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Pseudo-eigenvalues of W-operators on Hilbert modular forms

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Abstract. In this paper, we study pseudo-eigenvalues of W-operators on Hilbert modular forms. In particular, we show that they are roots of unity under a certain condition.

Key words: Hilbert modular forms.

1. Introduction

Let F be a totally real number field. For a positive integer k, an integral ideal $\mathfrak{N}$ of F and a (Hecke) character ψ, we consider the space $S_k^0(\mathfrak{N}, \psi)$ of new forms of weight k, level $\mathfrak{N}$ and character ψ (see §2 for the definition). We have operators $\eta_{\mathfrak{p}}$ (W-operator) on $S_k^0(\mathfrak{N}, \psi)$ for each prime $\mathfrak{p}$ dividing $\mathfrak{N}$ (see §2 for the definition and details). For a primitive form $\mathbf{f}$ of $S_k^0(\mathfrak{N}, \psi)$, we can write $\mathbf{f}|\eta_{\mathfrak{p}} = c\mathbf{g}$ with the corresponding primitive form $\mathbf{g}$ and call $c = c_{\mathfrak{p}}$ the pseudo-eigenvalue of $\eta_{\mathfrak{p}}$ associated to $\mathbf{f}$. In the case that the $\mathfrak{p}$-th Fourier coefficient of $\mathbf{f}$ does not vanish, the pseudo-eigenvalue of $\eta_{\mathfrak{p}}$ associated to $\mathbf{f}$ is expressed by the $\mathfrak{p}$-th Fourier coefficient of $\mathbf{f}$ and the local Gauss sum (Corollary 2.7). In the case that the $\mathfrak{p}$-th Fourier coefficient of $\mathbf{f}$ vanishes, we have the following theorem in the simplest case.

Theorem 1.1 *Let $\mathfrak{p}$ be a prime ideal of F, $\mathbf{f}$ a primitive form of $S_k^0(\mathfrak{p}^e, \psi)$ and $\mathfrak{p}^n$ the conductor of ψ. If $3n \leq e$, the pseudo-eigenvalue A associated to $\mathbf{f}$ satisfies $A^{2\alpha} = (\psi^{\alpha})^*(\mathfrak{p}^e)$, where α is the order of $\psi_{\mathfrak{p}}$ as the character of $(\mathfrak{o}_F/\mathfrak{p}^n)^{\times}$ and $(\psi^{\alpha})^*$ the ideal character associated with ψ^{α}. Moreover if ψ is of finite order, the pseudo-eigenvalue is a root of unity.*

In the next section, we will introduce necessary notations for adelic Hilbert modular forms. In §3, we study twisted forms, since primitive forms whose Fourier coefficients of level parts vanish may be twisted forms. In the last section, we prove above theorem in more general case (Theorem 4.3). We note that we consider a unitary Hecke character (possibly of infinite

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order) in this paper.

2. Space of Hilbert modular forms

Let F be a totally real algebraic number field, $F_{\mathbf{A}}$ the ring of adeles of F, and $F_{\mathbf{A}}^{\times}$ the group of ideles of F. For $\alpha \in F$, we write $\alpha \gg 0$ if α is totally positive and $\alpha \ll 0$ if α is totally negative. We denote by $\mathbf{a}$ (resp. $\mathbf{h}$) the set of all archimedean (resp. nonarchimedean) primes of F. We write $F_{\mathbf{a}}$ (resp. $F_{\mathbf{h}}$) for the archimedean (resp. nonarchimedean) factors of $F_{\mathbf{A}}$. For the prime ideal of F which corresponds to $\mathfrak{p} \in \mathbf{h}$, we also use the same symbol $\mathfrak{p}$. We denote by F_v the v-completion of F, and by x_v the v-component of $x \in F_{\mathbf{A}}$ for $v \in \mathbf{a} \cup \mathbf{h}$. For an ideal $\mathfrak{a}$ of F and $\mathfrak{p} \in \mathbf{h}$, we denote by $\mathfrak{a}_{\mathfrak{p}}$ the topological closure of $\mathfrak{a}$ in $F_{\mathfrak{p}}$. We denote by $\mathfrak{o}_F$ and $\mathfrak{d}_F$ the maximal order of F and the different of F over $\mathbf{Q}$, and by $\mathfrak{o}_{\mathfrak{p}}$ and $\mathfrak{d}_{\mathfrak{p}}$ $(\mathfrak{o}_F)_{\mathfrak{p}}$ and $(\mathfrak{d}_F)_{\mathfrak{p}}$. For every $a \in F_{\mathbf{A}}^{\times}$, we denote by $a\mathfrak{o}_F$ the fractional ideal of F associated with a. For an ideal $\mathfrak{a}$ of F and $\mathfrak{p} \in \mathbf{h}$, we denote by $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})$ the order of $\mathfrak{a}$ at $\mathfrak{p}$. For $a \in F_{\mathbf{A}}^{\times}$, we put $\mathrm{ord}_{\mathfrak{p}}(a) = \mathrm{ord}_{\mathfrak{p}}(a\mathfrak{o}_F)$. We denote by $\pi_{\mathfrak{p}}$ a prime element of $F_{\mathfrak{p}}$.

By a Hecke character ψ of F, we understand a continuous homomorphism of $F_{\mathbf{A}}^{\times}$ into $\{z \in \mathbf{C} \mid |z| = 1\}$ which is trivial on $F^{\times}$. For ψ we denote by ψ_v, $\psi_{\mathbf{a}}$, and $\psi_{\mathbf{h}}$ its restrictions to $F_v^{\times}$, $F_{\mathbf{a}}^{\times}$ and $F_{\mathbf{h}}^{\times}$, respectively. Given ψ, there exists a unique integral ideal $\mathfrak{f}$ with the following property: $\psi_v(x) = 1$ if $v \in \mathbf{h}$, $x \in \mathfrak{o}_v^{\times}$, and $x - 1 \in \mathfrak{f}_v$; if $\mathfrak{f}'$ is another integral ideal with this property, then $\mathfrak{f}' \subset \mathfrak{f}$. The ideal $\mathfrak{f}$ is called the conductor of ψ. Let $\mathfrak{c}$ be an integral ideal such that $\mathfrak{c} \subset \mathfrak{f}$. Given a fractional ideal $\mathfrak{a}$ prime to $\mathfrak{c}$, we take an element α of $F_{\mathbf{h}}^{\times}$ so that $\alpha_v \mathfrak{o}_v = \mathfrak{a}_v$ for every $v \in \mathbf{h}$ and $\alpha_v = 1$ for every $v|\mathfrak{c}$. We then put $\psi^{(\mathfrak{c})}(\mathfrak{a}) = \psi_{\mathbf{h}}(\alpha)$. This is well-defined. We put $\psi^{(\mathfrak{c})}(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not prime to $\mathfrak{c}$. We call $\psi^{(\mathfrak{c})}$ the ideal character mod $\mathfrak{c}$ associated with ψ. If $\mathfrak{c} = \mathfrak{f}$, we denote $\psi^{(\mathfrak{c})}$ by ψ^*.

We put $G = GL_2(F)$ and $G_v = GL_2(F_v)$ for every $v \in \mathbf{a} \cup \mathbf{h}$. Let $G_{\mathbf{A}}$ be the adelization of G, and $G_{\mathbf{a}}$ (resp. $G_{\mathbf{h}}$) its archimedean (resp. nonarchimedean) factors. For $v \in \mathbf{a} \cup \mathbf{h}$ and an element $x \in G_{\mathbf{A}}$, we denote by x_v its v-component and also by $x_{\mathbf{a}}$ its $\mathbf{a}$-component. For any set X, we write $X^{\mathbf{a}}$ for set of all indexed elements $(x_v)_{v \in \mathbf{a}}$ with $x_v \in X$. For each $v \in \mathbf{a}$, we take a corresponding injection τ_v of F into $\mathbf{R}$ and denote also by τ_v the isomorphism of F_v to $\mathbf{R}$. We put $G_+ = \{x \in G \mid \det(x) \gg 0\}$, $F_{\mathbf{a}+} = \{x \in F_{\mathbf{a}} \mid x_v^{\tau_v} > 0 \text{ for all } v \in \mathbf{a}\}$, $G_{\mathbf{a}+} = \{x \in G_{\mathbf{a}} \mid \det(x) \in F_{\mathbf{a}+}\}$

and $G_{\mathbf{A}+} = \{x \in G_{\mathbf{A}} \mid x_{\mathbf{a}} \in G_{\mathbf{a}+}\}$. Then we have $G_+ \subset G_{\mathbf{A}+}$ by the diagonal embedding of G to $G_{\mathbf{A}}$. For $x \in G_{\mathbf{A}}$, we put $x^{\iota} = \det(x)x^{-1}$ and $x^{-\iota} = (x^{\iota})^{-1}$.

We define the space S_k of 'classical' Hilbert cusp forms. Let H be the complex upper half plane. For

$$\alpha = (\alpha_v)_{v\in\mathbf{a}} = \left(\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}\right)_{v\in\mathbf{a}} \in G_{\mathbf{a}+},$$
$$z = (z_v)_{v\in\mathbf{a}} \in H^{\mathbf{a}}, \quad k = (k_v)_{v\in\mathbf{a}} \in \mathbf{Z}^{\mathbf{a}},$$

and a $\mathbf{C}$-valued function f on $H^{\mathbf{a}}$, we put

$$\alpha(z) = ((a_v^{\tau_v} z_v + b_v^{\tau_v})/(c_v^{\tau_v} z_v + d_v^{\tau_v}))_{v\in\mathbf{a}},$$
$$J_k(\alpha, z) = \prod_{v\in\mathbf{a}} \left((\det(\alpha_v)^{\tau_v})^{-k_v/2}(c_v^{\tau_v} z_v + d_v^{\tau_v})^{k_v}\right),$$
$$(f||_k\alpha)(z) = J_k(\alpha, z)^{-1} f(\alpha(z))$$

and denote by S_k the space of all holomorphic functions f on $H^{\mathbf{a}}$ satisfying the following conditions:

(A1) there exists a positive integer N such that $f||_k\gamma = f$ for all $\gamma \in SL_2(\mathfrak{o}_F) \cap (1_2 + N \cdot M_2(\mathfrak{o}_F))$,

(A2) f has a Fourier expansion

$$(f||_k\alpha)(z) = \sum_{\xi} c_\alpha(\xi)\mathbf{e}_F(\xi z),$$

with $c_\alpha(\xi) \in \mathbf{C}$ for every $\alpha \in G_+$, where ξ runs over all totally positive elements of a lattice L_α in F, and $\mathbf{e}_F(\xi z) = \exp\big(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \xi_v^{\tau_v} z_v\big)$.

We now define the space of (adelic) Hilbert cusp forms. We take an element δ of $F_{\mathbf{A}}^{\times}$ such that $\delta\mathfrak{o}_F = \mathfrak{d}_F$ and $\delta_{\mathbf{a}} = 1_{\mathbf{a}}$ and put $u = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$. For an element $\gamma \in G_{\mathbf{A}}$ and a $\mathbf{C}$-valued function $\mathbf{f}$ on $G_{\mathbf{A}}$, we put

$$(\mathbf{f}|\gamma)(x) = \mathbf{f}(x(u\gamma u^{-1})^{\iota}).$$

We have $\mathbf{f}|\gamma_1|\gamma_2 = \mathbf{f}|\gamma_1\gamma_2$ for any $\gamma_1, \gamma_2 \in G_{\mathbf{A}}$. Let $\mathfrak{c}$ be an integral ideal, $k \in \mathbf{Z}^{\mathbf{a}}$ and ψ a Hecke character whose conductor divides $\mathfrak{c}$. We denote by $S_k(\mathfrak{c}, \psi)$ the space of all $\mathbf{C}$-valued functions $\mathbf{f}$ on $G_{\mathbf{A}}$ satisfying the following conditions:

(B1) $\mathbf{f}(\alpha x) = \mathbf{f}(x)$ for $\alpha \in G$ and $x \in G_{\mathbf{a}}$;

(B2) $\mathbf{f}|\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} = \psi(s)\mathbf{f}$ for $s \in F_{\mathbf{A}}^{\times}$;

(B3) $\mathbf{f}|w = \psi_Y(w)\mathbf{f}$ for $w \in \Gamma_{\mathbf{h}}(\mathfrak{c})$;

(B4) for every $x \in G_{\mathbf{A}}$ with $x_{\mathbf{a}} = 1_{\mathbf{a}}$, there is an element g_x of S_k such that $\mathbf{f}|y(x) = \det(y)^{\mathbf{i}m} g_x||_k y^{\iota}(\mathbf{i})$ for all $y \in G_{\mathbf{a}+}$.

Here $\Gamma_{\mathbf{h}}(\mathfrak{c}) = \prod_{v \in \mathbf{h}} \Gamma_v(\mathfrak{c})$ and

$$\Gamma_v(\mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathfrak{o}_v) \mid c \in \mathfrak{c}_v \right\}; \quad \text{for} \quad w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathbf{h}},$$

we put

$$\psi_Y(w) = \psi_{\mathfrak{c}}(a_{\mathfrak{c}}) = \prod_{\mathfrak{p}|\mathfrak{c}} \psi_{\mathfrak{p}}(a_{\mathfrak{p}}); \quad \mathbf{i} = (\sqrt{-1}, \dots, \sqrt{-1}) \in H^{\mathbf{a}}; \quad m \in \mathbf{R}^{\mathbf{a}}$$

is determined by $\psi_{\mathbf{a}}(s) = s^{2\mathbf{i}m}$ for $s \in F_{\mathbf{a}+}$. We understand $s^l = \prod_{v \in \mathbf{a}} (s_v^{\tau_v})^{l_v}$ for $s \in F_{\mathbf{a}+}$ and $l \in \mathbf{C}^{\mathbf{a}}$.

We call elements of $S_k(\mathfrak{c}, \psi)$ (adelic) Hilbert cusp forms of weight k, level $\mathfrak{c}$ and character ψ. We note that the space $S_k(\mathfrak{c}, \psi)$ is independent of the choice of δ and coincides with that of [5] and [6]. We have

$$\psi(s) = \psi_{\mathfrak{c}}(s_{\mathfrak{c}}) \operatorname{sgn}(s_{\mathbf{a}})^k |s_{\mathbf{a}}|^{2\mathbf{i}m} \quad \text{for} \quad s \in F_{\mathbf{a}}^{\times} \times \prod_{v \in \mathbf{h}} \mathfrak{o}_v^{\times}.$$

So we see $S_k(\mathfrak{c}, \psi) = \{0\}$ unless $\psi_v(-1) = (-1)^{k_v}$ for all $v \in \mathbf{a}$. Moreover it is well known that $S_k = \{0\}$ unless $k_v > 0$ for all $v \in \mathbf{a}$ (cf. [5]).

Let $\mathbf{f}$ be an element of $S_k(\mathfrak{c}, \psi)$ and put $k_0 = \max_{v \in \mathbf{a}}\{k_v\}$. $\mathbf{f}$ has the expansion

$$\begin{aligned} &\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \\ &= \sum_{0 \ll \zeta \in F} C(\zeta y \mathfrak{o}_F, \mathbf{f}) N(\zeta y \mathfrak{o}_F)^{-k_0/2} (\zeta y_{\mathbf{a}})^{(k/2)+\mathbf{i}m} \mathbf{e}_F(\zeta \mathbf{i} y_{\mathbf{a}}) \mathbf{e}_{\mathbf{A}}(\zeta x), \end{aligned}$$

where $y \in F_{\mathbf{A}}^{\times}$, $y_{\mathbf{a}} \in F_{\mathbf{a}+}$ and $x \in F_{\mathbf{A}}$ and $\mathbf{e}_{\mathbf{A}}$ is the character of the additive group $F_{\mathbf{A}}/F$ such that $\mathbf{e}_{\mathbf{A}}(x_{\mathbf{a}}) = \mathbf{e}_F(x_{\mathbf{a}})$. (If $\zeta y \mathfrak{o}_F$ is not integral, then $C(\zeta y \mathfrak{o}_F, \mathbf{f}) = 0$.)

Now for an ideal $\mathfrak{q}$ we put

$$\mathbf{f}|B_{\mathfrak{q}} = N(\mathfrak{q})^{-k_0/2}\mathbf{f}|\begin{pmatrix}1 & 0\\ 0 & q^{-1}\end{pmatrix}, \tag{2.1}$$

where q is an element of $F_{\mathbf{A}}^{\times}$ such that $q\mathfrak{o}_F = \mathfrak{q}$ and $q_{\mathbf{a}} = 1_{\mathbf{a}}$. Then $\mathbf{f}|B_{\mathfrak{q}} \in S_k(\mathfrak{c}\mathfrak{q}, \psi)$ and $C(\mathfrak{m}, \mathbf{f}|B_{\mathfrak{q}}) = C(\mathfrak{m}\mathfrak{q}^{-1}, \mathbf{f})$. This is independent of a choice of q.

We define a Hecke operator $T'_{\mathfrak{c}}(\mathfrak{n})$ as that of [5]. We have

$$C(\mathfrak{m}, \mathbf{f}|T'_{\mathfrak{c}}(\mathfrak{n})) = \sum_{\mathfrak{m}+\mathfrak{n}\subset\mathfrak{a}} \psi^{(\mathfrak{c})}(\mathfrak{a})N(\mathfrak{a})^{k_0-1}C(\mathfrak{m}\mathfrak{n}\mathfrak{a}^{-2}, \mathbf{f}).$$

Under our notation, we have for a prime ideal $\mathfrak{p}$,

$$\mathbf{f}|T'_{\mathfrak{c}}(\mathfrak{p}) = \begin{cases} N(\mathfrak{p})^{(k_0/2)-1}\left\{ \sum_{l\in\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}} \mathbf{f}|\begin{pmatrix}1 & l\\ 0 & \pi_{\mathfrak{p}}\end{pmatrix}_{\mathfrak{p}} + \mathbf{f}|\begin{pmatrix}\pi_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}_{\mathfrak{p}}\right\} & \text{if } \mathfrak{p}\nmid\mathfrak{c}, \\ N(\mathfrak{p})^{(k_0/2)-1} \sum_{l\in\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}\mathfrak{o}_{\mathfrak{p}}} \mathbf{f}|\begin{pmatrix}1 & l\\ 0 & \pi_{\mathfrak{p}}\end{pmatrix}_{\mathfrak{p}} & \text{if } \mathfrak{p}\mid\mathfrak{c}. \end{cases} \tag{2.2}$$

We denote by $S_k^1(\mathfrak{c}, \psi)$ the subspace of $S_k(\mathfrak{c}, \psi)$ generated by the set $\cup_{\mathfrak{a}} \cup_{\mathfrak{b}} \{\mathbf{f}|B_{\mathfrak{b}} \mid \mathbf{f} \in S_k(\mathfrak{a}, \psi)\}$. Here $\mathfrak{a}$ runs over all integral ideals such that $\mathfrak{q}|\mathfrak{a}, \mathfrak{a}|\mathfrak{c}$ and $\mathfrak{a} \neq \mathfrak{c}$; $\mathfrak{b}$ runs over all divisors of $\mathfrak{c}\mathfrak{a}^{-1}$; $\mathfrak{q}$ is the conductor of ψ. Furthermore, we denote by $S_k^0(\mathfrak{c}, \psi)$ the orthogonal complement of $S_k^1(\mathfrak{c}, \psi)$ in $S_k(\mathfrak{c}, \psi)$ with respect to the (Petersson) inner product. (cf. [5], p. 651.)

We call an element $\mathbf{f} \in S_k^0(\mathfrak{c}, \psi)$ a primitive form if $\mathbf{f}$ is a common eigenfunction of all $T'_{\mathfrak{c}}(\mathfrak{n})$ ($(\mathfrak{n}, \mathfrak{c}) = 1$) and $C(\mathfrak{o}_F, \mathbf{f}) = 1$. We know that a primitive form is a common eigenfunction of all Hecke operators (and adjoint Hecke operators) and

$$\mathbf{f}|T'_{\mathfrak{c}}(\mathfrak{n}) = C(\mathfrak{n}, \mathbf{f})\mathbf{f}.$$

Definition 2.1 Let χ be a Hecke character of the conductor $\mathfrak{q}$. We fix $v \in F_{\mathfrak{p}}$ such that $v(\mathfrak{q}\mathfrak{d}_F)_{\mathfrak{p}} = \mathfrak{o}_{\mathfrak{p}}$. For a prime ideal $\mathfrak{p}$ dividing $\mathfrak{q}$, we put

$$\mathfrak{g}_{\mathfrak{p}}(\chi) = \sum_{b\in(\mathfrak{o}_{\mathfrak{p}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{p}})^{\times}} \chi_{\mathfrak{p}}(vb)^{-1}\mathbf{e}_{\mathfrak{p}}(vb)^{-1},$$

where $\mathbf{e}_\mathfrak{p}$ is the $\mathfrak{p}$-component of $\mathbf{e}_\mathbf{A}$. For a prime ideal $\mathfrak{p}$ not dividing $\mathfrak{q}$, we put $\mathfrak{g}_\mathfrak{p}(\chi) = 1$. We call $\mathfrak{g}_\mathfrak{p}(\chi)$ the local Gauss sum. (cf. [7], (A6.3.6).)

We note that $\mathfrak{g}_\mathfrak{p}(\chi)$ is independent of a choice of v. We have

$$\mathfrak{g}(\chi) = \chi^*\Big(\prod_{\mathfrak{p}\nmid\mathfrak{q}} \mathfrak{p}^{e(\mathfrak{p})}\Big)\prod_{\mathfrak{p}|\mathfrak{q}} \mathfrak{g}_\mathfrak{p}(\chi),$$

where $\mathfrak{d}_F = \prod_\mathfrak{p} \mathfrak{p}^{e(\mathfrak{p})}$. Here $\mathfrak{g}(\chi)$ is the (global) Gauss sum which is defined by [6], (9.31).

Let $\mathbf{f}$ be an element of $S_k(\mathfrak{c}, \psi)$ and χ a Hecke character of the conductor $\mathfrak{q}$. For each prime ideal $\mathfrak{p}$ dividing $\mathfrak{q}$, we fix $u_\mathfrak{p} \in F_\mathfrak{p}$ such that $u_\mathfrak{p}(\mathfrak{q}\mathfrak{d}_F)_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}$ and put

$$(\mathbf{f}|R_\chi)(x) = \frac{\chi(\det(x))}{\prod_{\mathfrak{p}|\mathfrak{q}} \mathfrak{g}_\mathfrak{p}(\overline{\chi})}\sum_{(v_\mathfrak{p})}\Big(\prod_{\mathfrak{p}|\mathfrak{q}} \chi_\mathfrak{p}(u_\mathfrak{p}v_\mathfrak{p})\Big)\Big(\mathbf{f}\Big|\prod_{\mathfrak{p}|\mathfrak{q}}\begin{pmatrix}1 & u_\mathfrak{p}v_\mathfrak{p}\delta_\mathfrak{p}\\ 0 & 1\end{pmatrix}_\mathfrak{p}\Big)(x),$$

where $(v_\mathfrak{p})$ runs over $\prod_{\mathfrak{p}|\mathfrak{q}}(\mathfrak{o}_\mathfrak{p}/\mathfrak{q}\mathfrak{o}_\mathfrak{p})^\times$.

We note that $\mathbf{f}|R_\chi$ is independent of a choice of $(u_\mathfrak{p})$ and

$$(\mathbf{f}|R_\chi)(x) = \frac{\chi(\det(x))}{\mathfrak{g}(\overline{\chi})}\sum_{v\in\mathfrak{q}^{-1}\mathfrak{d}_F^{-1}/\mathfrak{d}_F^{-1}} \overline{\chi^*}(v\mathfrak{q}\mathfrak{d}_F)\overline{\chi_\mathbf{a}}(v)\Big(\mathbf{f}\Big|\begin{pmatrix}1 & v\delta\\ 0 & 1\end{pmatrix}_\mathbf{h}\Big)(x),$$

where the subscript $\mathbf{h}$ indicates the projection to the nonarchimedean part. (cf. [5], p. 664 and [6], p. 354.)

Lemma 2.2 *Let $\mathbf{f}$ be an element of $S_k(\mathfrak{c}, \psi)$ and χ a Hecke character of the conducter $\mathfrak{q}$ and $\mathfrak{b}$ the conducter of $\psi\chi$. Suppose that $\mathfrak{a}$ is the least common multiple of $\mathfrak{c}$, $\mathfrak{q}\mathfrak{b}$ and $\mathfrak{q}\prod_{\mathfrak{p}|\mathfrak{q}}\mathfrak{p}$. Then $\mathbf{f}|R_\chi$ belongs to $S_k(\mathfrak{a}, \psi\chi^2)$ and $C(\mathfrak{m}, \mathbf{f}|R_\chi) = \chi^*(\mathfrak{m})C(\mathfrak{m}, \mathbf{f})$ for any integral ideal $\mathfrak{m}$. Moreover, for an integral ideal $\mathfrak{m}$ such that $(\mathfrak{m}, \mathfrak{q}) = 1$, the following diagram is commutative*:

$$\begin{array}{ccc} S_k(\mathfrak{c}, \psi) & \xrightarrow{\chi^*(\mathfrak{m})T'_\mathfrak{c}(\mathfrak{m})} & S_k(\mathfrak{c}, \psi) \\ {\scriptstyle R_\chi}\downarrow & & \downarrow{\scriptstyle R_\chi} \\ S_k(\mathfrak{c}, \psi\chi^2) & \xrightarrow{T'_\mathfrak{c}(\mathfrak{m})} & S_k(\mathfrak{c}, \psi\chi^2). \end{array} \tag{2.3}$$

Proof. It is known that the assertion holds if $\mathfrak{a}$ is the least common multiple of $\mathfrak{c}$, $\mathfrak{q}^2$ and $\mathfrak{q}\mathfrak{c}_0$. Here $\mathfrak{c}_0$ is the conductor of ψ. (cf. [6], Proposition 9.7.) We may assume $\mathfrak{q} = \mathfrak{p}^j$. We have to show that $(\mathbf{f}|R_\chi)|w = (\psi\chi^2)_\mathfrak{p}(a)\mathbf{f}|R_\chi$

for $w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathfrak{p}}(\mathfrak{a})$. Put $u = (\pi_{\mathfrak{p}}^{j}\delta_{\mathfrak{p}})^{-1}$. First suppose $\det(w) = 1$. For each $v \in (\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{j}))^{\times}$, we take $\varepsilon_v \in \mathfrak{o}_{\mathfrak{p}}^{\times}$ such that $d \equiv \varepsilon_v(a + cuv\delta_{\mathfrak{p}}) \bmod (\pi_{\mathfrak{p}}^{j})$. Then $v \mapsto \varepsilon_v v$ gives a bijection of $(\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{j}))^{\times}$ and

$$\begin{aligned}&\begin{pmatrix} 1 & uv\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & u\varepsilon_v v\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}^{-1} \\ &\qquad = \begin{pmatrix} a + cuv\delta_{\mathfrak{p}} & b + v\pi_{\mathfrak{p}}^{-j}(d - \varepsilon_v(a + cuv\delta_{\mathfrak{p}})) \\ c & d - cu\varepsilon_v v\delta_{\mathfrak{p}} \end{pmatrix}\end{aligned}$$

is in $\Gamma_{\mathfrak{p}}(\mathfrak{c})$. Hence

$$\begin{aligned}&(\mathbf{f}|R_{\chi}|w)(x) \\ &= \frac{\chi(\det(x))}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})} \sum_{v \in (\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{j}))^{\times}} \chi_{\mathfrak{p}}(uv)\left(\mathbf{f}\left|\begin{pmatrix} 1 & uv\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}\right|w\right)(x) \\ &= \frac{\chi(\det(x))}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})} \sum_{v \in (\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{j}))^{\times}} \chi_{\mathfrak{p}}(uv)\left(\psi_{\mathfrak{p}}(a + cuv\delta_{\mathfrak{p}})\mathbf{f}\left|\begin{pmatrix} 1 & u\varepsilon_v v\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}\right.\right)(x).\end{aligned}$$

So, we obtain $(\mathbf{f}|R_{\chi})|w(x) = (\psi\chi^2)_{\mathfrak{p}}(a)\mathbf{f}|R_{\chi}(x)$ since

$$\begin{aligned}\chi_{\mathfrak{p}}(\varepsilon_v)^{-1}\psi_{\mathfrak{p}}(a + cuv\delta_{\mathfrak{p}}) &= \chi_{\mathfrak{p}}(\varepsilon_v(a + cuv\delta_{\mathfrak{p}}))^{-1}(\chi\psi)_{\mathfrak{p}}(a + cuv\delta_{\mathfrak{p}}) \\ &= \chi_{\mathfrak{p}}(d)^{-1}(\chi\psi)_{\mathfrak{p}}(a) = (\psi\chi^2)_{\mathfrak{p}}(a).\end{aligned}$$

Next we suppose $w = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in \Gamma_{\mathfrak{p}}(\mathfrak{a})$.

$$\begin{aligned}&(\mathbf{f}|R_{\chi}|w)(x) \\ &= \frac{\chi(\det(x)d)}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})} \sum_{v \in (\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{j}))^{\times}} \chi_{\mathfrak{p}}(uv)\left(\mathbf{f}\left|\begin{pmatrix} 1 & uv\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}\right|w\right)(x) \\ &= \frac{\chi(\det(x)d)}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})} \sum_{v \in (\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{j}))^{\times}} \chi_{\mathfrak{p}}(uv)\left(\mathbf{f}\left|\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & udv\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}\right.\right)(x) \\ &= \frac{\chi(\det(x))}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})} \sum_{v \in (\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{j}))^{\times}} \chi_{\mathfrak{p}}(udv)\left(\mathbf{f}\left|\begin{pmatrix} 1 & udv\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}\right.\right)(x) \\ &= \mathbf{f}|R_{\chi}(x).\end{aligned}$$

Therefore we obtain our lemma. □

Let $\mathbf{f}$ be an element of $S_k(\mathfrak{c},\psi)$. For an ideal $\mathfrak{q}$ $((\mathfrak{q},\mathfrak{c}/\mathfrak{q})=1)$, take a Hecke character χ such that the conductor divides $\mathfrak{q}$ and $\chi_{\mathfrak{q}}=\psi_{\mathfrak{q}}$ on $(\mathfrak{o}_F/\mathfrak{q})^{\times}$. Then we put

$$(\mathbf{f}|\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})})(x)=\overline{\chi}(\det(x))(\mathbf{f}|b(\mathfrak{q}))(x),$$

where

$$b(\mathfrak{q})_v=\begin{cases}\begin{pmatrix}0 & -1\\ \pi_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{q}} & 0\end{pmatrix}, & \text{if } v=\mathfrak{p},\ \mathfrak{p}|\mathfrak{q},\ \pi_{\mathfrak{p}} \text{ is a prime element of } \mathfrak{p},\\[2ex] \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, & \text{otherwise.}\end{cases}$$

We note that $\mathbf{f}|\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})}$ is independent of a choice of $\pi_{\mathfrak{p}}$. However it may depend on δ. So, we sometimes use an operator $W_{\mathfrak{q},\chi}^{(\mathfrak{c})}$ putting $\mathbf{f}|W_{\mathfrak{q},\chi}^{(\mathfrak{c})}=\chi(\delta_{\mathfrak{q}})\mathbf{f}|\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})}$. To simplify notations, if we write $W_{\mathfrak{a},\chi}^{(\mathfrak{c})}$, we understand that $\mathfrak{a}|\mathfrak{c}$ and $W_{\mathfrak{a},\chi}^{(\mathfrak{c})}=W_{\mathfrak{q},\chi}^{(\mathfrak{c})}$ with $\mathfrak{q}=\prod_{\mathfrak{p}|\mathfrak{a}}\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{c}}$ and a suitable Hecke character χ. Moreover we omit $(\mathfrak{c})$ depending on the context.

Remark 2.3 Let χ be a Hecke character of the conductor $\mathfrak{o}_F$. Then

$$(\mathbf{f}|\eta_{\mathfrak{o}_F,\chi}^{(\mathfrak{c})})(x)=(\mathbf{f}|R_{\overline{\chi}})(x)=\overline{\chi}(\det(x))\mathbf{f}(x).$$

Lemma 2.4 *Let $\mathfrak{q}$ be an integral ideal such that $\mathfrak{q}|\mathfrak{c}$ and $(\mathfrak{q},\mathfrak{c}\mathfrak{q}^{-1})=1$. For an integral ideal $\mathfrak{m}$ such that $(\mathfrak{m},\mathfrak{q})=1$, we have the following diagram:*

$$\begin{array}{ccc} S_k(\mathfrak{c},\psi) & \xrightarrow{\overline{\chi}^*(\mathfrak{m})T_{\mathfrak{c}}'(\mathfrak{m})} & S_k(\mathfrak{c},\psi)\\ {\scriptstyle\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})}}\downarrow & & \downarrow{\scriptstyle\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})}}\\ S_k(\mathfrak{c},\psi\overline{\chi}^2) & \xrightarrow{T_{\mathfrak{c}}'(\mathfrak{m})} & S_k(\mathfrak{c},\psi\overline{\chi}^2). \end{array} \tag{2.4}$$

This is easily verified. We can prove the following property by a similar argument as in the proof of [3], Theorem 4.6.16.

Proposition 2.5 *Let $\mathbf{f}$ be an element of $S_k(\mathfrak{c},\psi)$.*

(1) $\mathbf{f}|\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})}|\eta_{\mathfrak{q},\overline{\chi}}^{(\mathfrak{c})}=\psi_{\mathfrak{q}}(-1)(\psi\overline{\chi})^*(\mathfrak{q})\mathbf{f}$.

(2) $\mathbf{f}|\eta_{\mathfrak{q}_1,\chi_1}^{(\mathfrak{c})}|\eta_{\mathfrak{q}_2,\chi_2}^{(\mathfrak{c})}=(\overline{\chi_1})^*(\mathfrak{q}_2)\mathbf{f}|\eta_{\mathfrak{q}_1\mathfrak{q}_2,\chi_1\chi_2}^{(\mathfrak{c})}$ *if* $(\mathfrak{q}_1,\mathfrak{q}_2)=1$.

(3) *By* $\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})}$, *we have the isomorphisms*:

$$S_k^0(\mathfrak{c},\psi)\simeq S_k^0(\mathfrak{c},\psi\overline{\chi}^2),\quad S_k^1(\mathfrak{c},\psi)\simeq S_k^1(\mathfrak{c},\psi\overline{\chi}^2).$$

(4) *Let* $\mathbf{f}$ *be a primitive form of* $S_k^0(\mathfrak{c},\psi)$ *and put* $\mathbf{f}|\eta_{\mathfrak{q},\chi}^{(\mathfrak{c})}=A\mathbf{g}$ *with* $C(\mathfrak{o}_F,\mathbf{g})=1$. *Then* $\mathbf{g}$ *is a primitive form of* $S_k^0(\mathfrak{c},\psi\overline{\chi}^2)$ *and*

$$C(\mathfrak{p},\mathbf{g})=\begin{cases}\overline{\chi}^*(\mathfrak{p})C(\mathfrak{p},\mathbf{f}) & \text{if } \mathfrak{p}\not|\,\mathfrak{q},\\ (\psi\overline{\chi})^*(\mathfrak{p})\overline{C(\mathfrak{p},\mathbf{f})} & \text{if } \mathfrak{p}\,|\,\mathfrak{q}.\end{cases}\tag{2.5}$$

Theorem 2.6 *Let* $\mathbf{f}$ *be a primitive form of* $S_k^0(\mathfrak{c},\psi)$, *and* $\mathfrak{c}_0$ *the conductor of* ψ. *For a prime ideal* $\mathfrak{p}$ *dividing* $\mathfrak{c}$, *we have*

(1) *If* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})=\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)$, *then* $|C(\mathfrak{p},\mathbf{f})|=N(\mathfrak{p})^{(k_0-1)/2}$.

(2) *If* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})=1$ *and* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)=0$, *then* $C(\mathfrak{p},\mathbf{f})^2=\psi^*(\mathfrak{p})N(\mathfrak{p})^{k_0-2}$.

(3) *Otherwise, namely, if* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})\geq 2$ *and* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})\neq\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)$, *then* $C(\mathfrak{p},\mathbf{f})=0$.

Proof. Suppose that $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})=1$ and $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)=0$ and χ is a trivial character. We see $\mathbf{f}|\eta_{\mathfrak{p},\chi}^{(\mathfrak{c})}=\mathbf{f}|b(\mathfrak{p})$. Putting $\mathbf{g}=\mathbf{f}|T_{\mathfrak{c}}'(\mathfrak{p})+N(\mathfrak{p})^{(k_0/2)-1}\mathbf{f}|b(\mathfrak{p})$, one can see $\mathbf{g}\in S_k(\mathfrak{c}\mathfrak{p}^{-1},\psi)$. Since $\mathbf{f}$ is a primitive form, we can write $\mathbf{g}=C(\mathfrak{p},\mathbf{f})\mathbf{f}+N(\mathfrak{p})^{(k_0/2)-1}A\mathbf{f}$. It follows that $\mathbf{g}=0$ and $C(\mathfrak{p},\mathbf{f})+N(\mathfrak{p})^{(k_0/2)-1}A=0$. We obtain the result since $A^2=\psi^*(\mathfrak{p})$ by the previous proposition. For other cases, we can also prove by a similar argument as in the proof of [3], Theorem 4.6.17. □

In the case that ψ is of finite order, similar results are given in [1] and [2]. We can obtain the following corollary by a similar argument as in the proof of [3], Corollary 4.6.18.

Corollary 2.7 *Under the same notation and assumptions of above, we have*

(1) *If* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})=\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)$, *then* $\mathbf{f}|W_{\mathfrak{p},\chi}=A\mathbf{g}$ *where*

$$A=\chi_{\mathfrak{p}}(-1)(\psi\overline{\chi})^*(\mathfrak{p}^e)N(\mathfrak{p}^e)^{(k_0-2)/2}\mathfrak{g}_{\mathfrak{p}}(\chi)C(\mathfrak{p},\mathbf{f})^{-e},$$

with a primitive form $\mathbf{g}$ *of* $S_k^0(\mathfrak{c},\psi\overline{\chi}^2)$ *and* $e=\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})$.

(2) *If* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c})=1$ *and* $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)=0$, *then* $\mathbf{f}|W_{\mathfrak{p},1}=A\mathbf{f}$ *where*

$$A=-N(\mathfrak{p})^{1-k_0/2}C(\mathfrak{p},\mathbf{f}).$$

We note that $\mathbf{f}|W_{\mathfrak{p},1}=A\mathbf{f}$ implies $C(\mathfrak{p},\mathbf{f})=\psi^*(\mathfrak{p})\overline{C(\mathfrak{p},\mathbf{f})}$ by Proposition 2.5. We conclude this section by stating kinds of multiplicity one theorem. They are proved by similar arguments as in the proof of [3], Corollary 4.6.20 and Corollary 4.6.22. See also [4].

Theorem 2.8 *Let* $\mathbf{f}$ *be a nonzero element of* $S_k(\mathfrak{c},\psi)$. *If* $\mathbf{f}|T'_{\mathfrak{c}}(\mathfrak{n}) = C(\mathfrak{n})\mathbf{f}$ *for all integral ideal* $\mathfrak{n}$ *prime to* $\mathfrak{c}$, *then there uniquely exist a divisor* $\mathfrak{a}$ *of* $\mathfrak{c}$ *and a primitive form* $\mathbf{g} \in S_k^0(\mathfrak{a},\psi)$ *such that* $\mathbf{g}|T'_{\mathfrak{a}}(\mathfrak{n}) = C(\mathfrak{n})\mathbf{g}$ *for all* $\mathfrak{n}$ *prime to* $\mathfrak{c}$. *Moreover* $\mathbf{f} \in \langle\, \mathbf{g}|B_{\mathfrak{q}} \mid \mathfrak{q}\mathfrak{a}|\mathfrak{c}\,\rangle$.

Theorem 2.9 *Let* $\mathbf{f}$ *be an element of* $S_k(\mathfrak{c},\psi)$ *with* $C(\mathfrak{o}_F,\mathbf{f}) = 1$, *and put* $\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{c},\psi} = c\cdot\mathbf{g}$, $C(\mathfrak{o}_F,\mathbf{g}) = 1$. *Then* $\mathbf{f}$ *is a primitive form of* $S_k^0(\mathfrak{c},\psi)$ *if and only if* $L(s;\mathbf{f})$ *and* $L(s;\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{c},\psi})$ *have the following Euler products*:

$$L(s;\mathbf{f}) = \prod_{\mathfrak{p}}(1 - C(\mathfrak{p},\mathbf{f})N(\mathfrak{p})^{-s} + \psi^{(\mathfrak{c})}(\mathfrak{p})N(\mathfrak{p})^{k_0-1-2s})^{-1},$$

$$L(s;\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{c},\psi}) = c\prod_{\mathfrak{p}}(1 - C(\mathfrak{p},\mathbf{g})N(\mathfrak{p})^{-s} + \overline{\psi}^{(\mathfrak{c})}(\mathfrak{p})N(\mathfrak{p})^{k_0-1-2s})^{-1},$$

where $\mathfrak{p}$ *runs over all prime ideals of* F.

Here $L(s;\mathbf{f}_1) = \sum_{\mathfrak{n}} C(\mathfrak{n},\mathbf{f}_1)N(\mathfrak{n})^{-s}$ for a form $\mathbf{f}_1$.

3. The twisted newform

Let $\mathbf{f}$ be a primitive form of $S_k^0(\mathfrak{c},\psi)$. We know that a twisted form $\mathbf{f}|R_\chi$ belongs to $S_k(\mathfrak{a},\psi\chi^2)$ where $\mathfrak{a}$ is an integral ideal as in Lemma 2.2. In this section, we consider whether it is primitive or not.

Theorem 3.1 *Let* χ *be a Hecke character of the conductor* $\mathfrak{o}_F$ *and* $\mathbf{f}$ *a primitive form of* $S_k^0(\mathfrak{c},\psi)$. *Then* $\mathbf{f}|R_\chi$ *is a primitive form of* $S_k^0(\mathfrak{c},\psi\chi^2)$.

Proof. Put $\mathbf{g} = \mathbf{f}|R_\chi$. Since

$$\mathbf{g}(x) = \chi(\det(x))\mathbf{f}(x), \quad \mathbf{g}|\eta^{(\mathfrak{c})}_{\mathfrak{c},\psi\chi^2} = \chi^*(\mathfrak{c})\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{c},\psi}|R_\chi.$$

It follows from Theorem 2.9 that $\mathbf{g}$ is primitive. □

Theorem 3.2 *Let* χ *be a Hecke character of the conductor* $\mathfrak{p}^\alpha$, $\mathbf{f}$ *a primitive form of* $S_k^0(\mathfrak{c},\psi)$ *and* $\mathfrak{c}_0$ *(resp.* $\mathfrak{b}$*) the conductor of* ψ *(resp.* $\psi\chi$*). Suppose* $e = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)$ *and put* $t = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b})$. *If* $t = 0$ *except for* $\alpha = 0$, *then* $\mathbf{f}|R_\chi = A^{-1}\{(\mathbf{f}|W_{\mathfrak{p},\overline{\chi}}) - (\mathbf{f}|W_{\mathfrak{p},\overline{\chi}}|T(p)|B_{\mathfrak{p}})\}$ *with* A *as in Corollary* 2.7.

If $t \neq 0$, then $\mathbf{f}|R_\chi$ is a primitive form of $S_k^0(\mathfrak{a}, \psi\chi^2)$, where $\mathfrak{a} = \mathfrak{c}\mathfrak{p}^{t+\alpha-e}$ (which is the same level as in Lemma 2.2). Moreover if $\alpha \geq e$, then

$$\begin{aligned}&\mathbf{f}|R_\chi|W_{\mathfrak{p},\varphi}\\ &= (\overline{\varphi}\chi)_{\mathfrak{p}}(-1)(\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^t)\frac{\mathfrak{g}_{\mathfrak{p}}(\varphi\overline{\chi})}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})}\cdot\frac{C(\mathfrak{p}^{\alpha-t},\mathbf{f})}{N(\mathfrak{p}^{\alpha-t})^{(k_0/2)-1}}\mathbf{f}|R_{\overline{\varphi}\chi},\end{aligned} \tag{3.1}$$

and if $\alpha < e$, then

$$\begin{aligned}&\mathbf{f}|R_\chi|W_{\mathfrak{p},\varphi}\\ &= (\overline{\varphi}\chi)_{\mathfrak{p}}(-1)(\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^e)\frac{\mathfrak{g}_{\mathfrak{p}}(\varphi\overline{\chi})}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})}\cdot\frac{\overline{C(\mathfrak{p}^{e-\alpha},\mathbf{f})}}{N(\mathfrak{p}^{e-\alpha})^{k_0/2}}\mathbf{f}|R_{\overline{\varphi}\chi}.\end{aligned} \tag{3.2}$$

Proof. The first assersion is proved by Lemma 2.2, Proposition 2.5 and Corollary 2.7. We shall prove (3.1). We fix a complete set A (resp. B, C) of representatives of $(\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^\alpha))^\times$ (resp. $(\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^t))^\times$, $\mathfrak{o}_{\mathfrak{p}}/(\pi_{\mathfrak{p}}^{\alpha-t})$). For each $v \in A$, take $(v', m) \in B \times C$ such that $1 + (v' + m\pi_{\mathfrak{p}}^t)v \equiv 0 \bmod (\pi_{\mathfrak{p}}^\alpha)$. Then $v \mapsto (v', m)$ gives a bijection of A onto $B \times C$. Putting $u = (\pi_{\mathfrak{p}}^\alpha\delta_{\mathfrak{p}})^{-1}$ and $u' = (\pi_{\mathfrak{p}}^t\delta_{\mathfrak{p}})^{-1}$, we have

$$\begin{aligned}&\begin{pmatrix}1 & uv\delta_{\mathfrak{p}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ \pi_{\mathfrak{p}}^{t+\alpha} & 0\end{pmatrix}\left\{\begin{pmatrix}\pi_{\mathfrak{p}}^t & 0\\ 0 & \pi_{\mathfrak{p}}^t\end{pmatrix}\begin{pmatrix}1 & m\\ 0 & \pi_{\mathfrak{p}}^{\alpha-t}\end{pmatrix}\begin{pmatrix}1 & u'v'\delta_{\mathfrak{p}}\\ 0 & 1\end{pmatrix}\right\}^{-1}\\ &\qquad = \begin{pmatrix}v & -\{1+(v'+m\pi_{\mathfrak{p}}^t)v\}/\pi_{\mathfrak{p}}^\alpha\\ \pi_{\mathfrak{p}}^\alpha & -(v'+m\pi_{\mathfrak{p}}^t)\end{pmatrix},\end{aligned}$$

which is in $\Gamma_{\mathfrak{p}}(\mathfrak{c})$. Therefore

$$\begin{aligned}&\mathbf{f}|R_\chi|W_{\mathfrak{p},\varphi}(x)\\ &= \varphi(\delta_{\mathfrak{p}})\overline{\varphi}(\det(x))\frac{\chi(\det(x)\pi_{\mathfrak{p}}^{t+\alpha})}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})}\\ &\quad\times\sum_{v\in A}\chi_{\mathfrak{p}}(uv)\left(\mathbf{f}|\begin{pmatrix}1 & uv\delta_{\mathfrak{p}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ \pi_{\mathfrak{p}}^{t+\alpha} & 0\end{pmatrix}\right)(x)\\ &= \varphi(\delta_{\mathfrak{p}})\overline{\varphi}(\det(x))\frac{\chi(\det(x)\pi_{\mathfrak{p}}^{t+\alpha})}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})}\\ &\quad\times\sum_{v\in A}\chi_{\mathfrak{p}}(uv)\left(\psi_{\mathfrak{p}}(v\pi_{\mathfrak{p}}^t)\mathbf{f}|\begin{pmatrix}1 & m\\ 0 & \pi_{\mathfrak{p}}^{\alpha-t}\end{pmatrix}\begin{pmatrix}1 & u'v'\delta_{\mathfrak{p}}\\ 0 & 1\end{pmatrix}\right)(x)\end{aligned}$$

$$= \frac{\overline{\varphi}\chi(\det(x))}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})}$$

$$\times \sum_{v \in A} \varphi(\delta_{\mathfrak{p}})\chi(\pi_{\mathfrak{p}}^{t+\alpha})\chi_{\mathfrak{p}}(uv)\psi_{\mathfrak{p}}(v)\psi(\pi_{\mathfrak{p}}^{t}) \left(\mathbf{f}| \begin{pmatrix} 1 & m \\ 0 & \pi_{\mathfrak{p}}^{\alpha-t} \end{pmatrix} \begin{pmatrix} 1 & u'v'\delta_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}\right)(x).$$

Here we have

$$\begin{aligned} &\varphi(\delta_{\mathfrak{p}})\chi(\pi_{\mathfrak{p}}^{t+\alpha})\chi_{\mathfrak{p}}(uv)\psi_{\mathfrak{p}}(v)\psi(\pi_{\mathfrak{p}}^{t})(\varphi\overline{\chi})_{\mathfrak{p}}(u'v') \\ &\quad = (\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^t)(\varphi\overline{\chi})_{\mathfrak{p}}(u'v'\pi_{\mathfrak{p}}^{t}\delta_{\mathfrak{p}})(\chi\psi)_{\mathfrak{p}}(uv\pi_{\mathfrak{p}}^{\alpha}\delta_{\mathfrak{p}}) \\ &\quad = (\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^t)(\varphi\overline{\chi})_{\mathfrak{p}}(v')(\chi\psi)_{\mathfrak{p}}(v) \\ &\quad = (\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^t)(\varphi\overline{\chi})_{\mathfrak{p}}(v')\overline{(\chi\psi)}_{\mathfrak{p}}(-(v'+m\pi_{\mathfrak{p}}^{t})) \\ &\quad = (\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^t)(\varphi\overline{\chi})_{\mathfrak{p}}(v')\overline{(\chi\psi)}_{\mathfrak{p}}(-v') \\ &\quad = (\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^t)(\varphi\overline{\chi})_{\mathfrak{p}}(-1)(\overline{\psi\chi^2}\varphi)_{\mathfrak{p}}(-v') \\ &\quad = (\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^t)(\varphi\overline{\chi})_{\mathfrak{p}}(-1) \end{aligned}$$

and

$$\begin{aligned} \sum_{m \in C} \mathbf{f}| \begin{pmatrix} 1 & m \\ 0 & \pi_{\mathfrak{p}}^{\alpha-t} \end{pmatrix} &= \frac{1}{N(\mathfrak{p}^{\alpha-t})^{(k_0/2)-1}} \mathbf{f}|T'_{\mathfrak{c}}(\mathfrak{p}^{\alpha-t}) \\ &= \frac{C(\mathfrak{p}^{\alpha-t}, \mathbf{f})}{N(\mathfrak{p}^{\alpha-t})^{(k_0/2)-1}} \mathbf{f}. \end{aligned}$$

So, we obtain (3.1). It follows from Proposition 2.5 and Theorem 2.9 that $\mathbf{f}|R_\chi$ is a primitive form of $S_k^0(\mathfrak{a}, \psi\chi^2)$ since $\mathbf{f}|R_{\overline{\varphi}\chi}|W_{\mathfrak{c}\mathfrak{p}^{-e}, \psi\chi^2\overline{\varphi}} = C \cdot \mathbf{f}|W_{\mathfrak{c}\mathfrak{p}^{-e}, \psi\chi^2\overline{\varphi}}|R_{\overline{\varphi}\chi}$ for some constant C. For another case, we can prove by a similar argument. □

Since the proof of (3.1) works even if $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}) > \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)$, we have also

Theorem 3.3 *Let χ be a Hecke character of the conductor $\mathfrak{p}^\alpha$, $\mathbf{f}$ a primitive form of $S_k^0(\mathfrak{c}, \psi)$ and $\mathfrak{c}_0$ the conductor of ψ. Suppose $e = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}) > \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)$. If $\alpha \geq e$, then $\mathbf{f}|R_\chi$ is a primitive form of $S_k^0(\mathfrak{a}, \psi\chi^2)$, where $\mathfrak{a} = \mathfrak{c}\mathfrak{p}^{2\alpha - e}$. Moreover*

$$\mathbf{f}|R_\chi|W_{\mathfrak{p},\varphi} = (\overline{\varphi}\chi)_{\mathfrak{p}}(-1)(\psi\chi^2\overline{\varphi})^*(\mathfrak{p}^\alpha)\frac{\mathfrak{g}_{\mathfrak{p}}(\varphi\overline{\chi})}{\mathfrak{g}_{\mathfrak{p}}(\overline{\chi})}\mathbf{f}|R_{\overline{\varphi}\chi}. \tag{3.3}$$

Theorem 3.4 *Let χ be a Hecke character of the conductor $\mathfrak{p}^\alpha$, $\mathbf{f}$ a primitive form of $S_k^0(\mathfrak{c}, \psi)$ and $\mathfrak{c}_0$ (resp. $\mathfrak{b}$) the conductor of ψ (resp. $\psi\chi$). Suppose $e = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}) > \mathrm{ord}_{\mathfrak{p}}(\mathfrak{c}_0)$ and put $t = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b})$. If $e \geq 2$, $t + \alpha < e$, $\mathbf{f}|R_\chi$*

is a primitive form of $S_k^0(\mathfrak{c}, \psi\chi^2)$.

Proof. Our assumption implies that $\mathbf{f}|R_\chi \in S_k(\mathfrak{c}, \psi\chi^2)$ by Lemma 2.2 and $C(\mathfrak{p}, \mathbf{f}) = 0$ by Theorem 2.6. If $t = 0$, by Lemma 2.2 and Proposition 2.5, we can write $\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\overline{\chi}} = A\mathbf{f}|R_\chi$ and $\mathbf{f}|R_\chi$ is primitive. Now we suppose $t \neq 0$. If $\mathbf{f}|R_\chi$ is not primitive, by Theorem 2.8, we can write $\mathbf{f}|R_\chi = \mathbf{g}_1 + \mathbf{g}_2|B_\mathfrak{p}$ with $\mathbf{g}_1, \mathbf{g}_2 \in S_k(\mathfrak{c}\mathfrak{p}^{-1}, \psi\chi^2)$. Thus $\mathbf{f}|R_\chi|R_{\overline{\chi}} = \mathbf{g}_1|R_{\overline{\chi}}$. However $\mathbf{f} = \mathbf{f}|R_\chi|R_{\overline{\chi}}$ and $\mathbf{g}_1|R_{\overline{\chi}} \in S_k(\mathfrak{c}\mathfrak{p}^{-1}, \psi)$ by Lemma 2.2 and our assumption. This is contradiction. □

4. Proof of main theorem

Let $\mathbf{f}$ be a primitive form of $S_k^0(\mathfrak{c}, \psi)$ and $\mathfrak{c}_0$ the conductor of ψ. We fix a prime ideal $\mathfrak{p}$ of F and put $e = \mathrm{ord}_\mathfrak{p}(\mathfrak{c})$ and $n = \mathrm{ord}_\mathfrak{p}(\mathfrak{c}_0)$. We fix a Hecke character φ such that the conductor divide $\mathfrak{p}^e$ (or $\mathfrak{p}^n$) and $\varphi_\mathfrak{p} = \psi_\mathfrak{p}$ on $(\mathfrak{o}_F/\mathfrak{p}^e)^\times$ (or $(\mathfrak{o}_F/\mathfrak{p}^n)^\times$). We assume $e \geq 2$, $e > n$.

We say that a Hecke character χ satisfies the condition (V) if χ satisfies that $t + \alpha < e$ where $\mathfrak{p}^\alpha$ (resp. $\mathfrak{p}^t$) is the conductor of χ (resp. $\varphi\chi$). For such χ, we put

$$\mathbf{f}|V_\chi = \mathbf{f}|R_\chi|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi\chi^2}|R_{\varphi\chi}.$$

This is well-defined and $\mathbf{f}|V_\chi = A_\chi\mathbf{f}$ for some constant A_χ by Theorem 3.4, Lemma 2.2 and Proposition 2.5. Especially for A_{id}, we have $\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi} = A_{id}\mathbf{f}|R_{\overline{\varphi}}$. We note that Corollary 2.7 tells nothing about A_{id}.

Lemma 4.1 *Under the above notation, let* χ_1 *(resp.* χ_2*) be a Hecke character of the conductor* $\mathfrak{p}^i$ *(resp.* $\mathfrak{p}^j$*) and* p^l *the conductor of* $\chi_1\chi_2\varphi$*. If* $e - i - l \geq \max\{i, l, n\}$*, then we have*

$$\mathbf{f}|V_{\chi_1}|V_{\chi_2} = \mathbf{f}|V_{id}|V_{\chi_1\chi_2}.$$

Proof. For $u = i$, $\max\{i, n\}$, $\max\{i, l, n\}$, $\max\{i, l\}$, $\max\{l, n\}$, and l, $\mathfrak{p}^u$ is divisible by the conductor of χ_1, $\chi_1\varphi$, χ_2, $\chi_2\varphi$, $\chi_1\chi_2$, and $\chi_1\chi_2\varphi$ respectively. Therefore χ_1, χ_2 and $\chi_1\chi_2$ satisfy the condition (V). We shall show

$$\mathbf{f}|R_{\chi_1}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi\chi_1^2}|R_{\varphi\chi_1\chi_2}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi\chi_2^2} = \mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi}|R_{\varphi\chi_1\chi_2}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi(\chi_1\chi_2)^2}|R_{\chi_1}.$$

We put $u = (\pi_\mathfrak{p}^l\delta_\mathfrak{p})^{-1}$ and $u' = (\pi_\mathfrak{p}^i\delta_\mathfrak{p})^{-1}$ and fix a complete set A (resp. A')

of all representatives of $(\mathfrak{o}_\mathfrak{p}/(\pi_\mathfrak{p}^l))^\times$ (resp. $(\mathfrak{o}_\mathfrak{p}/(\pi_\mathfrak{p}^i))^\times$). We have

$$\begin{aligned}
&\mathbf{f}|R_{\chi_1}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi\chi_1^2}|R_{\varphi\chi_1\chi_2}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi\chi_2^2}(x)\\
&\quad=\frac{\overline{\varphi\chi_2}(\det(x))\chi_1\chi_2(\pi_\mathfrak{p}^e)}{\mathfrak{g}_\mathfrak{p}(\overline{\varphi\chi_1\chi_2})\mathfrak{g}_\mathfrak{p}(\overline{\chi_1})}\\
&\qquad\times\sum_{\substack{v\in A\\ v'\in A'}}(\varphi\chi_1\chi_2)_\mathfrak{p}(uv)(\chi_1)_\mathfrak{p}(u'v')(\mathbf{f}|a'(v')b(\mathfrak{p}^e)a(v)b(\mathfrak{p}^e))(x)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi}|R_{\varphi\chi_1\chi_2}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi(\chi_1\chi_2)^2}|R_{\chi_1}(x)\\
&\quad=\frac{\overline{\varphi\chi_2}(\det(x))\chi_1\chi_2(\pi_\mathfrak{p}^e)}{\mathfrak{g}_\mathfrak{p}(\overline{\varphi\chi_1\chi_2})\mathfrak{g}_\mathfrak{p}(\overline{\chi_1})}\\
&\qquad\times\sum_{\substack{v\in A\\ v'\in A'}}(\varphi\chi_1\chi_2)_\mathfrak{p}(uv)(\chi_1)_\mathfrak{p}(u'v')(\mathbf{f}|b(\mathfrak{p}^e)a(v)b(\mathfrak{p}^e)a'(v'))(x)
\end{aligned}$$

where

$$a(v)=\begin{pmatrix}1 & uv\delta_\mathfrak{p}\\ 0 & 1\end{pmatrix},\quad a'(v')=\begin{pmatrix}1 & u'v'\delta_\mathfrak{p}\\ 0 & 1\end{pmatrix}.$$

We see that

$$\begin{aligned}
&a'(v')b(\mathfrak{p}^e)a(v)b(\mathfrak{p}^e)(b(\mathfrak{p}^e)a(v)b(\mathfrak{p}^e)a'(v'))^{-1}\\
&\qquad=\begin{pmatrix}(1-vv'\pi_\mathfrak{p}^{e-i-l})^2+vv'\pi_\mathfrak{p}^{e-i-l} & vv'^2\pi_\mathfrak{p}^{e-2i-l}\\ v^2v'\pi_\mathfrak{p}^{2e-i-2l} & 1+vv'\pi_\mathfrak{p}^{e-i-l}\end{pmatrix},
\end{aligned}$$

which is in $\Gamma_\mathfrak{p}(\mathfrak{c})$. Since $(1-vv'\pi_\mathfrak{p}^{e-i-l})^2+vv'\pi_\mathfrak{p}^{e-i-l}\equiv 1 \bmod (\pi_\mathfrak{p}^n)$,

$$\mathbf{f}|a'(v')b(\mathfrak{p}^e)a(v)b(\mathfrak{p}^e)=\mathbf{f}|b(\mathfrak{p}^e)a(v)b(\mathfrak{p}^e)a'(v').$$

It follows that

$$\mathbf{f}|R_{\chi_1}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi\chi_1^2}|R_{\varphi\chi_1\chi_2}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi\chi_2^2}=\mathbf{f}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi}|R_{\varphi\chi_1\chi_2}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi(\chi_1\chi_2)^2}|R_{\chi_1}.$$

□

Lemma 4.2 *For Hecke characters χ_1 and χ_2, we suppose that χ_1 satisfies the condition (V) and the conductor of χ_2 is $\mathfrak{o}_F$. Then we have for an* *element* $\mathbf{g}$ *of* $S_k^0(\mathfrak{c},\psi)$,

$$\mathbf{g}|V_{\chi_1\chi_2} = {\chi_2}^*(\mathfrak{p}^e)\mathbf{g}|V_{\chi_1}.$$

Proof. We see that

$$\begin{aligned}\mathbf{g}|V_{\chi_1\chi_2} &= \mathbf{g}|R_{\chi_1}|R_{\chi_2}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi{\chi_1}^2{\chi_2}^2}|R_{\chi_2}|R_{\varphi\chi}\\ &= {\chi_2}^*(\mathfrak{p}^e)\mathbf{g}|R_{\chi_1}|\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi{\chi_1}^2\chi_2}|R_{\chi_2}|R_{\varphi\chi}\\ &= {\chi_2}^*(\mathfrak{p}^e)\mathbf{g}|V_{\chi_1}.\end{aligned}$$

by Remark 2.3 and Proposition 2.5. □

Now we prove Theorem 1.1 in more general case.

Theorem 4.3 *Let* $\mathbf{f}$ *be a primitive form of* $S_k^0(\mathfrak{c},\psi)$. *If* $3n \le e$, $(A_{id})^{2\alpha} = (\psi^\alpha)^*(\mathfrak{p}^e)$, *where* α *is an order of* $\varphi_\mathfrak{p}$ *as the character of* $(\mathfrak{o}_F/\mathfrak{p}^n)^\times$.

Proof. We see by above lemma,

$$\mathbf{f}|(V_{\overline{\varphi}})^\alpha = \mathbf{f}|(V_{id})^{\alpha-1}|V_{\overline{\varphi}^\alpha} = (\overline{\varphi}^\alpha)^*(\mathfrak{p}^e)\mathbf{f}|(V_{id})^\alpha.$$

Hence

$$\begin{aligned}\mathbf{f}|(V_{id})^{2\alpha} &= (\varphi^\alpha)^*(\mathfrak{p}^e)\cdot\mathbf{f}|(V_{\overline{\varphi}})^\alpha|(V_{id})^\alpha = (\varphi^\alpha)^*(\mathfrak{p}^e)\cdot\mathbf{f}|(V_{id}V_{\overline{\varphi}})^\alpha\\ &= (\varphi^\alpha)^*(\mathfrak{p}^e)\cdot\mathbf{f}|(\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\varphi}\eta^{(\mathfrak{c})}_{\mathfrak{p}^e,\overline{\varphi}})^\alpha = (\varphi^\alpha)^*(\mathfrak{p}^e)\cdot\{\psi_\mathfrak{p}(-1)(\psi\overline{\varphi})^*(\mathfrak{p}^e)\}^\alpha\mathbf{f}\\ &= (\psi^\alpha)^*(\mathfrak{p}^e)\mathbf{f}\end{aligned}$$

by Proposition 2.5. Therefore we obtain the result. □

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