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# On the growth of solutions of $w^{(n)} + e^{-z}w' + Q(z)w = 0$ and some related extensions

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**Abstract.** In this paper, we show that if $Q(z)$ is a nonconstant polynomial, then every solution $w \not\equiv 0$ of the differential equation $w^{(n)} + e^{-z}w' + Q(z)w = 0$, has infinite order and we give an extension of this result. We will also show that if the equation $w^{(n)} + e^{-z}w' + cw = 0$, where $c \neq 0$ is a complex constant, possesses a solution $w \not\equiv 0$ of finite order, then $c = -k^n$ where $k$ is a positive integer. In the end, by study more general, we investigate the problem when $\sigma(Q) = 1$.

*Key words*: linear differential equations, entire functions, finite order of growth.

## 1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [10]). Let $\sigma(w)$ denote the order of an entire function $w$, that is,

$$\sigma(w) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,w)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,w)}{\log r}, \tag{1.1}$$

where $T(r,w)$ is the Nevanlinna characteristic function of $w$ (see [10]), and $M(r,w) = \max_{|z|=r} |w(z)|$.

Several authors have studied the particular differential equation

$$w'' + e^{-z}w' + B(z)w = 0, \tag{1.2}$$

where $B(z)$ is an entire function. For $B(z) \equiv c$ where $c$ is a nonzero constant, Frei [6] showed that if equation (1.2) possesses a solution $w \not\equiv 0$ of finite order, then $c = -k^2$ where $k$ is a positive integer. Conversely, for each positive integer $k$, the equation (1.2), with $B(z) \equiv c = -k^2$, possesses a solution $w$ which is a polynomial in $e^z$ of degree $k$. Other proofs of this result were given by Ozawa [14] and Wittich [15]. By completing results of Ozawa [14], Amemiya-Ozawa [1] and Gundersen [7], Langley showed in [12]

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that if $B(z)$ is nonconstant polynomial, then every solution $w \not\equiv 0$ of (1.2) has infinite order.

In this paper, we will extend these results to the differential equation

$$w^{(n)} + e^{-z}w' + B(z)w = 0, \tag{1.3}$$

where $n \geq 2$. In fact, we shall prove the following:

**Theorem 1.1** *If the equation*

$$w^{(n)} + e^{-z}w' + cw = 0, \tag{1.4}$$

*where $c \neq 0$ is a complex constant, possesses a solution $w \not\equiv 0$ of finite order, then $c = -k^n$ where $k$ is a positive integer. Conversely, for each positive integer $k$, the equation* (1.4), *with $c = -k^n$, possesses a solution $w$ which is a polynomial in $e^z$ of degree $k$.*

**Theorem 1.2** *If $Q(z)$ is nonconstant polynomial, then every solution $w \not\equiv 0$ of the differential equation*

$$w^{(n)} + e^{-z}w' + Q(z)w = 0, \tag{1.5}$$

*where $n \geq 2$, has infinite order.*

**Theorem 1.3** *Let $Q$ be nonconstant polynomial, $P_1, \ldots, P_{n-1}$ be polynomials and $\alpha_1, \ldots, \alpha_{n-1}$ be real constants. Suppose that there exists an $s \in \{1, \ldots, n-1\}$ such that $P_s(e^{\alpha_s z}) = e^{\alpha_s z}$ and either:*
(i) $\alpha_s > 0$ *and* $\alpha_k \leq 0$ *for all* $k = 1, \ldots, s-1, s+1, \ldots, n-1$ *or*
(ii) $\alpha_s < 0$ *and* $\alpha_k \geq 0$ *for all* $k = 1, \ldots, s-1, s+1, \ldots, n-1$.
*Then every solution $w \not\equiv 0$ of the differential equation*

$$\begin{aligned} w^{(n)} &+ P_{n-1}(e^{\alpha_{n-1}z})w^{(n-1)} + \cdots + P_s(e^{\alpha_s z})w^{(s)} + \cdots \\ &+ P_1(e^{\alpha_1 z})w' + Q(z)w = 0, \end{aligned} \tag{1.6}$$

*where $n \geq 2$, is of infinite order.*

**Remark 1.1** The following two theorems are natural extensions of [4, Theorem 1] and [4, Theorem 2].

**Theorem 1.4** *Let $P_1(z) = a_m z^m + \cdots$, $P_0(z) = b_m z^m + \cdots$ $(m \geq 1)$ be nonconstant polynomials such that $a_m = cb_m$ $(c > 1)$, and let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be entire functions with $\sigma(A_j) < m$ $(j = 0, 1)$. Then every* *solution $w \not\equiv 0$ of the differential equation*

$$w^{(n)} + A_1(z)e^{P_1(z)}w' + A_0(z)e^{P_0(z)}w = 0, \tag{1.7}$$

*where $n \geq 2$, is of infinite order.*

**Theorem 1.5** *Let $P_1(z) = a_m z^m + \cdots$, $P_0(z) = b_m z^m + \cdots$ $(m \geq 1)$ be nonconstant polynomials such that $a_m b_m \neq 0$ and either $\arg a_m \neq \arg b_m$ or $a_m = cb_m$ $(0 < c < 1)$, and let $A_k(z) \not\equiv 0$ $(k = 0, \ldots, n-1)$, $B_j(z)$ $(j = 0, 1)$ be entire functions such that $\sigma(A_k) < m$ $(k = 0, \ldots, n-1)$, $\sigma(B_j) < m$ $(j = 0, 1)$. Then every solution $w \not\equiv 0$ of the differential equation*

$$\begin{aligned} &w^{(n)} + A_{n-1}(z)w^{(n-1)} + \cdots + A_2(z)w'' \\ &\quad + (A_1(z)e^{P_1(z)} + B_1(z))w' + (A_0(z)e^{P_0(z)} + B_0(z))w = 0, \end{aligned} \tag{1.8}$$

*where $n \geq 2$, is of infinite order.*

**Remark 1.2** Using the same reasoning as in the proof of Theorem 1 [7] we obtain that if $B(z)$ is a transcendental entire function with $\sigma(B) \neq 1$, then every solution $w \not\equiv 0$ of the equation (1.3) has infinite order.

By combining Theorem 1.4 and Theorem 1.5 we get the following result which investigate the case when $\sigma(B) = 1$ in (1.3):

**Corollary 1.1** *If $B(z)$ is an entire function with $B(z) = h(z)e^{az}$ where $a \neq -1$ is a complex constant and $h(z)$ is an entire function with $\sigma(h) < 1$, then every solution $w \not\equiv 0$ of (1.3) is of infinite order.*

## 2. Lemmas for the proofs of theorems

Our proofs depend mainly upon the following Lemmas.

**Lemma 2.1** ([3]) *Suppose that $A_0(z), \ldots, A_{n-1}(z)$ with $A_0(z) \not\equiv 0$ are entire functions such that for real constants $\alpha$, $\beta$, $\theta_1$, $\theta_2$, where $\alpha > 0$, $\beta > 0$, and $\theta_1 < \theta_2$ we have*

$$|A_1(z)| \geq \exp\{(1 + o(1))\alpha|z|^\beta\} \tag{2.1}$$

*and*

$$|A_j(z)| \leq \exp\{o(1)\alpha|z|^\beta\} \quad (j = 0, 2, \ldots, n-1) \tag{2.2}$$

*as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Let $\varepsilon > 0$ be a given small constant, and let $S(\varepsilon)$ denote the set $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$.*

*If $w \not\equiv 0$ is a solution with $\sigma(w) < +\infty$ of the linear differential equation*

$$w^{(n)} + A_{n-1}(z)w^{(n-1)} + \cdots + A_1(z)w' + A_0(z)w = 0, \tag{2.3}$$

*then the following conditions hold:*

(i) *There exists a constant $b \neq 0$ such that $w \to b$ as $z \to \infty$ in $S(\varepsilon)$. Furthermore, as $z \to \infty$ in $S(\varepsilon)$,*

$$|w(z) - b| \leq \exp\{-(1 + o(1))\alpha|z|^{\beta}\}. \tag{2.4}$$

(ii) *For each integer $m \geq 1$, as $z \to \infty$ in $S(\varepsilon)$,*

$$|w^{(m)}(z)| \leq \exp\{-(1 + o(1))\alpha|z|^{\beta}\}. \tag{2.5}$$

**Remark 2.1** It should be noted that formula (2.5) is a special case of Theorem 1 in [9].

By using similar proof as in the proof of Theorem 2.1 in [11], we can obtain the following:

**Lemma 2.2** *Suppose that $A_0(z), \ldots, A_{n-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$ such that for real constants $\alpha$, $\beta$, $\theta_1$, $\theta_2$, $C$ where $\alpha > 0$, $\beta > 0$, $C > 0$, and $\theta_1 < \theta_2$ we have, for some integer $s$, $1 \leq s \leq n-1$,*

$$|A_s(z)| \geq \exp\{\alpha|z|^{\beta}\} \tag{2.6}$$

*and*

$$|A_j(z)| \leq C \tag{2.7}$$

*for all $j = 0, 1, \ldots, s-1, s+1, \ldots, n-1$ as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Given $\varepsilon > 0$ small enough, and let $S(\varepsilon)$ denote the set $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$.*

*If $w \not\equiv 0$ is a transcendental solution of (2.3) with $\sigma(w) < +\infty$, then the following conditions hold:*

(i) *There exists $j \in \{0, \ldots, s-1\}$ and a complex constant $b_j \neq 0$ such that $w^{(j)} \to b_j$ as $z \to \infty$ in $S(\varepsilon)$. Furthermore, as $z \to \infty$ in $S(\varepsilon)$,*

$$\left|w^{(j)}(z) - b_j\right| \leq \exp\{-(\alpha - \rho)|z|^{\beta}\}, \tag{2.8}$$

*where $0 < \rho < \alpha$.*

(ii) *For each integer $m \geq j+1$, as $z \to \infty$ in $S(\varepsilon)$,*

$$\left|w^{(m)}(z)\right| \leq \exp\{-(\alpha - \rho)|z|^{\beta}\}, \tag{2.9}$$

*where $0 < \rho < \alpha$.*

**Lemma 2.3** ([8, p. 89]) *Let $f$ be a transcendental entire function of finite order $\sigma$, let $\Gamma=\{(k_1,j_1),(k_2,j_2),\ldots,(k_m,j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i>j_i\ge 0$ $(i=1,\ldots,m)$, and let $\varepsilon>0$ be a given constant. Then there exists a set $E\subset[0,2\pi)$ that has linear measure zero, such that if $\psi_0\in[0,2\pi)-E$, then there is a constant $R_0=R_0(\psi_0)>1$ such that for all $z$ satisfying* $\arg z=\psi_0$ *and $|z|\ge R_0$, and for all $(k,j)\in\Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.10}$$

**Lemma 2.4** ([4], [11], [9, Lemma 3]) *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z=\theta$. Then there exists an infinite sequence of points $z_n=r_ne^{i\theta}$ $(n=1,2,\ldots)$, where $r_n\to+\infty$, such that $f^{(k)}(z_n)\to\infty$ and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right|\le\frac{1}{(k-j)!}(1+o(1))|z_n|^{k-j}\quad (j=0,\ldots,k-1). \tag{2.11}$$

**Lemma 2.5** ([2]) *Let $A_0(z),\ldots,A_{n-1}(z)$ be entire functions such that for real constants $\alpha$, $\beta$, $\mu$, $\theta_1$, $\theta_2$, where $\mu>0$, $0\le\beta<\alpha$ and $\theta_1<\theta_2$ we have*

$$|A_0(z)|\ge\exp\{\alpha|z|^{\mu}\} \tag{2.12}$$

*and*

$$|A_j(z)|\le\exp\{\beta|z|^{\mu}\}\quad (j=1,2,\ldots,n-1) \tag{2.13}$$

*as $z\to\infty$ in $\theta_1\le$ $\arg z$ $\le\theta_2$. Then every solution $w\not\equiv 0$ of (2.3) is of infinite order.*

**Lemma 2.6** ([4]) *Let $P(z)=a_mz^m+\cdots$, $(a_m=\alpha+i\beta\ne 0)$ be a polynomial with degree $m\ge 1$ and $A(z)$ $(\not\equiv 0)$ be an entire function with $\sigma(A)<m$. Set $f(z)=A(z)e^{P(z)}$, $z=re^{i\theta}$, $\delta(P,\theta)=\alpha\cos m\theta-\beta\sin m\theta$. Then for any given $\varepsilon>0$, there exists a set $H_1\subset[0,2\pi)$ that has linear measure zero, such that for any $\theta\in[0,2\pi)\setminus(H_1\cup H_2)$, where $H_2=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ is a finite set, there is $R>0$ such that for $|z|=r>R$, we have*

(i) *if $\delta(P,\theta)>0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^m\}\le|f(z)|\le\exp\{(1+\varepsilon)\delta(P,\theta)r^m\}, \tag{2.14}$$

(ii) *if* $\delta(P,\theta) < 0$, *then*

$$\exp\big\{(1+\varepsilon)\delta(P,\theta)r^m\big\} \le |f(z)| \le \exp\big\{(1-\varepsilon)\delta(P,\theta)r^m\big\}. \tag{2.15}$$

## 3. Proof of Theorem 1.1

Let $w_1, \dots, w_n$ be $n$ independent solutions of equation (1.4). We mention here that only at most one of solutions $w_1, \dots, w_n$ is of finite order of equation (1.4) (see [5]). We may suppose that $w_1(z)$ is a solution of finite order. Evidently $w_1(z + 2\pi i)$ is a solution of (1.4). Hence

$$w_1(z + 2\pi i) = \sum_{j=1}^{n} \alpha_j w_j(z). \tag{3.1}$$

Since $w_1(z + 2\pi i)$ is of finite order too, then $\alpha_j = 0$ for $j = 2, \dots, n$; and from Lemma 2.1 we have $w_1(z) \to b \neq 0$, $w_1(z + 2\pi i) \to b \neq 0$ as $z \to \infty$ in $S_1(\varepsilon)\colon \pi/2 + \varepsilon \le \arg z \le 3\pi/2 - \varepsilon$ $(\varepsilon > 0)$, which implies that $\alpha_1 = 1$ and $w_1(z + 2\pi i) = w_1(z)$. We deduce that there exist a regular function $f(\zeta)$ in $0 < |\zeta| < \infty$ such that $w_1(z) = f(e^z)$. If $f(\zeta)$ has an essential singularity at $\zeta = 0$, then $w_1(z) = f(e^z)$ does not have a limit as $z \to \infty$ in $S_1(\varepsilon)\colon \pi/2 + \varepsilon \le \arg z \le 3\pi/2 - \varepsilon$ $(\varepsilon > 0)$. Also, if $f(\zeta)$ has a pole at $\zeta = 0$, then $w_1(z) \to \infty$ as $z \to \infty$ in $S_1(\varepsilon)$. Hence $f(\zeta)$ is an entire function $(f(\zeta) = \sum_{k=0}^{+\infty} a_k \zeta^k$ and $w_1(z) = \sum_{k=0}^{+\infty} a_k e^{kz})$. Substituting this into (1.4), we obtain

$$\sum_{k=1}^{+\infty} k^n a_k e^{kz} + e^{-z} \sum_{k=1}^{+\infty} k a_k e^{kz} + c \sum_{k=0}^{+\infty} a_k e^{kz} = 0. \tag{3.2}$$

This gives

$$(k^n + c)a_k + (k+1)a_{k+1} = 0 \qquad (k \ge 1) \tag{3.3}$$

and

$$a_1 + c a_0 = 0. \tag{3.4}$$

We have $a_0 = b \neq 0$. If $c \neq -k^n$ for all $k \ge 1$, then

$$\lim_{k \to +\infty} \frac{|a_k|}{|a_{k+1}|} = 0, \tag{3.5}$$

which shows that the radius of convergence of $\sum_{k=0}^{+\infty} a_k \zeta^k$ is equal to zero. This is a contradiction. Hence there exists an integer $k_0 \ge 1$, such that $c =$

$-k_0^n$. Thus from (3.3), $a_k = 0$ for all $k \geq k_0 + 1$ and $w_1(z) = \sum_{j=0}^{k_0} a_j e^{jz}$. This proves Theorem 1.1. □

## 4. Proof of Theorem 1.2

Suppose that $w \not\equiv 0$ is a solution of (1.5) of finite order. The conditions of Lemma 2.1 are verified in the sector $S_2(\varepsilon)\colon \pi - \varepsilon \leq \arg z \leq \pi + \varepsilon$ $(0 < \varepsilon < \pi/2)$. Hence, there exists a constant $b \neq 0$ such that

$$|w(z) - b| \leq \exp\bigl\{-(1-\rho)\,|z|\cos\varepsilon\bigr\}, \tag{4.1}$$

and for each integer $n \geq 1$

$$|w^{(n)}(z)| \leq \exp\bigl\{-(1-\rho)\,|z|\cos\varepsilon\bigr\}, \tag{4.2}$$

where $0 < \rho < 1$ as $z \to \infty$ in $S_2'(\varepsilon)\colon \pi - \varepsilon < \arg z < \pi + \varepsilon$ $(0 < \varepsilon < \pi/2)$. Furthermore, we have

$$|e^{-z}| \leq e^{|z|} \tag{4.3}$$

and there exists a positive constant $c$ and a sufficiently large $r_0$, such that for $|z| \geq r_0$, we have

$$|Q(z)| \leq c|z|^q, \tag{4.4}$$

where $q = \deg Q(z)$. From (1.5), we can write

$$|Q(z)b| \leq |w^{(n)}(z)| + |e^{-z}|\,|w'(z)| + |Q(z)|\,|w(z) - b|. \tag{4.5}$$

By (4.1)–(4.5), we obtain

$$\begin{aligned} |Q(z)b| \leq{}& \exp\bigl\{-(1-\rho)\,|z|\cos\varepsilon\bigr\} + \exp\bigl\{|z|\,(1 - (1-\rho)\cos\varepsilon)\bigr\} \\ &+ c|z|^q \exp\bigl\{-(1-\rho)\,|z|\cos\varepsilon\bigr\}. \end{aligned} \tag{4.6}$$

Since (4.6) is verified for any arbitrary $0 < \varepsilon < \pi/2$ and $0 < \rho < 1$, it follows that there exists a positive constant $M$, such that for any $|z|$ very large, we can obtain, by taking $\varepsilon$ and $\rho$ small enough, that $|Q(z)| < M$. This contradicts that $Q(z)$ is a nonconstant polynomial. The proof of Theorem 1.2 is completed. □

## 5. Proof of Theorem 1.3

**Case.** $\alpha_s > 0$. If $\arg z = \theta \in S_3(\varepsilon) = \{z : -\varepsilon \le \arg z \le \varepsilon \ (0 < \varepsilon < \pi/2)\}$ and $|z|$ sufficiently large, then we have

$$\left|e^{\alpha_s z}\right| \ge \exp\{\alpha_s r \cos\varepsilon\} \tag{5.1}$$

and

$$|P_j(e^{\alpha_j z})| \le C \tag{5.2}$$

for all $j = 1, \dots, s-1, s+1, \dots, n-1$, where $C > 0$ is some real constant. Hence the conditions of Lemma 2.2 are verified. So, if we suppose that $w \not\equiv 0$ is solution of (1.6) with $\sigma(w) = \sigma < \infty$, then the following conditions hold:

(i) There exists $j \in \{0, \dots, s-1\}$ and a complex constant $b_j \neq 0$ such that $w^{(j)} \to b_j$ as $z \to \infty$ in $S_3'(\varepsilon)$: $-\varepsilon < \arg z < \varepsilon$ $(0 < \varepsilon < \pi/2)$ and more precisely,

$$\left|w^{(j)}(z) - b_j\right| \le \exp\{-(\alpha_s \cos\varepsilon - \rho)\,|z|\}, \tag{5.3}$$

where $0 < \rho < \alpha_s \cos\varepsilon$.

(ii) For each integer $m \ge j+1$, as $z \to \infty$ in $S_3'(\varepsilon)$,

$$\left|w^{(m)}(z)\right| \le \exp\{-(\alpha_s \cos\varepsilon - \rho)\,|z|\}, \tag{5.4}$$

where $0 < \rho < \alpha_s \cos\varepsilon$. From the condition (i), by $j$-fold iterated integration along the line segment $[0, z]$, we obtain

$$\begin{aligned} w(z) &= w(0) + w'(0)\,\frac{z}{1!} + w''(0)\,\frac{z^2}{2!} + \cdots + w^{(j-1)}(0)\,\frac{z^{j-1}}{(j-1)!} \\ &\quad + \int_0^z \cdots \int_0^\zeta \int_0^\xi w^{(j)}(t)\,dt\,d\xi \dots du \\ &= w(0) + w'(0)\,\frac{z}{1!} + w''(0)\,\frac{z^2}{2!} + \cdots + w^{(j-1)}(0)\,\frac{z^{j-1}}{(j-1)!} \\ &\quad + \int_0^z \cdots \int_0^\zeta \int_0^\xi (b_j + \lambda(t))\,dt\,d\xi \dots du \\ &= w(0) + w'(0)\frac{z}{1!} + w''(0)\frac{z^2}{2!} + \cdots + w^{(j-1)}(0)\frac{z^{j-1}}{(j-1)!} + \frac{b_j}{j!} z^j \\ &\quad + \int_0^z \cdots \int_0^\zeta \int_0^\xi \lambda(t)\,dt\,d\xi \dots du, \end{aligned} \tag{5.5}$$

where $\lambda(z) \to 0$ and $|\lambda(z)| \le \exp\{-(\alpha_s \cos\varepsilon - \rho)|z|\}$ $(0 < \rho < \alpha_s \cos\varepsilon)$ as $z \to \infty$ in $S_3'(\varepsilon)\colon -\varepsilon < \arg z < \varepsilon$ $(0 < \varepsilon < \pi/2)$. It then follows from (5.5)

$$\frac{w^{(l)}(z)}{z^j} \to 0 \quad \text{for all } l = 1, \ldots, j; \tag{5.6}$$

$$\frac{w(z)}{z^j} \to \frac{b_j}{j!} \neq 0 \tag{5.7}$$

and

$$\left|\frac{w(z)}{z^j} - \frac{b_j}{j!}\right| = O\left(\frac{1}{|z|}\right), \tag{5.8}$$

as $z \to \infty$ in $S_3'(\varepsilon)$. In the other hand, for $z \in S_3'(\varepsilon)$, we have

$$\left|e^{\alpha_s z}\right| \le \exp\{\alpha_s r\}. \tag{5.9}$$

We divide (1.6) over $z^j$ and write it as follow

$$\begin{aligned} |Q(z)|\frac{|b_j|}{j!} \le{}& \frac{|w^{(n)}(z)|}{|z|^j} + \frac{|P_{n-1}(e^{\alpha_{n-1}z})|\,|w^{(n-1)}(z)|}{|z|^j} + \cdots \\ &+ \frac{|P_s(e^{\alpha_s z})|\,|w^{(s)}(z)|}{|z|^j} + \cdots + \frac{|P_1(e^{\alpha_1 z})|\,|w'(z)|}{|z|^j} \\ &+ |Q(z)|\left|\frac{w(z)}{z^j} - \frac{b_j}{j!}\right|. \end{aligned} \tag{5.10}$$

By using (5.4) and (5.6)–(5.9), from (5.10) we get a contradiction as $z \to \infty$, $\varepsilon \to 0$ and $\rho \to 0$.
**Case.** $\alpha_s < 0$. We take the sector $S_2(\varepsilon)\colon \pi - \varepsilon \le \arg z \le \pi + \varepsilon$ $(0 < \varepsilon < \pi/2)$ and we use the same argument as above.

## 6. Proof of Theorem 1.4

Assume $w(z)$ is a transcendental solution of (1.7) with $\sigma(w) < \infty$. By Lemma 2.3, for any given $\varepsilon > 0$, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0(\theta) = R_0 > 1$ such that for all $z$ satisfying $\arg z = \theta$ and $|z| = r \ge R_0$, we have

$$\left|\frac{w^{(n)}(z)}{w'(z)}\right| \le |z|^{(n-1)(\sigma-1+\varepsilon)}. \tag{6.1}$$

Let $P_1(z) = a_m z^m + \cdots$, $(a_m = \alpha + i\beta \neq 0)$, $\delta(P_1, \theta) = \alpha \cos m\theta - \beta \sin m\theta$. By Lemma 2.6 we have for any given $0 < \varepsilon < 1$, there exists a set $H_1 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$ $(H_2 = \{\theta \in [0, 2\pi) : \delta(P_1, \theta) = 0\})$, there is $R_1 > 0$ such that for $|z| = r > R_1$, we have
(i) if $\delta(P_1, \theta) < 0$, then

$$\begin{aligned} \left|A_1(z)e^{P_1(z)}\right| &\leq \exp\{(1-\varepsilon)\delta(P_1,\theta)r^m\}, \\ \left|A_0(z)e^{P_0(z)}\right| &\leq \exp\Big\{(1-\varepsilon)\,\frac{1}{c}\,\delta(P_1,\theta)r^m\Big\}; \end{aligned} \tag{6.2}$$

(ii) if $\delta(P_1, \theta) > 0$, then

$$\begin{aligned} \left|A_1(z)e^{P_1(z)}\right| &\geq \exp\{(1-\varepsilon)\delta(P_1,\theta)r^m\}, \\ \left|A_0(z)e^{P_0(z)}\right| &\leq \exp\Big\{(1+\varepsilon)\,\frac{1}{c}\,\delta(P_1,\theta)r^m\Big\}. \end{aligned} \tag{6.3}$$

Now we take $\theta \in [0, 2\pi) \setminus (E_1 \cup H_1 \cup H_2)$, such that the linear measure of $E_1 \cup H_1 \cup H_2$ is zero, then $\theta$ satisfies $\delta(P_1, \theta) < 0$ or $\delta(P_1, \theta) > 0$. We divide it into two cases to prove.

**Case 1.** $\delta(P_1, \theta) < 0$. By $a_m = cb_m$, $\delta(P_0, \theta) = (1/c)\delta(P_1, \theta) < 0$. From (1.7), we get

$$1 \leq \left|A_1(z)e^{P_1(z)}\right| \left|\frac{w'(z)}{w^{(n)}(z)}\right| + \left|A_0(z)e^{P_0(z)}\right| \left|\frac{w(z)}{w^{(n)}(z)}\right|. \tag{6.4}$$

If $|w^{(n)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $\{z_p = r_p e^{i\theta}\}$, where $r_p \to +\infty$ such that $w^{(n)}(z_p) \to \infty$ and

$$\begin{aligned} \left|\frac{w'(r_p e^{i\theta})}{w^{(n)}(r_p e^{i\theta})}\right| &\leq \frac{1}{(n-1)!}\,(1+o(1))r_p^{n-1}, \\ \left|\frac{w(r_p e^{i\theta})}{w^{(n)}(r_p e^{i\theta})}\right| &\leq \frac{1}{(n)!}\,(1+o(1))r_p^{n}. \end{aligned} \tag{6.5}$$

Substituting (6.2) and (6.5) into (6.4) we get for any $\theta \in [0, 2\pi) \setminus (E_1 \cup H_1 \cup H_2)$ and $r_p > \max(R_0, R_1)$,

$$\begin{aligned} 1 \leq{} & \frac{1}{(n-1)!}\,r_p^{n-1}(1+o(1))\exp\{(1-\varepsilon)\delta(P_1,\theta)r_p^m\} \\ & + \frac{1}{(n)!}\,r_p^{n}(1+o(1))\exp\Big\{(1-\varepsilon)\,\frac{1}{c}\,\delta(P_1,\theta)r_p^m\Big\}, \end{aligned} \tag{6.6}$$

which gives a contradiction as $r_p \to +\infty$. Hence $w^{(n)}(re^{i\theta})$ is bounded on $\arg z = \theta$, i.e.

$$\left|w^{(n)}(re^{i\theta})\right| \le M_1, \tag{6.7}$$

where $M_1 > 0$ is a constant. From (6.7) and by $n$-fold iterated integration along the line segment $[0, z]$, we obtain

$$|w(re^{i\theta})| \le |w(0)| + |w'(0)|\frac{|z|}{1!} + |w''(0)|\frac{|z|^2}{2!} + \cdots + M_1\frac{|z|^n}{n!}, \tag{6.8}$$

on the ray $\arg z = \theta$.

**Case 2.** $\delta(P_1, \theta) > 0$. Then $\delta(P_0, \theta) = (1/c)\delta(P_1, \theta) > 0$. By (1.7) we have

$$\left|A_1(z)e^{P_1(z)}\right| \le \left|\frac{w^{(n)}(z)}{w'(z)}\right| + \left|A_0(z)e^{P_0(z)}\right| \left|\frac{w(z)}{w'(z)}\right|. \tag{6.9}$$

If $|w'(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists an infinite sequence of points $\{z_p = r_p e^{i\theta}\}$ where $r_p \to +\infty$ such that $w'(z_p) \to \infty$ and

$$\left|\frac{w(r_p e^{i\theta})}{w'(r_p e^{i\theta})}\right| \le (1 + o(1))r_p. \tag{6.10}$$

Substituting (6.1), (6.3) and (6.10) into (6.9) we get

$$\begin{aligned} \exp\bigl\{(1-\varepsilon)\delta(P_1,\theta)r_p^m\bigr\} &\le r_p^{(n-1)(\sigma-1+\varepsilon)} \\ &\quad + (1+o(1))r_p \exp\Bigl\{(1+\varepsilon)\,\frac{1}{c}\,\delta(P_1,\theta)r_p^m\Bigr\} \\ &\le (1+o(1))\,2r_p^{\alpha} \exp\Bigl\{(1+\varepsilon)\,\frac{1}{c}\,\delta(P_1,\theta)r_p^m\Bigr\}, \end{aligned} \tag{6.11}$$

where $\alpha = \max\{1, (n-1)(\sigma - 1 + \varepsilon)\}$.

If we choose $\varepsilon$ such that $0 < \varepsilon < (c-1)/(c+1)$ in (6.11), then we get a contradiction. Hence $\left|w'(re^{i\theta})\right|$ is bounded on $\arg z = \theta$, i.e. there exists a constant $M_2 > 0$, such that

$$\left|w'(re^{i\theta})\right| \le M_2.$$

As above, we get

$$|w(re^{i\theta})| = \left|w(0) + \int_0^z w'(u)\,du\right| \le |w(0)| + M_2|z|$$

on the ray $\arg z = \theta$. In the two cases, we have

$$|w(re^{i\theta})| \le |w(0)| + |w'(0)|\frac{|z|}{1!} + |w''(0)|\frac{|z|^2}{2!} + \cdots + M\frac{|z|^n}{n!} \quad (M > 0) \tag{6.12}$$

on any ray $\arg z = \theta \in [0, 2\pi) \setminus (E_1 \cup H_1 \cup H_2)$. By Phragmén-Lindelöf Theorem [**13**], (6.12) holds in the whole plane. So, $w(z)$ is a polynomial. But $w(z)$ is a transcendental, hence every transcendental solution of (1.7) is of infinite order.

Now we prove that (1.7) cannot have nonzero polynomial solution. Assume $w(z)$ is nonzero polynomial of degree $d$. We can take a ray $\arg z = \theta$ such that $\delta(P_1, \theta) > 0$. From (1.7), we can write

$$\left|A_1(z)e^{P_1(z)}\right| \left|w'(z)\right| \le \left|w^{(n)}(z)\right| + \left|A_0(z)e^{P_0(z)}\right| \left|w(z)\right|, \tag{6.13}$$

and by using Lemma 2.6, we obtain

$$\begin{aligned}&(1 + o(1))r^{d-1} \exp\left\{(1-\varepsilon)\delta(P_1, \theta)r^m\right\}\\ &\quad \le (1 + o(1))\lambda r^d \exp\left\{(1+\varepsilon)\,\frac{1}{c}\,\delta(P_1, \theta)r^m\right\},\end{aligned}$$

where $\lambda > 0$ is some real constant, this is absurd by taking $\varepsilon$ such that $0 < \varepsilon < (c-1)/(c+1)$. Hence every solution $w \not\equiv 0$ of (1.7) is of infinite order.

## 7. Proof of Theorem 1.5

Suppose that $\arg a_m \ne \arg b_m$. By Lemma 2.6, there exists a ray $\arg z = \theta$ such that $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, where $H_1$ and $H_2$ are defined as in Lemma 2.6, $H_1 \cup H_2$ is of linear measure zero, and $\delta(P_0, \theta) > 0$, $\delta(P_1, \theta) < 0$ and for sufficiently large $|z| = r$, we have

$$\left|A_0(z)e^{P_0(z)} + B_0(z)\right| \ge (1 + o(1)) \exp\left\{(1-\varepsilon)\delta(P_0, \theta)r^m\right\}, \tag{7.1}$$

and

$$\begin{aligned}\left|A_1(z)e^{P_1(z)} + B_1(z)\right| &\le \exp\left\{(1-\varepsilon)\delta(P_1, \theta)r^m\right\} \exp\left\{r^{\sigma(B_1)+\frac{\varepsilon}{2}}\right\}\\ &\le \exp\left\{r^{\sigma(B_1)+\varepsilon}\right\}.\end{aligned} \tag{7.2}$$

If we take $\varepsilon$ in (7.1) and (7.2) such that $\sigma(B_1) + \varepsilon < m$, then the conditions (2.12), (2.13) of Lemma 2.5 are satisfied. Hence every solution $w \not\equiv 0$ of (1.8) is of infinite order.

Now suppose that $a_m = cb_m$ $(0 < c < 1)$. Then $\delta(P_1, \theta) = c\delta(P_0, \theta)$. Using the same reasoning as above, there exists a ray $\arg z = \theta$ satisfying $\delta(P_1, \theta) = c\delta(P_0, \theta) > 0$ and for sufficiently large $|z| = r$

$$\left|A_0(z)e^{P_0(z)} + B_0(z)\right| \geq (1 + o(1)) \exp\left\{(1 - \varepsilon)\delta(P_0, \theta)r^m\right\} \qquad (7.3)$$

and

$$\left|A_1(z)e^{P_1(z)} + B_1(z)\right| \leq \exp\left\{(1 + \varepsilon)c'\delta(P_0, \theta)r^m\right\}, \qquad (7.4)$$

where $0 < c < c' < 1$. By taking $\varepsilon$ in (7.3), (7.4) such that $0 < \varepsilon < \frac{1-c'}{1+c'}$, then from Lemma 2.5 we get the result.

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