class is that the class $\mathcal{SSP}_N$ is the collection of functions $f \in \mathcal{A}$ such that for any r close to 1, $r < 1$, the angular velocity of $f(z)$ about the point $M_{f,N}(z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction.

The case $N = 1$ gives a well-known subclass $\mathcal{S}^*$ of univalent functions in $\mathcal{A}$ such that $f(\Delta)$ is a starlike domain with respect to the origin, i.e., $t\omega \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. For $N = 2$ we get back to the class $\mathcal{SSP}$.

A closely related class to $\mathcal{SSP}_N$ is defined as follows. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{C}_N$ if there exists a function g in $\mathcal{SSP}_N$ such that

$$\operatorname{Re}\left(e^{i\tau}\frac{zf'(z)}{g_N(z)}\right) > 0, \quad z \in \Delta, \quad \text{for some } |\tau| < \pi/2.$$

We remark that replacing $g_N(z)$ by $g(z)$ results in a different class, which was studied in [7]. The elements of the class $\mathcal{C}_N$ are called close-to-convex functions with respect to N-symmetric points. The case $N = 1$ gives the usual class $\mathcal{C}$ of all functions in $\mathcal{A}$ that are univalent and close-to-convex in Δ.

If we substitute $\varepsilon^j z$ for z in the above analytic characterization for the class $\mathcal{SSP}_N$, then we see that $f \in \mathcal{SSP}_N$ implies that

$$\mathrm{Re}\left(\frac{zf_N'(z)}{f_N(z)}\right) > 0, \quad z \in \Delta,$$

and, therefore, $f_N(z) \in \mathcal{S}^*$. Thus, every function in $\mathcal{SSP}_N$ is close-to-convex in the unit disc. In [8], the Maclaurin coefficients of $f \in \mathcal{SSP}_N$ for $N = 2$ are shown to be bounded by 1. In general case, the coefficient estimates for $f \in \mathcal{SSP}_N$ are obtained in [10].

In [10], P. Singh and R. Chand defined some generalizations of the class $\mathcal{SSP}_N$ and found two-sided estimates for $|f(z)|$ on it. In particular, they have proved the following

Theorem 1.1 ([10, Theorem 2.3]) *If $f \in \mathcal{SSP}_N$, then*

$$\int_0^r \frac{1-t}{(1+t)(1+t^N)^{2/N}}dt \le |f(z)| \le \int_0^r \frac{1+t}{(1-t)(1-t^N)^{2/N}}dt, \quad |z| = r < 1. \tag{1.2}$$

While the upper estimate in (1.2) appears to be sharp, the lower one is sharp only if N is odd. Similar comments apply for their estimates of $|f'(z)|$. Therefore, our main aim is to obtain sharp estimates of the modulus both for $f(z) \in \mathcal{SSP}_N$ and its derivative $f'(z)$. Now, we state the distortion theorems for the class $\mathcal{SSP}_N$.

Theorem 1.3 *Let $f \in \mathcal{SSP}_N$.*
(1) *If $N \ge 1$ is odd, then we have*

$$\frac{1-r}{(1+r)(1+r^N)^{2/N}} \le |f'(z)| \le \frac{1+r}{(1-r)(1-r^N)^{2/N}}, \quad 0 \le |z| = r < 1. \tag{1.4}$$

(2) *For any even $N \ge 2$ the upper estimate in* (1.4) *holds, while the lower one should be replaced either by*

$$|f'(z)| \ge \frac{1-r}{(1+r)(1-r^N)^{2/N}}, \quad \text{for } 0 \le r \le r_N, \tag{1.5}$$

or by

$$|f'(z)| \ge \frac{1-r^2}{(1+2r\cos\theta_N + r^2)(1-2r^N\cos N\theta_N + r^{2N})^{1/N}}, \quad \text{for } r_N < r < 1. \tag{1.6}$$

Here r_N *is a unique root of the equation*

$$1 + r^{2N} - Nr^{N-1}(1 + r^2) - (N + 1)r^N = 0 \tag{1.7}$$

in the interval $0 < r < 1$, *and* θ_N *is a unique root of the equation*

$$\sin\theta(1 + r^{2N}) - r^{N-1}(1 + r^2)\sin N\theta - r^N \sin(N + 1)\theta = 0 \tag{1.8}$$

in the interval $0 < \theta < \pi/N$, *provided that* $r > r_N$. *All the above estimates are sharp.*

According to a result of Privalov [3, Vol. I, p. 67], we can integrate the estimates for $|f'(z)|$ to derive the following

Corollary 1.9 *If* $f \in \mathcal{SSP}_N$, N *being odd, then*

$$\int_0^r \frac{1-t}{(1+t)(1+t^N)^{2/N}}dt \le |f(z)| \le \int_0^r \frac{1+t}{(1-t)(1-t^N)^{2/N}}dt, \tag{1.10}$$

for all $|z| = r < 1$. *If* N *is even, then the upper estimate holds, whereas the lower estimate is given either by*

$$|f(z)| \ge \int_0^r \frac{1-t}{(1+t)(1-t^N)^{2/N}}dt, \quad \text{for } 0 \le r \le r_N, \tag{1.11}$$

or by

$$|f(z)| \ge \int_0^{r_N} \frac{1-t}{(1+t)(1-t^N)^{2/N}}dt + \int_{r_N}^r \frac{1-t^2}{(1+2t\cos\theta_N + t^2)(1-2t^N\cos N\theta_N + t^{2N})^{1/N}}dt \quad \text{for } r_N \le r < 1. \tag{1.12}$$

The estimates in (1.10) *and* (1.11) *are sharp.*

Since $\mathcal{SSP}_1 = \mathcal{S}^*$, it is therefore interesting to know whether there exists an inclusion result between the classes $\mathcal{SSP}_N$ and $\mathcal{S}^*$ for $N \ge 2$. It is easy to construct examples of functions in $\mathcal{SSP}_N$ $(N \ge 2)$ but not in $\mathcal{S}^*$.

Theorem 1.13 *If* $N \ge 2$, *then the inclusion* $\mathcal{SSP}_N \subset \mathcal{S}^*$ *does not hold.*

In a particular case, for $N = 2$, Theorem 1.3 and Corollary 1.9 are obtained in [6] while Theorem 1.13 is proved in [5].

2. Structural Formulae for $\mathcal{SSP}_N$ and $\mathcal{C}_N$

To prove Theorem 1.3 we need the following structural formula for functions in $\mathcal{SSP}_N$, and this theorem is obtained by the second author in his Ph.D thesis work [7, Chapter V]. Since the following lemmas are unavailable elsewhere, we recall its proofs from there.

Lemma 2.1 *A function $f \in \mathcal{A}$ is in $\mathcal{SSP}_N$ if and only if*

$$f(z) = \int_0^z p(t) \exp\{q(t)\}\, dt, \quad q(t) = \int_0^t \frac{1}{N\eta}\left(\sum_{j=0}^{N-1} p(\varepsilon^j \eta) - N\right) d\eta \tag{2.2}$$

where $\varepsilon = e^{2\pi i/N}$, and $p \in \mathcal{P}$. Here $\mathcal{P}$ denotes the class of all analytic functions in Δ with $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$ for $z \in \Delta$.

Proof. We first prove the necessity of (2.2). Suppose that $f \in \mathcal{SSP}_N$. Then by the definition it follows that

$$\frac{zf'(z)}{f_N(z)} = p(z) \tag{2.3}$$

where $p \in \mathcal{P}$. Writing (2.3) as $f_N(z) = zf'(z)/p(z)$ and then differentiating it, we obtain

$$f_N'(z) = -\frac{p'(z)}{p^2(z)} zf'(z) + \frac{zf''(z) + f'(z)}{p(z)} \tag{2.4}$$

Replacing z by $\varepsilon^j z$ in (2.3), we get

$$f'(\varepsilon^j z) = \frac{f_N(\varepsilon^j z)}{z} p(\varepsilon^j z) = \frac{f'(z)}{p(z)} p(\varepsilon^j z)$$

and therefore

$$f_N'(\varepsilon^j z) = \frac{1}{N} \sum_{j=0}^{N-1} f'(\varepsilon^j z) = \frac{f'(z)}{p(z)} \left(\frac{1}{N} \sum_{j=0}^{N-1} p(\varepsilon^j z)\right).$$

Comparing this equation with (2.4) we find that

$$\frac{f''(z)}{f'(z)} = \frac{p'(z)}{p(z)} + \frac{1}{Nz}\left(\sum_{j=0}^{N-1} p(\varepsilon^j z) - N\right).$$

Integrating this equation and then exponentiating both sides of the resulting equation we obtain the desired integral representation:

$$f'(z) = p(z) \exp\{q(z)\}, \quad q(z) = \int_0^z \frac{1}{N\eta}\left(\sum_{j=0}^{N-1} p(\varepsilon^j \eta) - N\right) d\eta. \tag{2.5}$$

The structural formula (2.2) easily follows from (2.5).

Next we prove the sufficiency. Suppose that (2.2) holds for some $p \in \mathcal{P}$. Then the function f defined by (2.2) is obviously in $\mathcal{A}$. Differentiation of (2.2) gives the representation (2.5), so $f'(z)$ is nonzero in Δ. From (2.2) and the fact that $\varepsilon^N = 1$, it can be easily seen by change of variables that

$$f_N(z) = \int_0^z \frac{1}{N} \sum_{j=0}^{N-1} p(\varepsilon^j \zeta) q(\zeta) d\zeta,$$

$$q(\zeta) = \exp\left\{\int_0^\zeta \frac{1}{N\eta}\left(\sum_{j=0}^{N-1} p(\varepsilon^j \eta) - N\right) d\eta\right\}. \tag{2.6}$$

The following identity can be verified by differentiation

$$z \exp\{q(z)\} = \int_0^z \frac{1}{N} \sum_{j=0}^{N-1} p(\varepsilon^j \zeta) q(\zeta) d\zeta. \tag{2.7}$$

In view of (2.5) and (2.7), the formula (2.6) is equivalent to $f_N(z) = zf'(z)/p(z)$, thus proving the sufficiency of (2.2). □

The case $N = 1$ of Lemma 2.1 gives a well-known representation for functions in $\mathcal{S}^*$ while the case $N = 2$ yields the structural formula obtained by Stankiewicz [9].

Next we prove the following structural formula for functions in $\mathcal{C}_N$.

Lemma 2.8 *A function f belongs to the class $\mathcal{C}_N$ with respect to $g \in \mathcal{SSP}_N$ if and only if there exist two functions p_1, p_2 in $\mathcal{P}$ such that*

$$f(z) = \int_0^z [\cos\tau\, p_1(t) + i \sin\tau] \exp\{q(t)\}\, dt \quad \textit{and}$$

$$g(z) = \int_0^z p_2(t) \exp\{q(t)\}\, dt, \tag{2.9}$$

where

$$q(t) = \int_0^t \frac{1}{N\eta}\left(\sum_{j=0}^{N-1} p_2(\varepsilon^j \eta) - N\right) d\eta, \quad \varepsilon = e^{2\pi i/N}.$$

Proof. If f belongs to $\mathcal{C}_N$ with respect to $g \in \mathcal{SSP}_N$, then by definition it follows that

$$e^{i\tau}\frac{zf'(z)}{g_N(z)} = \cos\tau\, p_1(z) + i\sin\tau \tag{2.10}$$

and

$$\frac{zg'(z)}{g_N(z)} = p_2(z), \tag{2.11}$$

where $p_1, p_2 \in \mathcal{P}$. From (2.10) it follows that

$$g_N(z) = e^{i\tau}\frac{zf'(z)}{\cos\tau\, p_1(z) + i\sin\tau} \tag{2.12}$$

and

$$e^{i\tau}\frac{f'(z)}{g'(z)} = \frac{\cos\tau\, p_1(z) + i\sin\tau}{p_2(z)}. \tag{2.13}$$

As in the proof of Lemma 2.1, the equation (2.11) implies that

$$g_N'(z) = \frac{g'(z)}{p_2(z)}\left(\frac{1}{N}\sum_{j=0}^{N-1} p_2(\varepsilon^j z)\right). \tag{2.14}$$

Differentiating both sides of (2.12), we find that

$$g_N'(z) = -\frac{e^{i\tau}\cos\tau\, p_1'(z)}{(\cos\tau\, p_1(z) + i\sin\tau)^2} zf'(z) + \frac{e^{i\tau}(zf''(z) + f'(z))}{\cos\tau\, p_1(z) + i\sin\tau}.$$

By (2.13), we have from (2.14)

$$g_N'(z) = \frac{e^{i\tau} f'(z)}{\cos\tau\, p_1(z) + i\sin\tau}\left(\frac{1}{N}\sum_{j=0}^{N-1} p_2(\varepsilon^j z)\right).$$

Comparing the last two equations, we deduce that

$$\frac{f''(z)}{f'(z)} = \frac{\cos\tau p_1'(z)}{\cos\tau\, p_1(z) + i\sin\tau} + \frac{1}{Nz}\left(\sum_{j=0}^{N-1} p_2(\varepsilon^j z) - N\right).$$

Repeated integration yields the desired integral representation.

Sufficiency part of this theorem can be proved on the same lines as those of Lemma 2.1. So we omit its proof. □

3. Convolution theorems

For $-1 \le B < A \le 1$, we define

$$\mathcal{P}(A,B) = \left\{ p \in \mathcal{A} : p(0 = 1 \text{ and } p(z) \prec \frac{1+Az}{1+Bz}, \ z \in \Delta \right\},$$

where $\prec$ denotes the usual subordination [3, Vol. I, p. 85]. Note that $\mathcal{P}(1,-1) = \mathcal{P}$. Using this, we define

$$\mathcal{SSP}_N(A,B) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f_N(z)} \in \mathcal{P}(A,B) \right\},$$

$\mathcal{SSP}_N(1-2\beta,-1) = \mathcal{SSP}_N(\beta)$, and $\mathcal{SSP}_N(0) = \mathcal{SSP}_N$. Further, set

$$\mathcal{C}_N(A,B) = \left\{ f \in \mathcal{S} : \frac{1}{\cos\tau} \left(e^{i\tau} \frac{zf'(z)}{g_N(z)} - i \sin\tau \right) \in \mathcal{P}(A,B), \right.$$
$$\left. \text{for some } \ g \in \mathcal{SSP}_N \right\}.$$

In fact, one can define a more general class than $\mathcal{C}_N(A,B)$ by allowing g to belong to $\mathcal{SSP}_N(A',B')$ with $-1 \le B' < A' \le 1$. In such cases we say that f belongs to $\mathcal{C}_N(A,B,A',B')$, but we shall avoid the use of too many parameters. In any case, the above two definitions generalize several well-known subclasses of $\mathcal{SSP}$ studied, for instance, in [9, 10, 11]. It would be interesting to note that the proof of Lemmas 2.1 and 2.8 immediately yields the following structural formula for functions in $\mathcal{SSP}_N(A,B)$ and $\mathcal{C}_N(A,B,A',B')$ respectively.

Theorem 3.1 *A function $f \in \mathcal{A}$ is in $\mathcal{SSP}_N(A,B)$ if and only if there exists a $p \in \mathcal{P}(A,B)$ such that* (2.2) *holds.*

Theorem 3.2 *A function $f \in \mathcal{A}$ is in the class $\mathcal{C}_N(A,B,A',B')$ with respect to $g \in \mathcal{SSP}_N(A',B')$ if and only if there exist two functions $p_1 \in \mathcal{P}(A,B)$ and $p_2 \in \mathcal{P}(A',B')$ such that* (2.9) *holds.*

Next, we give the following simple characterization of functions in $\mathcal{SSP}_N(A,B)$ in terms of Hadamard product/convolution (see [3, Vol. II,

p. 122]).

Theorem 3.3 *A function f is in $\mathcal{SSP}_N(A,B)$ if and only if,*

$$\frac{1}{z}\left[f(z) * \frac{z+\{Ax+(1+Bx)A_{N-1}(z)\}(B-A)^{-1}x^{-1}z^2}{(1-z)^2 A_N(z)}\right] \neq 0,$$

$$z \in \Delta, \quad |x|=1,$$

where $A_N(z) = (1-z^N)/(1-z)$.

Proof. A function f is in $\mathcal{SSP}_N(A,B)$ if and only if

$$\frac{zf'(z)}{f_N(z)} \neq \frac{1+Ax}{1+Bx} \quad \text{for all } \ z \in \Delta \ \text{ and } \ |x|=1,$$

which, because of the normalization of f, is equivalent to the condition that

$$\frac{1}{z}[(1+Bx)zf'(z) - f_N(z)(1+Ax)] \neq 0, \quad z \in \Delta.$$

Since

$$zf'(z) = f(z) * \frac{z}{(1-z)^2} \quad \text{and} \quad f_N(z) = f(z) * \frac{z}{1-z^N},$$

the last relation reduces to the desired convolution condition. □

In particular, for $N-1 = A = -B = 1$, the last theorem gives a convolution characterization for univalent functions starlike with respect to symmetric points:

$$f \in \mathcal{SSP} \iff \frac{1}{z}\left[f(z) * \frac{z-x^{-1}z^2}{(1-z)^2(1+z)}\right] \neq 0, \quad z \in \Delta, \quad |x|=1.$$

Finally, it is natural to ask whether the results related to $\mathcal{SSP}_1(A,B)$ and $\mathcal{C}_1(A,B,A',B')$ can be generalized to these classes for all $N \in \mathbb{N}$. In particular, we conclude our paper with the following

Problem 3.4 State and prove the counterparts of Theorems 1.3, 1.13 and Corollary 1.9 for the above general classes.

4. Proofs of Main Results

4.1. Proof of Theorem 1.3

Consider the following extremal problem: find the minimum of $|f'(z)|$, where $z \in \Delta$ is fixed and $f \in \mathcal{SSP}_N$. It is easy to show that the class $\mathcal{SSP}_N$

is compact (cf., e.g., [4, Theorem 4.1]), so the problem has a solution, say, $f_0 \in \mathcal{SSP}_N$. Choosing a suitable $\gamma \in \mathbb{R}$, we have

$$\mathrm{Re}\left(\frac{e^{i\gamma}}{f_0'(z)}\right) = \frac{1}{|f_0'(z)|} \geq \frac{1}{|f'(z)|} \geq \mathrm{Re}\left(\frac{e^{i\gamma}}{f'(z)}\right), \quad f \in \mathcal{SSP}_N.$$

Thus, $f_0 \in \mathcal{SSP}_N$ also gives the maximum to the functional $\mathrm{Re}[e^{i\gamma}/f'(z)]$ on $\mathcal{SSP}_N$. We apply a variational method of Goluzin [2, p.504–506].

To this end, we first use the Herglotz integral formula to deduce from Lemma 2.1 the following representations

$$p(z) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t), \quad q(z) = -\frac{2}{N}\int_{-\pi}^{\pi} \log(1-e^{-iNt}z^N)d\mu(t), \tag{4.2}$$

$\mu(t)$ being a probability measure on $[-\pi, \pi)$.

Now, by applying the Goluzin variation of the first type, we get

$$\begin{aligned}\mathrm{Re}&\left(\frac{e^{i\gamma}}{f_*'(z)}\right)\\ &= \mathrm{Re}\,\frac{e^{i\gamma}}{f'(z)} - 2i\lambda \int_{t_1}^{t_2} \mathrm{Re}\left\{\frac{e^{i\gamma}}{f'(z)}\left[\frac{e^{it}z}{p(z)(e^{it}-z)^2}\right.\right.\\ &\qquad\qquad \left.\left. + \frac{z^N}{e^{iNt}-z^N}\right]\right\}|\mu(t)-c|dt + O(\lambda^2),\end{aligned}$$

where $-1 \leq \lambda \leq 1$, $-\pi \leq t_1 < t_2 < \pi$, and c is a constant. Consequently, if $f(z)$ is an extremal function and the equation

$$\mathrm{Re}\left\{\frac{e^{i\gamma}}{f'(z)}\left[\frac{e^{it}z}{p(z)(e^{it}-z)^2} + \frac{z^N}{e^{iNt}-z^N}\right]\right\} = 0 \tag{4.3}$$

has no root on the interval $t_1 < t < t_2$, then $\mu(t) \equiv c$ for $t_1 < t < t_2$. Since (4.3) can be reduced to an algebraic equation of $2(N+1)$-th degree with respect to e^{it}, it follows that $\mu(t)$ is a step function with at most $2(N+1)$ jumps.

Moreover, after considering the Goluzin variation of the second type, we conclude that there is a root of (4.3) between any two of the jumps of $\mu(t)$. Hence, the required extremum is attained when $\mu(t)$ is a convex combination of at most $N+1$ point masses on $[-\pi, \pi)$.

We observe that, for any $f \in \mathcal{SSP}_N$ and $\varphi \in \mathbb{R}$, the rotation $e^{-i\varphi}f(e^{i\varphi}z)$ is also in $\mathcal{SSP}_N$, and, therefore, it suffices to study the case when $z = r \in$

$(0,1)$. Assuming that f is an extremal function, we can write that

$$f'(r) = \prod_{j=1}^{N+1} \frac{1}{(1-e^{-iNt_j}r^N)^{\mu_j}} \sum_{j=1}^{N+1} \mu_j \frac{e^{it_j}+r}{e^{it_j}-r}, \tag{4.4}$$

where $-\pi \le t_j < \pi$, $\mu_j \ge 0$, $1 \le j \le N+1$, $\sum_{j=1}^{N+1} \mu_j = 1$. Evidently,

$$|f'(r)| \ge \prod_{j=1}^{N+1} a_j^{\mu_j} \sum_{j=1}^{N+1} \mu_j b_j =: G(\mu, a, b), \tag{4.5}$$

with

$$a_j = \left| \frac{1}{1-e^{-iNt_j}r^N} \right|, \quad b_j = \operatorname{Re}\left(\frac{e^{it_j}+r}{e^{it_j}-r} \right), \quad 1 \le j \le N+1.$$

Supposing that all $a_j \ne a_{N+1}$, $1 \le j \le N$, we can deduce that

$$\frac{\partial^2}{\partial \mu_j^2} (\log G) < 0.$$

Therefore, if all the variables except for μ_j are fixed, then $G(\mu, a, b)$ as a function of μ_j attains its minimum on the interval $0 \le \mu_j \le 1$ at either of the endpoints. Thus,

$$|f'(r)| \ge \min_t \frac{1-r^2}{(1-2r\cos t + r^2)(1-2r^N \cos Nt + r^{2N})^{1/N}},$$

and we have to maximize the function

$$g(t) = (1-2r\cos t + r^2)^N (1-2r^N \cos Nt + r^{2N})$$

over the interval $-\pi \le t \le \pi$. Since $g(t)$ is even, assume that $0 \le t \le \pi$. Note that the second factor of $g(t)$ is $2\pi/N$-periodic, increasing on $[\pi - 2\pi/N, \pi - \pi/N]$ and decreasing on $[\pi - \pi/N, \pi]$, whereas the first factor increases everywhere on $[0, \pi]$. Consequently, the global maximum for $g(t)$ on the latter interval is attained somewhere on the interval $[\pi - \pi/N, \pi]$.

Logarithmic differentiation of g yields

$$\frac{g'(t)}{g(t)} = \frac{2Nrh(t)\sin t}{(1-2r\cos t + r^2)(1-2r^N \cos Nt + r^{2N})},$$

where

$$h(t) = 1 + r^{2N} - 2r^N \frac{\sin(N+1)t}{\sin t} + r^{N-1}(1+r^2)\frac{\sin Nt}{\sin t}.$$

In view of a well-known estimate $|\sin Nt| \leq N|\sin t|$, we have

$$h(t) \geq 1 + r^{2N} - 2(N+1)r^N - Nr^{N-1}(1+r^2).$$

Clearly, the right-hand side polynomial of the last inequality has a unique root r_N on the interval $(0,1)$. Therefore, if $r \leq r_N$, then $h(t) \geq 0$, and the maximum of $g(t)$ is attained at $t = \pi$ so that

$$g(t) \leq g(\pi) = (1 - r^N)^2(1+r)^{2N}.$$

Otherwise, let $r_N < r < 1$. If we put $t = \pi - \theta$, where $0 < \theta < \pi/N$, then we have

$$h(t) = 1 + r^{2N} - 2r^N \frac{\sin(N+1)\theta}{\sin\theta} - r^{N-1}(1+r^2)\frac{\sin N\theta}{\sin\theta} =: \kappa(\theta).$$

Next, we show that both functions $\mu_N(\theta) = \sin N\theta/\sin\theta$ and $\mu_{N+1}(\theta)$ decrease on the interval $(0, \pi/N)$. It is easy to verify when $N = 1, 2$. If $N \geq 3$, then we get

$$\lambda_N(\theta) = \mu_N'(\theta)\sin^2\theta = N\cos N\theta\sin\theta - \sin N\theta\cos\theta,$$

so that

$$\lambda_N'(\theta) = -(N^2-1)\sin N\theta\sin\theta.$$

Clearly, $\lambda_N'(\theta) < 0$ for $0 < \theta < \pi/N$ and $\lambda_N'(\theta) > 0$ for $\pi/N < \theta < \pi/(N-1) \leq 2\pi/N$. Since $\lambda_N(0) = 0$ and

$$\lambda_N\left(\frac{\pi}{N-1}\right) = -(N+1)\sin\left(\frac{\pi}{N-1}\right)\cos\left(\frac{\pi}{N-1}\right) \leq 0 \quad \text{for } N \geq 3,$$

we find that $\lambda_N(\theta) \leq 0$, and $\mu_N(\theta)$ decreases on $[0, \pi/(N-1)]$. Hence, there is a unique $\theta_N = \theta_N(r)$ such that $0 < \theta_N < \pi/N$ and $\kappa(\theta_N) = 0$, which corresponds to the maximum of $g(t)$.

In order to prove the sharpness of the estimates (1.5) and (1.6), we construct an appropriate example of a function from $\mathcal{SSP}_N$, N being even. Set

$$\mu(t) = \frac{1}{2}[\delta_\alpha(t) + \delta_{-\alpha}(t)],$$

where $\delta_\alpha(t)$ is the point mass at $t = \alpha$, $0 \leq \alpha \leq \pi$. Then from Lemma 2.1

and the relation (4.2) we obtain that

$$f'_{N,\alpha}(z) = \frac{1-z^2}{(1+2z\cos\alpha+z^2)(1-2z^N\cos N\alpha+z^{2N})^{1/N}}. \tag{4.6}$$

If $0 \le r \le r_N$, then the estimate (1.4) is attained by this function for $\alpha = 0$, $z = r$, otherwise we take $\alpha = \theta_N$, $z = r$. □

Here it is easy to give an example proving the sharpness of all estimates in (1.10)–(1.11), while for (1.12) this is not the case.

Example 4.7 Integration of (4.6) yields

$$f_{N,\alpha}(z) = \int_0^z \frac{1-t^2}{(1+2t\cos\alpha+t^2)(1-2t^N\cos N\alpha+t^{2N})^{1/N}}\,dt,$$
$$0 < \alpha < \frac{\pi}{N},$$

and by selecting a suitable α we can obtain the upper bounds for Koebe constants of $\mathcal{SSP}_N$, N being even.

We present the following tables, the first of them containing the precise values of the Koebe constants, denoted by $k(\mathcal{SSP}_N)$, in the case of odd N. The second table gives the lower k_N^- and the upper k_N^+ estimates of the constants together with the values of α for which the latter are attained. We also present the lower estimates $k_{N,0}^-$ obtained in [10].

N	1	3	5	7	9	11	13	15
$k(\mathcal{SSP}_N)$	0.25	0.3700	0.3817	0.3844	0.3853	0.3857	0.3860	0.3861

N	2	4	6	8	10	12	14	16
k_N^-	0.4038	0.3876	0.3863	0.3862	0.3862	0.3862	0.3862	0.3862
k_N^+	0.4142	0.3935	0.3893	0.3879	0.3872	0.3869	0.3867	0.3866
α	0.7856	0.3363	0.1860	0.1148	0.0760	0.0515	0.0362	0.0260
$k_{N,0}^-$	0.3466	0.3782	0.3834	0.3849	0.3856	0.3859	0.3860	0.3861

Let $\mathcal{R}(\beta)$ be the class of functions from $\mathcal{A}$ such that $\operatorname{Re} f'(z) > \beta$, $z \in \Delta$. From the above tables one may conjecture that the constant $k(\mathcal{SSP}_N)$ tends to $k(\mathcal{R}(0)) = 2\log 2 - 1 = 0.38629\ldots$ (see [1]). Note that for a fixed f, f_N converges to z as $N \to \infty$. This observation shows that the condition for f to be in $\mathcal{R}(0)$ is the limiting case of that for $\mathcal{SSP}_N$ as $N \to \infty$.

We can also use the representation (2.5) to investigate relations between the class $\mathcal{SSP}_N$ and some other well-known subclasses of univalent functions. Choose the measure in (2.5) to be equal to

$$\mu(t) = \frac{1+a}{2}\delta_0(t) + \frac{1-a}{2}\delta_{-\pi}(t)$$

with $-1 < a < 1$. Then we obtain

$$g_{N,a}(z) = \int_0^z \frac{(1+2at+t^2)\,dt}{(1-t^2)(1-t^N)^{(1+a)/N}(1-(-1)^N t^N)^{(1-a)/N}}, \quad (4.8)$$

$g_{N,a}(z)$ being in $\mathcal{SSP}_N$. It is a well-known fact that $\mathcal{R}(0) \subset \mathcal{C}$. As observed in the introduction (see also [8]), for each $N \geq 1$, $\mathcal{SSP}_N$ is also contained in $\mathcal{C}$. Assuming, without loss of generality, that N is even, we get from (4.8) that $g'_{N,a}(z) \approx (1-z)^{-1-2/N}$ as $z \to 1$, therefore, the real part of $g'_{N,a}(z)$ cannot be bounded from below in Δ. This observation shows that for no real β the class $\mathcal{SSP}_N$ is contained in $\mathcal{R}(\beta)$.

4.9. Proof of Theorem 1.13

It is easy to verify that $g_{N,a}(z)$ is a Schwarz-Christoffel integral mapping Δ onto a polygonal domain. For even N, here the angle at $f(-1)$ equals $-2\pi/N$, since this vertex is at the infinity. Assume that $0 < \alpha < \pi/N$. Since the function $g_{N,a}(z)$ is univalent in Δ and is real-valued on the real axis, the image of the arc $z = e^{i\phi}$, $0 < \phi < \alpha$, is a ray beginning at $g_{N,a}(e^{i\alpha})$ and forming an angle π/N with the real axis. The image domain $g_{N,a}(\Delta)$ has the interior angle 2π at $g_{N,a}(e^{i\alpha})$, so it can be starlike with respect to the origin only if $\arg g_{N,a}(e^{i\alpha}) = \pi/N$. However, if we set $\alpha \to 0$, then the left-hand side is infinitesimal, so the identity is not possible here and for even N the inclusion $\mathcal{SSP}_N \subset \mathcal{S}^*$ fails.

Assume now that $N \geq 3$ is an odd integer. We consider the mapping given by the function (4.8) where α is determined as above but lies between $[N/2]\frac{\pi}{N}$ and $([N/2]+1)\frac{\pi}{N}$, $[x]$ standing for the entire part of x. Moreover, suppose that $\alpha = \pi/2 + \gamma$ with $\gamma \to 0$. Again, $g_{N,a}(z)$ maps the unit disc onto a polygonal domain, the angle at the vertex $g_{N,a}(e^{i\alpha})$ being equal to 2π. Therefore, the image domain has a slit starting at $g_{N,a}(e^{i\alpha})$, and the function (4.8) is not starlike provided that the angle σ_N between the slit and the positive real axis is not equal to $\arg g_{N,a}(e^{i\alpha})$.

For odd N we deduce from (4.8) that the point $z = 1$ is mapped into infinity, the corresponding ray forming the angle $-(1+a)\pi/N$ with the real axis. The points π/N, $3\pi/N$, $5\pi/N, \dots$ (called points of the first kind) are mapped into vertices whose interior angles equal $\pi(1-(1-a)/N)$, whereas the points $2\pi/N$, $4\pi/N, \dots$ (points of the second kind) go into vertices with interior angles $\pi(1-(1+a)/N)$.

If $N = 4m+1$, $m \in \mathbb{N}$, then there are m points of the first and the second kind lying between $t=0$ and $t=\alpha=\pi/2+\gamma$, provided that γ is small enough. Then we have $\sigma_N = \pi/2+\pi a/(2N) = \pi/2+\pi\gamma/(2N)+o(\gamma)$, $\gamma \to 0$.

On the other hand, by a substitution $t = e^{i\alpha}\rho$, we get from (4.8) that

$$\arg g_{N,a}(e^{i\alpha}) = \frac{\pi+2\gamma}{2} + \arg\int_0^1 \frac{(1-\rho)(1+\rho e^{2i\gamma})}{(1+\rho^2 e^{2i\gamma})(1+\rho^{2N}e^{2iN\gamma})^{1/N}}\left(\frac{1+i\rho^N e^{Ni\gamma}}{1-i\rho^N e^{iN\gamma}}\right)^{\frac{\sin\gamma}{N}} d\rho. \tag{4.10}$$

For $\gamma \to 0$ the following asymptotic formula can be derived from (4.10).

$$\arg g_{N,a}(e^{i\alpha}) = \frac{\pi}{2} + \gamma\left(1-2\frac{I_1}{I_2}\right) + o(\gamma), \tag{4.11}$$

where

$$I_1 = \int_0^1 g_N(\rho)h_N(\rho)d\rho, \quad I_2 = \int_0^1 g_N(\rho)d\rho, \tag{4.12}$$

the integrands being defined as follows:

$$g_N(\rho) = \frac{1-\rho^2}{(1+\rho^2)(1+\rho^{2N})^{1/N}}, \quad h_N(\rho) = h_{N,1}(\rho)+h_{N,2}(\rho), \tag{4.13}$$

with

$$h_{N,1}(\rho) = \frac{\rho^2(1-\rho)}{(1+\rho)(1+\rho^2)}, \quad \text{and} \quad h_{N,2}(\rho) = \frac{\rho^{2N}}{1+\rho^{2N}} - \frac{1}{N}\arctan\rho^N.$$

In view of the mean value theorem from the integral calculus, it suffices to prove that

$$h_N(\rho) \le h_N(1) = \frac{1}{2} - \frac{\pi}{4N}, \quad \text{on } 0 \le \rho \le 1. \tag{4.14}$$

In fact, for ρ close to 0 the bound is not attained, therefore, the ratio of the integrals in (4.11) is strictly less that $h_N(1)$. Hence, the coefficients at the first order terms in the asymptotic expansions for σ_N and $\arg g_{N,a}(e^{i\alpha})$ do not coincide.

Since

$$h'_{N,1}(\rho) = -\frac{2\rho(\rho^3+\rho^2+\rho-1)}{(1+\rho)^2(1+\rho^2)^2}, \tag{4.15}$$

it is clear that $h_{N,1}(\rho)$ attains its maximum value on $0 \le \rho \le 1$ at $\rho_0 = 0.5437\ldots$, and $h_{N,1}(\rho_0) = \delta_0 = 0.0674\ldots$. Thus, if $\rho^{2N}/(1+\rho^{2N}) \le 0.5 - \delta_0 - \pi/20 = \varepsilon_0 = 0.2755\ldots$, or, equivalently,

$$\rho^N \le \eta_0 = \sqrt{\frac{\varepsilon_0}{1-\varepsilon_0}} = 0.6167\ldots, \tag{4.16}$$

then (4.14) clearly holds. On the other hand, if $\rho^N > \eta_0$, then we have the following estimates

$$h'_{N,2}(\rho) = \frac{\rho^{N-1}(2N\rho^N - 1 - \rho^{2N})}{(1+\rho^{2N})^2} \ge \frac{1}{2}\eta_0(N\eta_0 - 1) \ge 0.6424\ldots, \tag{4.17}$$

and $h'_{N,1}(\rho) \ge -0.25$ for all $0 \le \rho \le 1$. Therefore, $h'_N(\rho) \ge 0$, and the inequality (4.14) is verified.

The case $N = 4m+3$ is studied in a similar way, so, the proof is complete. □

Acknowledgment This work was completed during the visit of the first author to Indian Institute of Technology (IIT-Madras), Chennai, and the visit was supported by a grant (No. 43/3/99/R& D-II/559) from National Board for Higher Mathematics, India. The authors also thank Commission on Development and Exchanges, International Mathematical Union, for its travel support to IIT-Madras for the first author.

References

[1] Gal'perin I.M., *The theory of univalent functions with bounded rotation (in Russian)*. Izv. Vyss. Uchebn. Zaved. Matematika **3** (4) (1958), 50–61.

[2] Goluzin G.M., *Geometric theory of functions of a complex variable*. [Engl. transl.] Providence, RI: Amer. Math. Soc., 1969.

[3] Goodman A.W., *Univalent functions*, in 2 vols. Washington, NJ: Polygonal Publishers, 1983.

[4] Hallenbeck D. and MacGregor T., *Linear problems and convexity technique*. Boston, MA: Pitman Publishers, 1984.

[5] Nezhmetdinov I., *On the order of starlikeness of the class UST*. J. Math. Anal. Appl. **234** (1999), 559–566.

[6] Nezhmetdinov I., *A sharp lower bound in the distortion theorem for the Sakaguchi class*. J. Math. Anal. Appl. **242** (2000), 129–134.

[7] Ponnusamy S., *Some applications of differential subordination and convolution techniques to univalent functions theory*. Ph. D. thesis, I. I. T. Kanpur, India, 1988.

[8] Sakaguchi K., *On a certain univalent mapping*. J. Math. Soc. Japan **11** (1959), 72–75.

[9] Stankiewicz J., *Some remarks on functions starlike with respect to symmetric points*. Ann. Univ. Mariae-Curie Sklodowska **19** (1965), 53–59.

[10] Singh P. and Chand R., *On certain schlicht mappings*. Indian. J. Pure Appl. Math. **10** (9) (1979), 1167–1174.

[11] Singh P. and Tygel M., *On some univalent functions in the unit disc*. Indian. J. Pure Appl. Math. **12** (4) (1981), 513–520.

I.R. Nezhmetdinov
Faculty of Mechanics and Mathematics
Kazan State University
Kazan 420008, Russia
E-mail: nezh@ksu.ru
inezhmet@acer.iitm.ernet.in

S. Ponnusamy
Department of Mathematics
Indian Institute of Technology
IIT-Madras, Chennai 600 036
India
E-mail: samy@acer.iitm.ernet.in