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Distributors on a tensor category

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Abstract. Let $\mathcal{A}$ be a tensor category and let $\mathcal{V}$ denote the category of vector spaces. A distributor on $\mathcal{A}$ is a functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$. We are concerned with distributors with two-sided $\mathcal{A}$-action. Those distributors form a tensor category, which we denoted by ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$. The functor category $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ is also a tensor category and has the center $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$. We show that if $\mathcal{A}$ is rigid, then ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ and $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ are equivalent as tensor categories.

Key words: tensor category, distributor, center.

Introduction

Let $\mathcal{A}$ be a tensor category over a field k and let $\mathcal{V}$ denote the category of vector spaces over k. A distributor on $\mathcal{A}$ is a functor $L\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{V}$ ([1]). We say L admits two-sided $\mathcal{A}$-action if maps

$$L(X, Y) \to L(A \otimes X, A \otimes Y), \quad L(X, Y) \to L(X \otimes A, Y \otimes A)$$

are given for all objects $A, X, Y \in \mathcal{A}$ so that they satisfy certain conditions. Distributors with two-sided $\mathcal{A}$-action form a tensor category, which we denote by ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$.

Such distributors arise in studying extensions of a tensor category. Given a tensor functor $\mathcal{A} \to \mathcal{B}$, set $L(X, Y) = \mathrm{Hom}_{\mathcal{B}}(X, Y)$ for $X, Y \in \mathcal{A}$. Then L is a monoid object of ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$. Conversely a monoid object of ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ produces a tensor category having the same objects as $\mathcal{A}$.

On the other hand there is a notion of the center of a tensor category ([3], [4], [5]). The center $\mathbf{Z}(\mathcal{A})$ of $\mathcal{A}$ is the category consisting of objects $X \in \mathcal{A}$ equipped with isomorphisms $X \otimes Y \to Y \otimes X$ for all $Y \in \mathcal{A}$ satisfying certain conditions. The center is a braided tensor category. When $\mathcal{A}$ is the category of representations of a Hopf algebra H, $\mathbf{Z}(\mathcal{A})$ is the category of representations of the double Hopf algebra $D(H)$ ([4]).

Now the category $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ of functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a tensor category ([2]). So it has the center $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$. We assume that $\mathcal{A}$ is

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rigid, that is, every object of $\mathcal{A}$ has left and right dual objects. Our result is as follows.

Theorem *We have an equivalence of tensor categories*

$$_{\mathcal{A}}\mathbf{D}(\mathcal{A},\,\mathcal{A})_{\mathcal{A}} \simeq \mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V})).$$

The equivalence is sketched as follows. Let $L \in {}_{\mathcal{A}}\mathbf{D}(\mathcal{A},\,\mathcal{A})_{\mathcal{A}}$. For $Y \in \mathcal{A}$ let Y^c be a left dual of Y and Y^d a right dual of Y. It is proved that the left $\mathcal{A}$-action on L yields an isomorphism $L(X,\,Y) \cong L(Y^c \otimes X,\,I)$ with I unit object, and the right $\mathcal{A}$-action on L yields $L(X,\,Y) \cong L(X \otimes Y^d,\,I)$. Hence $L(Y^c \otimes X,\,I) \cong L(X \otimes Y^d,\,I)$. Thus the functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ given by $F(X) = L(X,\,I)$ admits isomorphisms $F(Y^c \otimes X) \cong F(X \otimes Y^d)$ for $X,\,Y \in \mathcal{A}$. This makes F an object of $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V}))$. The correspondence $L \mapsto F$ gives the equivalence of the theorem.

The paper is organized as follows. Sections 1 and 2 contain basic definitions about tensor categories, tensor linear functors, and distributors. In Section 3 we show the isomorphisms $L(X,\,A \otimes Y) \cong L(A^c \otimes X,\,Y)$, $L(X,\,Y \otimes A) \cong L(X \otimes A^d,\,Y)$ for a distributor L with $\mathcal{A}$-action. In Section 4 we consider the centralizer $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V}))$ of $\mathcal{A}$ in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V})$. This category is isomorphic to the center $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V}))$. An object of $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V}))$ is described in two ways: as a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ equipped with isomorphisms $F(A^c \otimes X) \cong F(X \otimes A^d)$, and as a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ equipped with morphisms $F(X) \to F(A \otimes X \otimes A^d)$. In Section 5 we prove the equivalence ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A},\,\mathcal{A})_{\mathcal{A}} \simeq \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V}))$ as plain categories.

In the remaining sections we consider tensor structures. The tensor product (composition product) in ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A},\,\mathcal{A})_{\mathcal{A}}$ is defined in Section 6, and the tensor product (Day's product) in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V})$ is defined in Section 7. The tensor product in $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V}))$ is described in Section 8. Then we prove in Section 9 that the equivalence ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A},\,\mathcal{A})_{\mathcal{A}} \simeq \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\,\mathcal{V}))$ preserves tensor products.

1. Tensor categories and tensor linear functors

Throughout the paper categories and functors are linear over a field k. The category of k-vector spaces is denoted by $\mathcal{V}$. The category of functors $\mathcal{X} \to \mathcal{Y}$ is denoted by $\mathrm{Hom}(\mathcal{X},\,\mathcal{Y})$.

In this section we review basic definitions for tensor categories, tensor linear functors, and centralizers.

Let $\mathcal{A}$ be a tensor category. The tensor product of objects X and Y of $\mathcal{A}$ is denoted by XY. The tensor product of morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ of $\mathcal{A}$ is denoted by $fg\colon XY \to X'Y'$, while the composition of $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $g \circ f\colon X \to Z$. The unit object of $\mathcal{A}$ is denoted by I. The identity morphism on an object X is denoted by 1_X, and often abbreviated as 1.

For simplicity we assume that $\mathcal{A}$ is a strict tensor category, that is, the equalities

$$(XY)Z = X(YZ), \quad XI = X = IX$$

for objects and the equalities

$$(fg)h = f(gh), \quad f1_I = f = 1_I f$$

for morphisms hold.

We review the language of modules over tensor categories ([6]). A left $\mathcal{A}$-module is a category $\mathcal{X}$ equipped with a bilinear functor $\mathcal{A} \times \mathcal{X} \to \mathcal{X}$, called an $\mathcal{A}$-action, satisfying the axiom of associativity and unitality analogous to the axiom for a module over a ring. We write the $\mathcal{A}$-action as $(A,\, X) \mapsto AX$ for objects and $(e,\, f) \mapsto ef$ for morphisms. Then the axiom says

$$\begin{aligned} (AA')X &= A(A'X), & IX &= X, \\ (ee')f &= e(e'f), & 1_I f &= f \end{aligned}$$

for objects A, A' of $\mathcal{A}$ and X of $\mathcal{X}$, and morphisms e, e' of $\mathcal{A}$ and f of $\mathcal{X}$.

A right $\mathcal{A}$-module is similarly defined.

Let $\mathcal{A}$ and $\mathcal{B}$ be strict tensor categories. An $(\mathcal{A},\, \mathcal{B})$-bimodule is a category $\mathcal{X}$ equipped with bilinear functors $\mathcal{A} \times \mathcal{X} \to \mathcal{X}$ and $\mathcal{X} \times \mathcal{B} \to \mathcal{X}$, called actions, satisfying the axiom analogous to the axiom for a usual bimodule. With the notation for the actions similar to the above, the axiom consists of the equalities

$$\begin{aligned} &(AA')X = A(A'X), \quad IX = X, \\ &(AX)B = A(XB), \\ &X(BB') = (XB)B', \quad XI = X \end{aligned}$$

for objects A, A' of $\mathcal{A}$, X of $\mathcal{X}$, and B, B' of $\mathcal{B}$, and the corresponding equalities for morphisms. The tensor category $\mathcal{A}$ itself is an $(\mathcal{A},\,\mathcal{A})$-bimodule in which AX, XA are tensor products in $\mathcal{A}$.

Let $\mathcal{X}$, $\mathcal{Y}$ be left $\mathcal{A}$-modules. An $\mathcal{A}$-linear functor $\mathcal{X} \to \mathcal{Y}$ is a functor $F\colon \mathcal{X} \to \mathcal{Y}$ equipped with a family of isomorphisms $\lambda_{A,\,X}\colon F(AX) \to AF(X)$ for all $A \in \mathcal{A}$ and $X \in \mathcal{X}$ satisfying the following conditions.

(1.1.i) $\lambda_{A,\,X}$ is natural in A and X.

(1.1.ii) The diagram

$$\begin{tikzcd} F(AA'X) \arrow[r, "\lambda_{A,\,A'X}"] \arrow[dr, "\lambda_{AA',\,X}"'] & AF(A'X) \arrow[d, "1\lambda_{A',\,X}"] \\ & AA'F(X) \end{tikzcd}$$

commutes for all $A, A' \in \mathcal{A}$ and $X \in \mathcal{X}$.

(1.1.iii) $\lambda_{I,\,X} = 1$ for all $X \in \mathcal{X}$.

We call the family of $\lambda_{A,\,X}$ the *left $\mathcal{A}$-linear structure* of F.

If $\mathcal{X}$, $\mathcal{Y}$ are right $\mathcal{B}$-modules, a $\mathcal{B}$-linear functor $\mathcal{X} \to \mathcal{Y}$ is a functor $F\colon \mathcal{X} \to \mathcal{Y}$ equipped with a family of isomorphisms $\rho_{X,\,B}\colon F(XB) \to F(X)B$, called the *right $\mathcal{B}$-linear structure*, satisfying similar conditions.

If $\mathcal{X}$, $\mathcal{Y}$ are $(\mathcal{A},\,\mathcal{B})$-bimodules, an $(\mathcal{A},\,\mathcal{B})$-linear functor $\mathcal{X} \to \mathcal{Y}$ is a functor $F\colon \mathcal{X} \to \mathcal{Y}$ equipped with a family of isomorphisms $\lambda_{A,\,X}\colon F(AX) \to AF(X)$ and $\rho_{X,\,B}\colon F(XB) \to F(X)B$ satisfying (1.1.i)–(1.1.iii) for λ, the corresponding conditions for ρ, and the following:

(1.2) The diagram

$$\begin{tikzcd} F(AXB) \arrow[r, "\lambda_{A,\,XB}"] \arrow[d, "\rho_{AX,\,B}"'] & AF(XB) \arrow[d, "1\rho_{X,\,B}"] \\ F(AX)B \arrow[r, "\lambda_{A,\,X}1"'] & AF(X)B \end{tikzcd}$$

commutes for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $X \in \mathcal{X}$.

If $\mathcal{X}$, $\mathcal{Y}$ are left $\mathcal{A}$-modules and F, G are $\mathcal{A}$-linear functors $\mathcal{X} \to \mathcal{Y}$, an $\mathcal{A}$-linear natural transformation $F \to G$ is a natural transformation $F \to G$ commuting with the left $\mathcal{A}$-linear structure of F and G. We then have the category $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},\,\mathcal{Y})$ whose objects are $\mathcal{A}$-linear functors $\mathcal{X} \to \mathcal{Y}$ and whose morphisms are $\mathcal{A}$-linear natural transformations.

Similarly, for $(\mathcal{A},\,\mathcal{B})$-bimodules $\mathcal{X}$ and $\mathcal{Y}$ we have the category of $(\mathcal{A},\,\mathcal{B})$-linear functors $\mathcal{X} \to \mathcal{Y}$, which we denote by $\mathrm{Hom}_{\mathcal{A},\,\mathcal{B}}(\mathcal{X},\,\mathcal{Y})$.

The following is an analogue of the isomorphism $\mathrm{Hom}_R(R, M) \cong M$ for an R-module M, and can be proved easily.

Proposition 1.3 *Let $\mathcal{X}$ be a right $\mathcal{A}$-module. We have an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{X}) \simeq \mathcal{X}$$

which takes an object $X \in \mathcal{X}$ to an object $G \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ as follows. We have

$$G(A) = XA$$

for $A \in \mathcal{A}$, and the right $\mathcal{A}$-linear structure

$$\rho_{A,B} \colon G(AB) \to G(A)B$$

for $A, B \in \mathcal{A}$ is the identity

$$XAB \to XAB.$$

For an $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{X}$, the centralizer $\mathbf{Z}_{\mathcal{A}}(\mathcal{X})$ is the category defined as follows. An object of $\mathbf{Z}_{\mathcal{A}}(\mathcal{X})$ is an object $X \in \mathcal{X}$ equipped with a family of isomorphisms

$$\omega_A \colon AX \to XA \quad \text{for all} \quad A \in \mathcal{A}$$

satisfying the following conditions.

(1.4.i) ω_A is natural in A.

(1.4.ii) The diagram

$$\begin{tikzcd} ABX \arrow[r, "1\omega_B"] \arrow[dr, "\omega_{AB}"'] & AXB \arrow[d, "\omega_A 1"] \\ & XAB \end{tikzcd}$$

commutes for all $A, B \in \mathcal{A}$ and $X \in \mathcal{X}$.

(1.4.iii) ω_I is the identity.

We call the family of ω_A the *central structure.*

A morphism of $\mathbf{Z}_{\mathcal{A}}(\mathcal{X})$ is a morphism of $\mathcal{X}$ commuting with central structures.

The following is also an analogue of the well-known isomorphism for a usual bimodule.

Proposition 1.5 *We have an equivalence of categories*

$$\mathrm{Hom}_{\mathcal{A},\mathcal{A}}(\mathcal{A},\,\mathcal{X}) \simeq \mathbf{Z}_{\mathcal{A}}(\mathcal{X})$$

which takes an object $X \in \mathbf{Z}_{\mathcal{A}}(\mathcal{X})$ *to an object* $G \in \mathrm{Hom}_{\mathcal{A},\mathcal{A}}(\mathcal{A},\,\mathcal{X})$ *as follows. We have*

$$G(A) = XA$$

for $A \in \mathcal{A}$. *The right* $\mathcal{A}$*-linear structure* $\rho_{A,B}\colon G(AB) \to G(A)B$ *for* $A,\,B \in \mathcal{A}$ *is the identity*

$$XAB \to XAB.$$

The left $\mathcal{A}$*-linear structure* $\lambda_{B,A}\colon G(BA) \to BG(A)$ *is*

$$\omega_B^{-1}1\colon XBA \to BXA,$$

where ω_B *is the central structure of* X.

For the $(\mathcal{A},\,\mathcal{A})$-bimodule $\mathcal{A}$, the centralizer $\mathbf{Z}_{\mathcal{A}}(\mathcal{A})$ is called the center of $\mathcal{A}$, and denoted by $\mathbf{Z}(\mathcal{A})$. This is a tensor category: the tensor product of $X,\,Y \in \mathbf{Z}(A)$ is the object XY of $\mathcal{A}$ with central structure given by the composite

$$AXY \xrightarrow{\omega_A 1} XAY \xrightarrow{1\omega_A} XYA.$$

For details see [4, p. 330].

2. Distributors with tensor action

Let $\mathcal{X}$ and $\mathcal{Y}$ be categories. Let $\mathcal{V}$ denote the category of k-vector spaces. A distributor from $\mathcal{X}$ to $\mathcal{Y}$ is a bilinear functors $\mathcal{X}^{\mathrm{op}} \times \mathcal{Y} \to \mathcal{V}$ ([1, Chapter 7]). Namely a distributor L from $\mathcal{X}$ to $\mathcal{Y}$ consists of vector spaces $L(X,\,Y)$ for all objects X of $\mathcal{X}$ and Y of $\mathcal{Y}$, and linear maps $L(f,\,g)\colon L(X,\,Y) \to L(X',Y')$ for all morphisms $f\colon X' \to X$ of $\mathcal{X}$ and $g\colon Y \to Y'$ of $\mathcal{Y}$ satisfying the following conditions.

(2.1.i) For morphisms $f\colon X' \to X$, $f'\colon X'' \to X'$ of $\mathcal{X}$ and $g\colon Y \to Y'$ and $g'\colon Y' \to Y''$ of $\mathcal{Y}$, we have

$$L(f \circ f',\, g' \circ g) = L(f',\, g') \circ L(f,\, g).$$

(2.1.ii) $L(1,\,1) = 1$.

(2.1.iii) $L(f, g)$ is bilinear in f and g.

An easy consequence of (2.1.i) is

$$L(f, g) = L(f, 1) \circ L(1, g) = L(1, g) \circ L(f, 1).$$

We also denote $L(f, 1) = f^*$, $L(1, g) = g_*$.

We denote by $\mathbf{D}(\mathcal{X}, \mathcal{Y})$ the category of distributors from $\mathcal{X}$ to $\mathcal{Y}$.

Let $\mathcal{A}$ be a tensor category and let $\mathcal{X}$, $\mathcal{Y}$ be left $\mathcal{A}$-modules. A distributor from $\mathcal{X}$ to $\mathcal{Y}$ with left $\mathcal{A}$-action is a distributor L from $\mathcal{X}$ to $\mathcal{Y}$ equipped with linear maps

$$A!\colon L(X, Y) \to L(AX, AY)$$

for all objects A of $\mathcal{A}$, X of $\mathcal{X}$, and Y of $\mathcal{Y}$, satisfying the following conditions.

(2.2.i) For morphisms $f\colon X' \to X$ of $\mathcal{X}$ and $g\colon Y \to Y'$ of $\mathcal{Y}$, we have a commutative diagram

$$\begin{tikzcd} L(X, Y) \arrow[r, "A!"] \arrow[d, "L(f,g)"'] & L(AX, AY) \arrow[d, "L(1f, 1g)"] \\ L(X', Y') \arrow[r, "A!"'] & L(AX', AY'). \end{tikzcd}$$

(2.2.ii) For a morphism $e\colon A \to A'$ of $\mathcal{A}$, we have a commutative diagram

$$\begin{tikzcd} L(X, Y) \arrow[r, "A!"] \arrow[d, "A'!"'] & L(AX, AY) \arrow[d, "L(1, e1)"] \\ L(A'X, A'Y) \arrow[r, "L(e1, 1)"'] & L(AX, A'Y). \end{tikzcd}$$

(2.2.iii) For objects A, A' of $\mathcal{A}$, we have a commutative diagram

$$\begin{tikzcd} L(X, Y) \arrow[r, "A!"] \arrow[dr, "(A'A)!"'] & L(AX, AY) \arrow[d, "A'!"] \\ & L(A'AX, A'AY). \end{tikzcd}$$

(2.2.iv) For the unit object I, $I!\colon L(X, Y) \to L(X, Y)$ is the identity.

We denote by ${}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{Y})$ the category of distributors from $\mathcal{X}$ to $\mathcal{Y}$ with left $\mathcal{A}$-action.

Let $\mathcal{A}$ and $\mathcal{B}$ be tensor categories and let $\mathcal{X}$, $\mathcal{Y}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. A distributor from $\mathcal{X}$ to $\mathcal{Y}$ with $(\mathcal{A}, \mathcal{B})$-action is a distributor L equipped with linear maps

$$A!\colon L(X, Y) \to L(AX, AY)$$
$$!B\colon L(X, Y) \to L(XB, YB)$$

for all objects A of $\mathcal{A}$, B of $\mathcal{B}$, X of $\mathcal{X}$, and Y of $\mathcal{Y}$, satisfying (2.2.i)–(2.2.iv) for $A!$, the analogous conditions for $!B$, and the following:

(2.3) For objects A of $\mathcal{A}$, B of $\mathcal{B}$, X of $\mathcal{X}$, and Y of $\mathcal{Y}$, we have a commutative diagram

$$\begin{tikzcd} L(X, Y) \arrow[r, "A!"] \arrow[d, "!B"'] & L(AX, AY) \arrow[d, "!B"] \\ L(XB, YB) \arrow[r, "A!"'] & L(AXB, AXB). \end{tikzcd}$$

We denote by ${}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{Y})_{\mathcal{B}}$ the category of distributors from $\mathcal{X}$ to $\mathcal{Y}$ with $(\mathcal{A}, \mathcal{B})$-action.

3. Duality isomorphism

In this section we show that if $\mathcal{A}$ is rigid, distributors with $\mathcal{A}$-action can be identified with $\mathcal{A}$-linear functors.

Let $\mathcal{A}$ be a tensor category. We call a quadruple (A, A', ϵ, η) a *duality* if A, A' are objects of $\mathcal{A}$ and $\epsilon\colon AA' \to I$, $\eta\colon I \to A'A$ are morphisms of $\mathcal{A}$ such that the composites

$$A \xrightarrow{1\eta} AA'A \xrightarrow{\epsilon 1} A, \quad A' \xrightarrow{\eta 1} A'AA' \xrightarrow{1\epsilon} A'$$

are the identity morphisms.

It is well-known that a duality (A, A', ϵ, η) gives rise to the adjoint isomorphism

$$\mathrm{Hom}(AX, Y) \cong \mathrm{Hom}(X, A'Y)$$

for $X, Y \in \mathcal{A}$. We will show that Hom of the both sides may be replaced by any object L of ${}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{Y})$.

Proposition 3.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be left $\mathcal{A}$-modules and let $L \in {}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{Y})$. Suppose that (A, A', ϵ, η) is a duality in $\mathcal{A}$. Then we have an isomorphism*

$$L(AX, Y) \cong L(X, A'Y)$$

for any $X \in \mathcal{X}$, $Y \in \mathcal{Y}$. This is given by the maps

$$\sigma\colon L(AX, Y) \xrightarrow{A'!} L(A'AX, A'Y) \xrightarrow{L(\eta 1, 1)} L(X, A'Y),$$
$$\tau\colon L(X, A'Y) \xrightarrow{A!} L(AX, AA'Y) \xrightarrow{L(1, \epsilon 1)} L(AX, Y),$$

which are inverse to each other.

Proof. We will show that σ, τ are inverse to each other. (a) By (2.2.i) we have a commutative diagram

$$\begin{tikzcd}
L(AX, Y) \arrow[r, "A'!"] \arrow[dr, "\sigma"'] & L(A'AX, A'Y) \arrow[r, "A!"] \arrow[d, "L(\eta 1, 1)"'] & L(AA'AX, AA'Y) \arrow[d, "L(1\eta 1, 1)"] \\
& L(X, A'Y) \arrow[r, "A!"'] \arrow[dr, "\tau"'] & L(AX, AA'Y) \arrow[d, "L(1, \epsilon 1)"] \\
& & L(AX, Y).
\end{tikzcd}$$

Hence by (2.2.iii) and (2.1.i)

$$\begin{tikzcd}
L(AX, Y) \arrow[r, "(AA')!"] \arrow[d, "\tau\circ\sigma"'] & L(AA'AX, AA'Y) \arrow[d, "L(1, \epsilon 1)"] \\
L(AX, Y) & L(AA'AX, Y) \arrow[l, "L(1\eta 1, 1)"]
\end{tikzcd}$$

is commutative. On the other hand, by (2.2.ii) applied for the morphism $\epsilon\colon AA' \to I$ and (2.2.iv),

$$\begin{tikzcd}
L(AX, Y) \arrow[r, "(AA')!"] \arrow[dr, "L(\epsilon 11, 1)"'] & L(AA'AX, AA'Y) \arrow[d, "L(1, \epsilon 1)"] \\
& L(AA'AX, Y)
\end{tikzcd}$$

is commutative. Then we have

$$\tau \circ \sigma = L(1\eta 1, 1) \circ L(\epsilon 11, 1) = 1.$$

(b) We have a commutative diagram

$$\begin{tikzcd}
L(X,\, A'Y) \arrow[r, "A!"] \arrow[dr, "\tau"'] & L(AX,\, AA'Y) \arrow[r, "A'!"] \arrow[d, "L(1,\epsilon 1)"'] & L(A'AX,\, A'AA'Y) \arrow[d, "L(1,1\epsilon 1)"] \\
& L(AX,\, Y) \arrow[r, "A'!"'] \arrow[dr, "\sigma"'] & L(A'AX,\, A'Y) \arrow[d, "L(\eta 1,1)"] \\
& & L(X,\, A'Y).
\end{tikzcd}$$

Hence

$$\begin{tikzcd}
L(X,\, A'Y) \arrow[r, "(A'A)!"] \arrow[d, "\sigma\circ\tau"'] & L(A'AX,\, A'AA'Y) \arrow[d, "L(\eta 1,1)"] \\
L(X,\, A'Y) & L(X,\, A'AA'Y) \arrow[l, "L(1,1\epsilon 1)"']
\end{tikzcd}$$

is commutative. But

$$\begin{tikzcd}
L(X,\, A'Y) \arrow[r, "(A'A)!"] \arrow[dr, "L(1,\eta 11)"'] & L(A'AX,\, A'AA'Y) \arrow[d, "L(\eta 1,1)"] \\
& L(X,\, A'AA'Y)
\end{tikzcd}$$

is commutative. Hence

$$\sigma \circ \tau = L(1,\, 1\epsilon 1) \circ L(1,\, \eta 11) = 1.$$

This proves the proposition. $\square$

Proposition 3.2 *Under the assumption of the previous proposition, the diagrams*

$$\begin{tikzcd}
L(X,\, Y) \arrow[d, "A!"'] \arrow[dr, "L(1,\eta 1)"] & \\
L(AX,\, AY) & L(X,\, A'AY) \arrow[l, "\tau"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L(X,\, Y) \arrow[d, "A'!"'] \arrow[dr, "L(\epsilon 1,1)"] & \\
L(A'X,\, A'Y) & L(AA'X,\, Y) \arrow[l, "\sigma"']
\end{tikzcd}$$

are commutative.

Proof. The first one follows from the commutative diagram

$$\begin{tikzcd} L(X,\,Y) \arrow[r, "L(1,\eta 1)"] \arrow[d, "A!"'] & L(X,\,A'AY) \arrow[r, "\tau"] \arrow[d, "A!"'] & L(AX,\,AY) \\ L(AX,\,AY) \arrow[r, "L(1,1\eta 1)"'] & L(AX,\,AA'AY) \arrow[ur, "L(1,\epsilon 11)"'] & \end{tikzcd}$$

and the equality $\epsilon 1 \circ 1\eta = 1$.

The second follows from the commutative diagram

$$\begin{tikzcd} L(X,\,Y) \arrow[r, "L(\epsilon 1,1)"] \arrow[d, "A'!"'] & L(AA'X,\,Y) \arrow[r, "\sigma"] \arrow[d, "A'!"'] & L(A'X,\,A'Y) \\ L(A'X,\,A'Y) \arrow[r, "L(1\epsilon 1,1)"'] & L(A'AA'X,\,A'Y) \arrow[ur, "L(\eta 11,1)"'] & \end{tikzcd}$$

and the equality $1\epsilon \circ \eta 1 = 1$. □

For the convenience in later use we record the right-sided version of Proposition 3.1.

Proposition 3.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be right $\mathcal{A}$-modules and let $L \in \mathbf{D}(\mathcal{X},\,\mathcal{Y})_{\mathcal{A}}$. Suppose that $(A,\,A',\,\epsilon,\,\eta)$ is a duality in $\mathcal{A}$. Then we have an isomorphism*

$$L(XA',\,Y) \cong L(X,\,YA)$$

for any $X \in \mathcal{X}$, $Y \in \mathcal{Y}$. This is given by the maps

$$\sigma\colon L(XA',\,Y) \xrightarrow{!A} L(XA'A,\,YA) \xrightarrow{L(1\eta,1)} L(X,\,YA),$$
$$\tau\colon L(X,\,YA) \xrightarrow{!A'} L(XA',\,YAA') \xrightarrow{L(1,1\epsilon)} L(XA',\,Y),$$

which are inverse to each other.

We assume that $\mathcal{A}$ is left rigid, that is, for every object $A \in \mathcal{A}$ there exists a duality $(A',\,A,\,\epsilon,\,\eta)$. We choose such a duality for each $A \in \mathcal{A}$ and denote it by

$$(A^c,\,A,\,\epsilon_A\colon A^cA \to I, \eta_A\colon I \to AA^c).$$

Then the assignment $A \mapsto A^c$ becomes a contravariant functor $\mathcal{A} \to \mathcal{A}$. For a morphism $f\colon A \to B$ one has a morphism $f^c\colon B^c \to A^c$ so that the following diagrams are commutative.

$$\begin{array}{ccc} B^cA & \xrightarrow{f^c1} & A^cA \\ {\scriptstyle 1f}\downarrow & & \downarrow{\scriptstyle \epsilon_A} \\ B^cB & \xrightarrow[\epsilon_B]{} & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\eta_B} & BB^c \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle 1f^c} \\ AA^c & \xrightarrow[f1]{} & BA^c. \end{array}$$

We have also natural isomorphisms $(AB)^c \cong B^cA^c$, $I^c \cong I$. For simplicity we assume that $(AB)^c = B^cA^c$, $I^c = I$ and the natural isomorphisms are the identities. This means that the diagrams

$$\begin{array}{ccc} B^cA^cAB & = & (AB)^cAB \\ {\scriptstyle 1\epsilon_A1}\downarrow & & \downarrow{\scriptstyle \epsilon_{AB}} \\ B^cB & \xrightarrow[\epsilon_B]{} & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\eta_A} & AA^c \\ {\scriptstyle \eta_{AB}}\downarrow & & \downarrow{\scriptstyle 1\eta_B1} \\ AB(AB)^c & = & ABB^cA^c \end{array}$$

commute for $A, B \in \mathcal{A}$.

Proposition 3.4 *Let $L \in \mathbf{D}(\mathcal{X}, \mathcal{Y})$. There is a one-to-one correspondence between the following two objects*:

- *a family of maps*

  $$A!\colon L(X, Y) \to L(AX, AY)$$

  for all $A \in \mathcal{A}$, $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ satisfying (2.2.i)–(2.2.iv).
- *a family of isomorphisms*

  $$\tau_A\colon L(X, AY) \to L(A^cX, Y)$$

  for all $A \in \mathcal{A}$, $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ satisfying the following conditions:

  (i) *The maps τ_A are natural in X, Y.*

  (ii) *The maps τ_A are natural in A in the sense that for any morphism $f\colon A \to B$ of $\mathcal{A}$ we have a commutative diagram*

  $$\begin{array}{ccc} L(X, AY) & \xrightarrow{\tau_A} & L(A^cX, Y) \\ {\scriptstyle L(1,f1)}\downarrow & & \downarrow{\scriptstyle L(f^c1,1)} \\ L(X, BY) & \xrightarrow[\tau_B]{} & L(B^cX, Y). \end{array}$$

(iii) *The diagram*

$$\begin{tikzcd}
L(X,\, ABY) \arrow[r, "\tau_{AB}"] \arrow[d, "\tau_A"'] & L((AB)^c X,\, Y) \arrow[d, equal] \\
L(A^c X,\, BY) \arrow[r, "\tau_B"'] & L(B^c A^c X,\, Y)
\end{tikzcd}$$

is commutative.

(iv) $\tau_I = 1$.

Proof. (a) Construction of $A! \mapsto \tau_A$. Suppose that the maps $A!$ are given. We define τ_A to be the isomorphism τ of Proposition 3.1 for the duality $(A^c,\, A,\, \epsilon_A,\, \eta_A)$:

$$\tau_A \colon L(X,\, AY) \xrightarrow{A^c!} L(A^c X,\, A^c A Y) \xrightarrow{L(1,\, \epsilon_A 1)} L(A^c X,\, Y).$$

Its inverse is given by

$$\sigma_A \colon L(A^c X,\, Y) \xrightarrow{A!} L(A A^c X,\, AY) \xrightarrow{L(\eta_A 1,\, 1)} L(X,\, AY).$$

Let us verify (i)–(iv). (i) and (iv) are obvious.

Proof of (ii): Let $f \colon A \to B$ be a morphism. We have a commutative diagram

$$\begin{tikzcd}
 & & L(X,\, AY) \arrow[r, "A^c!"] \arrow[dll, "L(1,f1)"'] \arrow[d, "B^c!"] & L(A^c X,\, A^c A Y) \arrow[d, "L(f^c 1,1)"'] \arrow[dr, "L(1,\epsilon_A 1)"] & \\
L(X,\, BY) \arrow[d, "B^c!"'] & & L(B^c X,\, B^c A Y) \arrow[r, "L(1,\, f^c 11)"'] \arrow[dll, "L(1,1f1)"] & L(B^c X,\, A^c A Y) \arrow[dr, "L(1,\epsilon_A 1)"'] & L(A^c X,\, Y) \arrow[d, "L(f^c 1,1)"] \\
L(B^c X,\, B^c B Y) \arrow[rrrr, "L(1,\epsilon_B 1)"'] & & & & L(B^c X,\, Y).
\end{tikzcd}$$

The commutativity of the bottom quadrangle follows from that of the diagram

$$\begin{tikzcd}
B^c A \arrow[r, "f^c 1"] \arrow[d, "1f"'] & A^c A \arrow[d, "\epsilon_A"] \\
B^c B \arrow[r, "\epsilon_B"'] & I.
\end{tikzcd}$$

Hence, looking at the surrounding arrows, we obtain the commutative diagram

$$\begin{tikzcd}
L(X,\, AY) \arrow[r, "\tau_A"] \arrow[d, "L(1,\, f1)"'] & L(A^cX,\, Y) \arrow[d, "L(f^c1,\, 1)"] \\
L(X,\, BY) \arrow[r, "\tau_B"'] & L(B^cX,\, Y).
\end{tikzcd}$$

Proof of (iii): Let $A,\, B \in \mathcal{A}$. We have a commutative diagram

$$\begin{tikzcd}
L(X,\, ABY) \arrow[r, "A^c!"] \arrow[dr, "\tau_A"'] & L(A^cX,\, A^cABY) \arrow[r, "B^c!"] \arrow[d, "L(1{,}\epsilon_A11)"] & L(B^cA^cX,\, B^cA^cABY) \arrow[d, "L(1,\, 1\epsilon_A11)"] \\
& L(A^cX,\, BY) \arrow[r, "B^c!"] \arrow[dr, "\tau_B"'] & L(B^cA^cX,\, B^cBY) \arrow[d, "L(1{,}\epsilon_B1)"] \\
& & L(B^cA^cX, Y).
\end{tikzcd}$$

The upper horizontal arrows yield $(B^cA^c)!$ and the right vertical arrows yield $L(1,\, \epsilon_{AB}1)$, and the composition of these is τ_{AB}. Hence $\tau_{AB} = \tau_B \circ \tau_A$.

(b) Construction of $\tau_A \mapsto A!$. Suppose that the maps τ_A are given. Let $\sigma_A = \tau_A^{-1}$. Define $A!$ to be the composite

$$L(X,\, Y) \xrightarrow{L(\epsilon_A 1,\, 1)} L(A^cAX,\, Y) \xrightarrow{\sigma_A} L(AX,\, AY).$$

Let us verify (2.2.i)–(2.2.iv). (2.2.i) and (2.2.iv) are obvious.

Proof of (2.2.ii): Let $f\colon A \to B$ be a morphism. We have a commutative diagram

$$\begin{tikzcd}
L(X,\, Y) \arrow[r, "L(\epsilon_A1,\, 1)"] \arrow[d, "L(\epsilon_B1,\, 1)"'] & L(A^cAX,\, Y) \arrow[d, "L(f^c11,\, 1)"'] \arrow[dr, "\sigma_A"] & \\
L(B^cBX,\, Y) \arrow[r, "L(1f1,\, 1)"'] \arrow[dr, "\sigma_B"'] & L(B^cAX,\, Y) \arrow[dr, "\sigma_B"] & L(AX,\, AY) \arrow[d, "L(1,\, f1)"] \\
& L(BX,\, BY) \arrow[r, "L(f1,\, 1)"'] & L(AX,\, BY).
\end{tikzcd}$$

Hence

$$\begin{tikzcd} L(X,\,Y) \arrow[r, "A!"] \arrow[d, "B!"'] & L(AX,\,AY) \arrow[d, "L(1,f1)"] \\ L(BX,\,BY) \arrow[r, "L(f1,\,1)"'] & L(AX,\,BY) \end{tikzcd}$$

is commutative.

Proof of (2.2.iii): Let $A,\, B \in \mathcal{A}$. We have a commutative diagram

$$\begin{tikzcd} L(X,\,Y) \arrow[r, "L(\epsilon_A 1,\,1)"] \arrow[dr, "A!"'] & L(A^cAX,\,Y) \arrow[r, "L(1\epsilon_B 11,\,1)"] \arrow[d, "\sigma_A"] & L(A^cB^cBAX,\,Y) \arrow[d, "\sigma_A"] \\ & L(AX,\,AY) \arrow[r, "L(\epsilon_B 11,\,1)"] \arrow[dr, "B!"'] & L(B^cBAX,\,AY) \arrow[d, "\sigma_B"] \\ & & L(BAX, BAY). \end{tikzcd}$$

The upper horizontal arrows yield $L(\epsilon_{BA}1,\,1)$, and the right vertical arrows yield σ_{BA}, and the composition of these is $(BA)!$. Hence $(BA)! = B! \circ A!$.

(c) Let us verify that the constructions of (a) and (b) are inverse to each other.

Firstly let $A! \mapsto \tau_A$ by construction (a). Proposition 3.2 tells us that $\tau_A \mapsto A!$.

Secondly let $\tau_A \mapsto A!$ by construction (b). We have a commutative diagram

$$\begin{tikzcd} & L(AA^cX,\,AY) \arrow[r, "L(\eta 1,\,1)"] \arrow[d, "\tau_A"] & L(X,\,AY) \arrow[d, "\tau_A"] \\ L(A^cX,\,Y) \arrow[ur, "A!"] \arrow[r, "L(\epsilon 11,\,1)"'] & L(A^cAA^cX,\,Y) \arrow[r, "L(1\eta 1,\,1)"'] & L(A^cX,\,Y). \end{tikzcd}$$

Since the lower horizontal composite is the identity, we have

$$\sigma_A = \tau_A^{-1} = L(\eta 1,\,1) \circ A!.$$

This means that $A! \mapsto \tau_A$. This concludes the proof. □

For any categories $\mathcal{X}$ and $\mathcal{Y}$ we have an isomorphism of categories

$$\mathbf{D}(\mathcal{X},\,\mathcal{Y}) \cong \mathrm{Hom}(\mathcal{Y},\,\mathrm{Hom}(\mathcal{X}^{\mathrm{op}},\,\mathcal{V})).$$

Here $\mathcal{V}$ denotes the category of k-vector spaces, and $\mathrm{Hom}(-,\,-)$ means the functor category. This isomorphism connects $L \in \mathbf{D}(\mathcal{X}, \mathcal{Y})$ and $K \in \mathrm{Hom}(\mathcal{Y}, \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V}))$ so that

$$L(X, Y) = K(Y)(X).$$

Let $\mathcal{X}$ be a left $\mathcal{A}$-module. Then the category $\mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V})$ becomes a left $\mathcal{A}$-module with action

$$\mathcal{A} \times \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V}) \to \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V})\colon (A, F) \mapsto AF$$

defined by

$$(AF)(X) = F(A^c X).$$

Let $\mathcal{X}$, $\mathcal{Y}$ be left $\mathcal{A}$-modules. Let

$$L \in \mathbf{D}(\mathcal{X}, \mathcal{Y}) \quad \text{and} \quad K \in \mathrm{Hom}(\mathcal{Y}, \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V}))$$

correspond under the above isomorphism. Then a map

$$L(X, AY) \to L(A^c X, Y)$$

is rewritten as

$$K(AY)(X) \to (A(K(Y)))(X).$$

A family of isomorphisms

$$\tau_A\colon L(X, AY) \to L(A^c X, Y)$$

for all A, X, Y satisfying (i)–(iv) of Proposition 3.4 is the same thing as a family of isomorphisms

$$\lambda_{A,Y}\colon K(AY) \to A(K(Y))$$

for all A, Y satisfying (1.1.i)–(1.1.iii) for K. So Proposition 3.3 says that there is a one-to-one correspondence between a family of maps $A!$ giving L a left $\mathcal{A}$-action and a family of isomorphisms $\lambda_{A,Y}$ giving K a left $\mathcal{A}$-linear structure. Thus we obtain

Proposition 3.5 *We have an isomorphism of categories*

$${}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{Y}) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{Y}, \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V})).$$

Under this isomorphism objects $L \in {}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{Y})$ *and*

$K \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{Y}, \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V}))$ *correspond in the following way: We have*

$$L(X, Y) = K(Y)(X)$$

and the map

$$A!\colon L(X, Y) \to L(AX, AY)$$

equals

$$K(Y)(X) \overset{K(Y)(\epsilon_A 1)}{\longrightarrow} K(Y)(A^c AX)$$
$$= (A(K(Y)))(AX) \overset{\lambda^{-1}_{A,Y}}{\longrightarrow} K(AY)(AX),$$

where λ is the left $\mathcal{A}$-linear structure of K.

Let $\mathcal{B}$ be a tensor category. Assume that $\mathcal{B}$ is right rigid, namely for every object $B \in \mathcal{B}$ there is a duality (B, B', ϵ, η). We choose such a duality for each B and denote it by

$$(B, B^d, \epsilon_B\colon BB^d \to I, \eta_B\colon I \to B^d B).$$

Then the assignment $B \mapsto B^d$ becomes a functor $\mathcal{B}^{\mathrm{op}} \to \mathcal{B}$. We assume that the natural isomorphisms $(AB)^d \cong B^d A^d$ and $I^d \cong I$ are identities.

Let $\mathcal{X}$, $\mathcal{Y}$ be right $\mathcal{B}$-modules. The category $\mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V})$ becomes a right $\mathcal{B}$-module with action $(F, B) \mapsto FB$ defined by

$$(FB)(X) = F(XB^d).$$

For the sake of later use we state versions of the previous proposition for right modules and bimodules.

Proposition 3.6 *We have an isomorphism of categories*

$$\mathbf{D}(\mathcal{X}, \mathcal{Y})_{\mathcal{B}} \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{Y}, \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V})).$$

Under this isomorphism objects $L \in \mathbf{D}(\mathcal{X}, \mathcal{Y})_{\mathcal{B}}$ and $K \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{Y}, \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V}))$ correspond in the following way: We have

$$L(X, Y) = K(Y)(X)$$

and the map

$$!B\colon L(X, Y) \to L(XB, YB)$$

equals

$$K(Y)(X) \xrightarrow{K(Y)(1\epsilon_B)} K(Y)(XBB^d) =$$

$$(K(Y)B)(XB) \xrightarrow{\rho_{Y,B}^{-1}} K(YB)(XB),$$

where ρ is the right $\mathcal{A}$-linear structure of K.

Let $\mathcal{X}$, $\mathcal{Y}$ be $(\mathcal{A}, \mathcal{B})$-bimodules. The category $\mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V})$ becomes an $(\mathcal{A}, \mathcal{B})$-bimodule.

Proposition 3.7 *We have an isomorphism of categories*

$${}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{Y})_{\mathcal{B}} \cong \mathrm{Hom}_{\mathcal{A},\mathcal{B}}(\mathcal{Y}, \mathrm{Hom}(\mathcal{X}^{\mathrm{op}}, \mathcal{V})).$$

Later we will use this in the case $\mathcal{A} = \mathcal{B} = \mathcal{X} = \mathcal{Y}$.

4. Centralizer $\mathbf{Z}_{\mathcal{A}}(\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$

Let $\mathcal{A}$ be a tensor category. The functor category $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ becomes a tensor category (Section 7) and has the center $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$. When $\mathcal{A}$ is rigid, the center is isomorphic to the centralizer $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ (Section 8). Our purpose is to show the equivalence

$${}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}} \simeq \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})).$$

In what follows we assume that $\mathcal{A}$ is left and right rigid, and we choose for each $A \in \mathcal{A}$ dualities

$$(A^c, A, \epsilon_A\colon A^cA \to I, \eta_A\colon I \to AA^c)$$

and

$$(A, A^d, \epsilon_A\colon AA^d \to I, \eta_A\colon I \to A^dA).$$

Though we use the same letters ϵ_A, η_A for different morphisms, it will not cause confusion. We further assume that the natural isomorphisms

$$(AB)^c \cong B^cA^c, \quad I^c \cong I, \quad (AB)^d \cong B^dA^d, \quad I^d \cong I$$

are all identities.

As $\mathcal{A}$ is an $(\mathcal{A}, \mathcal{A})$-bimodule, the category $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ becomes an $(\mathcal{A}, \mathcal{A})$-bimodule by the recipe of Section 3. So we have the centralizer

$\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$. In this section we describe an object of $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ in two ways.

For $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ and a morphism f of $\mathcal{A}$, we write $f^* = F(f)$. Recall from Section 3 that for $A \in \mathcal{A}$ and $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ the objects AF, $FA \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ are defined by

$$(AF)(X) = F(A^c X), \quad (FA)(X) = F(XA^d)$$

for $X \in \mathcal{A}$. Recall also that an object of $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ is an object $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ equipped with a family of isomorphisms $\omega_A \colon AF \to FA$ for all A satisfying (1.4.i)–(1.4.iii). The isomorphism ω_A is in itself a family of isomorphisms

$$(\omega_A)_X \colon (AF)(X) = F(A^c X) \to F(XA^d) = (FA)(X)$$

for all X which are natural in X. (1.4.i)–(1.4.iii) are rephrased into the following:

(4.1.i) For a morphism $f \colon A \to B$ of $\mathcal{A}$ the diagram

$$\begin{tikzcd} F(A^c X) \arrow[r, "(\omega_A)_X"] \arrow[d, "(f^c 1)^*"'] & F(XA^d) \arrow[d, "(1 f^d)^*"] \\ F(B^c X) \arrow[r, "(\omega_B)_X"'] & F(XB^d) \end{tikzcd}$$

is commutative.

(4.1.ii) The diagram

$$\begin{tikzcd} F(B^c A^c X) \arrow[r, "(\omega_B)_{A^c X}"] \arrow[dr, "(\omega_{AB})_X"'] & F(A^c X B^d) \arrow[d, "(\omega_A)_{XB^d}"] \\ & F(XB^d A^d) \end{tikzcd}$$

is commutative.

(4.1.iii) $(\omega_I)_X = 1$.

Thus we may say an object of $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ is an object $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ equipped with a family of isomorphisms $(\omega_A)_X \colon F(A^c X) \to F(XA^d)$ which are natural in X and satisfy (4.1.i)–(4.1.iii).

Let us give another description of $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$. Let $F \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$

be a functor. For $A \in \mathcal{A}$ define the functor $F^A \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ by

$$F^A(X) = F(AXA^d).$$

Proposition 4.2 *Let $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$. There is a one-to-one correspondence between the following two objects.*

- *A family of isomorphisms $\omega_A \colon AF \to FA$ in* $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ *for all $A \in \mathcal{A}$ satisfying* (4.1.i)–(4.1.iii).
- *A family of morphisms $\gamma_A \colon F \to F^A$ in* $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ *for all $A \in \mathcal{A}$ satisfying the following conditions.*
  (i) *For a morphism $f \colon A \to B$ of $\mathcal{A}$ the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "(\gamma_A)_X"] \arrow[d, "(\gamma_B)_X"'] & F(AXA^d) \arrow[d, "(11f^d)^*"] \\ F(BXB^d) \arrow[r, "(f11)^*"'] & F(AXB^d) \end{tikzcd}$$

  commutes.
  (ii) *For A, $B \in \mathcal{A}$ the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "(\gamma_A)_X"] \arrow[dr, "(\gamma_{BA})_X"'] & F(AXA^d) \arrow[d, "(\gamma_B)_{AXA^d}"] \\ & F(BAXA^dB^d) \end{tikzcd}$$

  commutes.
  (iii) $\gamma_I = 1$.

We call the family of such γ_A the *conjugate structure*. The proposition says that there is a one-to-one correspondence between central structures and conjugate structures on F.

Proof. (a) Construction of $\omega \mapsto \gamma$: Suppose that a family ω is given. Define $(\gamma_A)_X$ to be the composite

$$F(X) \xrightarrow{(\epsilon_A 1)^*} F(A^c AX) \xrightarrow{(\omega_A)_{AX}} F(AXA^d).$$

Let us verify (i)–(iii).
Proof of (i): Let $f \colon A \to B$ be a morphism. We have commutative dia-

grams

$$\begin{tikzcd} F(X) \arrow[r, "(\epsilon_A 1)^*"] \arrow[dr, "\gamma_A"'] & F(A^c AX) \arrow[r, "(f^c 11)^*"] \arrow[d, "\omega_A"] & F(B^c AX) \arrow[d, "\omega_B"] \\ & F(AXA^d) \arrow[r, "(11f^d)^*"'] & F(AXB^d), \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "(\epsilon_B 1)^*"] \arrow[dr, "\gamma_B"'] & F(B^c BX) \arrow[r, "(1f1)^*"] \arrow[d, "\omega_B"] & F(B^c AX) \arrow[d, "\omega_B"] \\ & F(BXB^d) \arrow[r, "(f11)^*"'] & F(AXB^d), \end{tikzcd}$$

and

$$\begin{tikzcd} F(X) \arrow[r, "(\epsilon_A 1)^*"] \arrow[d, "(\epsilon_B 1)^*"'] & F(A^c AX) \arrow[d, "(f^c 11)^*"] \\ F(B^c BX) \arrow[r, "(1f1)^*"'] & F(B^c AX). \end{tikzcd}$$

It follows that

$$\begin{tikzcd} F(X) \arrow[r, "\gamma_A"] \arrow[d, "\gamma_B"'] & F(AXA^d) \arrow[d, "(11f^d)^*"] \\ F(BXB^d) \arrow[r, "(f11)^*"'] & F(AXB^d) \end{tikzcd}$$

is commutative.

Proof of (ii): We have a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "(\epsilon_B 1)^*"] \arrow[dr, "\gamma_B"'] & F(B^c BX) \arrow[r, "(1\epsilon_A 11)^*"] \arrow[d, "\omega_B"'] & F(B^c A^c ABX) \arrow[d, "\omega_B"] \\ & F(BXB^d) \arrow[r, "(\epsilon_A 111)^*"] \arrow[dr, "\gamma_A"'] & F(A^c ABXB^d) \arrow[d, "\omega_A"] \\ & & F(ABXB^d A^d). \end{tikzcd}$$

The composition of the upper horizontal arrows equals $(\epsilon_{AB} 1)^*$, and the composition of the right vertical arrows equals ω_{AB}. The composition of these equals γ_{AB}. Hence $\gamma_A \circ \gamma_B = \gamma_{AB}$.

(iii) is obvious.

(b) Construction of $\gamma \mapsto \omega$. Suppose that a family γ is given. Define $(\omega_A)_X$, $(\omega'_A)_X$ to be the composites

$$\begin{aligned}
(\omega_A)_X \colon F(A^cX) &\xrightarrow{\gamma_A} F(AA^cXA^d) \xrightarrow{(\eta_A 11)^*} F(XA^d),\\
(\omega'_A)_X \colon F(XA^d) &\xrightarrow{\gamma_{A^c}} F(A^cXA^d(A^c)^d)\\
&\qquad\qquad = F(A^cX(A^cA)^d) \xrightarrow{(11\epsilon_A^d)^*} F(A^cX).
\end{aligned}$$

Let us verify (4.1.i)–(4.1.iii) and that ω_A and ω'_A are inverse to each other.

Proof of (4.1.i): Let $f\colon A \to B$ be a morphism. We have a commutative diagram

$$\begin{tikzcd}
F(A^cX) \arrow[rrr, "\gamma_A"] \arrow[d, "(f^c1)^*"'] \arrow[dr, "\gamma_B"] & & & F(AA^cXA^d) \arrow[dl, "(111f^d)^*"] \arrow[d, "(\eta_A 11)^*"] \\
F(B^cX) \arrow[dr, "\gamma_B"'] & F(BA^cXB^d) \arrow[r, "(f111)^*"] \arrow[d, "(1f^c11)^*"] & F(AA^cXB^d) \arrow[d, "(\eta_A 11)^*"] & F(XA^d) \arrow[dl, "(1f^d)^*"] \\
& F(BB^cXB^d) \arrow[r, "(\eta_B 11)^*"'] & F(XB^d). &
\end{tikzcd}$$

Hence the composites

$$F(A^cX) \xrightarrow{\gamma_A} F(AA^cXA^d) \xrightarrow{(\eta_A 11)^*} F(XA^d) \xrightarrow{(1f^d)^*} F(XB^d)$$

and

$$F(A^cX) \xrightarrow{(f^c1)^*} F(B^cX) \xrightarrow{\gamma_B} F(BB^cXB^d) \xrightarrow{(\eta_B 11)^*} F(XB^d)$$

are equal.

By the definition of ω, this means that

$$\begin{array}{ccc}
F(A^cX) & \xrightarrow{\omega_A} & F(XA^d) \\
{\scriptstyle (f^c1)^*}\big\downarrow & & \big\downarrow{\scriptstyle (1f^d)^*} \\
F(B^cX) & \xrightarrow[\omega_B]{} & F(XB^d)
\end{array}$$

is commutative.

Proof of (4.1.ii): We have a commutative diagram

$$\begin{tikzcd}
F(B^cA^cX) \arrow[r, "\gamma_B"] \arrow[dr, "\omega_B"'] & F(BB^cA^cXB^d) \arrow[r, "\gamma_A"] \arrow[d, "(\eta_B 111)^*"'] & F(ABB^cA^cXB^dA^d) \arrow[d, "(1\eta_B 1111)^*"] \\
 & F(A^cXB^d) \arrow[r, "\gamma_A"'] \arrow[dr, "\omega_A"'] & F(AA^cXB^dA^d) \arrow[d, "(\eta_A 111)^*"] \\
 & & F(XB^dA^d).
\end{tikzcd}$$

The composition of the upper horizontal arrows equals γ_{AB} and the composition of the right vertical arrows equals $(\eta_{AB}111)^*$. The composition of these equals ω_{AB}. Hence $(\omega_A)_{XB^d} \circ (\omega_B)_{A^cX} = (\omega_{AB})_X$.

(4.1.iii) is obvious.

Proof of $\omega'_A \circ \omega_A = 1$: We have a commutative diagram

$$\begin{tikzcd}
F(XA^d) \arrow[r, "\gamma_{A^c}"] \arrow[dr, "\omega'_A"'] & F(A^cXA^d(A^c)^d) \arrow[r, "\gamma_A"] \arrow[d, "(11\epsilon^d_A)^*"'] & F(AA^cXA^d(A^c)^dA^d) \arrow[d, "(111\epsilon^d_A 1)^*"] \\
 & F(A^cX) \arrow[r, "\gamma_A"'] \arrow[dr, "\omega_A"'] & F(AA^cXA^d) \arrow[d, "(\eta_A 11)^*"] \\
 & & F(XA^d).
\end{tikzcd}$$

By (ii) this results in a commutative diagram

$$\begin{tikzcd}
F(XA^d) \arrow[r, "\gamma_{AA^c}"] \arrow[d, "\omega_A \circ \omega'_A"'] & F(AA^cXA^d(A^c)^dA^d) \arrow[d, "(\eta_A 1111)^*"] \\
F(XA^d) & F(XA^d(A^c)^dA^d). \arrow[l, "(1\epsilon^d_A 1)^*"]
\end{tikzcd}$$

By (i) and (iii)

$$\begin{tikzcd}
F(XA^d) \arrow[r, "\gamma_{AA^c}"] \arrow[dr, "(11\eta^d_A)^*"'] & F(AA^cXA^d(A^c)^dA^d) \arrow[d, "(\eta_A 1111)^*"] \\
 & F(XA^d(A^c)^dA^d)
\end{tikzcd}$$

is commutative. Hence

$$\omega_A \circ \omega'_A = (1\epsilon_A^d 1)^* \circ (11\eta_A^d)^* = 1.$$

Proof of $\omega_A \circ \omega'_A = 1$: We have a commutative diagram

$$\begin{tikzcd}
F(A^cX) \arrow[r, "\gamma_A"] \arrow[dr, "\omega_A"'] & F(AA^cXA^d) \arrow[r, "\gamma_{A^c}"] \arrow[d, "(\eta_A 11)^*"'] & F(A^cAA^cXA^d(A^c)^d) \arrow[d, "(1\eta_A 111)^*"] \\
& F(XA^d) \arrow[r, "\gamma_{A^c}"'] \arrow[dr, "\omega'_A"'] & F(A^cXA^d(A^c)^d) \arrow[d, "(11\epsilon_A^d)^*"] \\
& & F(A^cX).
\end{tikzcd}$$

By (ii) this results in a commutative diagram

$$\begin{tikzcd}
F(A^cX) \arrow[r, "\gamma_{A^cA}"] \arrow[d, "\omega'_A \circ \omega_A"'] & F(A^cAA^cXA^d(A^c)^d) \arrow[d, "(1111\epsilon_A^d)^*"] \\
F(A^cX) & F(A^cAA^cX). \arrow[l, "(1\eta_A 1)^*"]
\end{tikzcd}$$

By (i) and (iii) the diagram

$$\begin{tikzcd}
F(A^cX) \arrow[r, "\gamma_{A^cA}"] \arrow[dr, "(\epsilon_A 11)^*"'] & F(A^cAA^cXA^d(A^c)^d) \arrow[d, "(1111\epsilon_A^d)^*"] \\
& F(A^cAA^cX)
\end{tikzcd}$$

is commutative. Hence

$$\omega'_A \circ \omega_A = (1\eta_A 1)^* \circ (\epsilon_A 11)^* = 1.$$

(c) Let us verify that the constructions of (a) and (b) are inverse to each other.

Firstly suppose that ω is given. Let $\omega \mapsto \gamma$. We have a commutative diagram

$$\begin{tikzcd}
F(A^cX) \arrow[r, "(\epsilon_A 11)^*"] \arrow[dr, "\gamma_A"'] & F(A^cAA^cX) \arrow[r, "(1\eta_A 1)^*"] \arrow[d, "\omega_A"] & F(A^cX) \arrow[d, "\omega_A"] \\
& F(AA^cXA^d) \arrow[r, "(\eta_A 11)^*"'] & F(XA^d).
\end{tikzcd}$$

Since the composition of the horizontal arrows is the identity, we have

$$\omega_A = (\eta_A 11)^* \circ \gamma_A.$$

This means that $\gamma \mapsto \omega$.

Secondly suppose that γ is given. Let $\gamma \mapsto \omega$. We have a commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "(\epsilon_A 1)^*"] \arrow[d, "\gamma_A"'] & F(A^c AX) \arrow[d, "\gamma_A"] \arrow[dr, "\omega_A"] & \\
F(AXA^d) \arrow[r, "(1\epsilon_A 11)^*"'] & F(AA^c AXA^d) \arrow[r, "(\eta_A 111)^*"'] & F(AXA^d).
\end{tikzcd}$$

Since the composition of the horizontal arrows is the identity, we have

$$\gamma_A = \omega_A \circ (\epsilon_A 1)^*.$$

This means that $\omega \mapsto \gamma$.

The proof is completed. □

5. Equivalence $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})) \simeq {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$

Theorem 5.1 *We have an equivalence*

$$\Delta\colon \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})) \to {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}.$$

Under this equivalence an object $F \in \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ *is mapped to an object* $L \in {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ *defined as follows: We have*

$$L(X, Y) = F(XY^d).$$

The operation $!A\colon L(X, Y) \to L(XA, YA)$ *is given by*

$$(1\epsilon_A 1)^*\colon F(XY^d) \to F(XAA^dY^d) = F(XA(YA)^d).$$

The operation $A!\colon L(X, Y) \to L(AX, AY)$ *is given by*

$$(\gamma_A)_{XY^d}\colon F(XY^d) \to F(AXY^dA^d),$$

where $\gamma_A\colon F \to F^A$ *is the conjugate structure of* F.

Proof. Applying Proposition 3.7 to the $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{X} = \mathcal{Y} = \mathcal{A}$, we have the isomorphism

$$ {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}} \cong \mathrm{Hom}_{\mathcal{A},\mathcal{A}}(\mathcal{A}, \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})). \tag{1} $$

Applying Proposition 1.5 to the $(\mathcal{A}, \mathcal{A})$-bimodule $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$, we have the equivalence

$$ \mathrm{Hom}_{\mathcal{A},\mathcal{A}}(\mathcal{A}, \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})) \simeq \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})). \tag{2} $$

Combining these, we obtain the equivalence

$$ {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}} \simeq \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})). $$

Suppose that an object $F \in \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ is mapped to an object $K \in \mathrm{Hom}_{\mathcal{A},\mathcal{A}}(\mathcal{A}, \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ under (2), and K is mapped to an object $L \in {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ under (1). Then we have

$$ K(Y) = FY $$

for $Y \in \mathcal{A}$, and

$$ L(X, Y) = K(Y)(X) $$

for $X, Y \in \mathcal{A}$, so

$$ L(X, Y) = (FY)(X) = F(XY^d). $$

By Proposition 1.5 the right $\mathcal{A}$-linear structure $\rho_{Y,A}\colon K(YA) \to K(Y)A$ of K is the identity $FYA \to FYA$. By Proposition 3.6 the operation

$$!A\colon L(X, Y) \to L(XA, YA) $$

is the map

$$ \begin{aligned} K(Y)(X) &\xrightarrow{K(Y)(1\epsilon_A)} K(Y)(XAA^d) \\ &= (K(Y)A)(XA) \xrightarrow{\rho_{Y,A}^{-1}} K(YA)(XA). \end{aligned} $$

This equals the map

$$ (1\epsilon_A 1)^*\colon F(XY^d) \to F(XAA^dY^d) = F(XA(YA)^d). $$

By Proposition 1.5 the inverse $\lambda_{A,Y}^{-1}\colon AK(Y) \to K(AY)$ of the left

$\mathcal{A}$-linear structure is given by

$$\omega_A 1 \colon AFY \to FAY,$$

where $\omega_A \colon AF \to FA$ is the central structure of F. By Proposition 3.5 the operation $A! \colon L(X, Y) \to L(AX, AY)$ is the map

$$K(Y)(X) \xrightarrow{K(Y)(\epsilon_A 1)} K(Y)(A^c AX)$$
$$= (A(K(Y)))(AX) \xrightarrow{\lambda^{-1}_{A, Y}} K(AY)(AX).$$

This equals the map

$$F(XY^d) \xrightarrow{(\epsilon_A 111)^*} F(A^c AXY^d) \xrightarrow{\omega_A} F(AXY^d A^d),$$

which is identical to the map $(\gamma_A)_{XY^d}$ by (a) of the proof of Proposition 4.2.

The proof is completed. $\square$

6. Tensor product in $\mathbf{D}(\mathcal{X}, \mathcal{X})$

We first review the definition of the tensor product (called also the composition) of distributors ([1]). Let $\mathcal{X}$ be a category. Let $L, M, N \in \mathbf{D}(\mathcal{X}, \mathcal{X})$. A *bilinear morphism* $\pi \colon (L, M) \to N$ is a family of linear maps

$$\pi_{X, Y, Z} \colon L(X, Y) \otimes M(Y, Z) \to N(X, Z)$$

for all $X, Y, Z \in \mathcal{X}$ satisfying the following conditions.

(i) $\pi_{X, Y, Z}$ is natural in X, Y, and Z.

(ii) If $g \colon Y \to Y'$ is a morphism of $\mathcal{X}$, then the digram

$$\begin{array}{ccc}
L(X, Y) \otimes M(Y', Z) & \xrightarrow{L(1, g) \otimes 1} & L(X, Y') \otimes M(Y', Z) \\
{\scriptstyle 1 \otimes M(g, 1)} \downarrow & & \downarrow {\scriptstyle \pi_{X, Y', Z}} \\
L(X, Y) \otimes M(Y, Z) & \xrightarrow[\pi_{X, Y, Z}]{} & N(X, Z)
\end{array}$$

is commutative.

Given $L, M \in \mathbf{D}(\mathcal{X}, \mathcal{X})$, there is a bilinear morphism $\pi \colon (L, M) \to N$ having the universal property: if $\pi' \colon (L, M) \to N'$ is a bilinear morphism, there exists a unique morphism $f \colon N \to N'$ such that $\pi'_{X, Y, Z} = f_{X, Z} \circ \pi_{X, Y, Z}$ for all X, Y, Z. One may construct such an N as

$$N(X, Z) = \text{Coequalizer}\Big(\bigoplus_{g\colon Y\to Y'} L(X, Y)\otimes M(Y', Z) \rightrightarrows \bigoplus_{Y} L(X, Y)\otimes M(Y, Z)\Big),$$

where the two arrows have components $L(1, g)\otimes 1$ and $1\otimes M(g, 1)$. We choose a universal bilinear morphism $\pi\colon (L, M)\to N$ and write $L\otimes M = N$.

The hom-functor

$$\mathrm{Hom}\colon \mathcal{X}^{\mathrm{op}}\times\mathcal{X}\to\mathcal{V}\colon (X, Y)\mapsto \mathrm{Hom}(X, Y)$$

is a distributor on $\mathcal{X}$. This has the property

$$L\otimes\mathrm{Hom}\cong L\cong \mathrm{Hom}\otimes L$$

for any $L\in\mathbf{D}(\mathcal{X}, \mathcal{X})$. These isomorphisms are given by

$$\pi(x\otimes 1_Y)\leftrightarrow x\leftrightarrow \pi(1_X\otimes x)$$

for $x\in L(X, Y)$.

With the above tensor product and the unit object Hom, the category $\mathbf{D}(\mathcal{X}, \mathcal{X})$ becomes a tensor category.

Let $\mathcal{X}$ be an $(\mathcal{A}, \mathcal{A})$-bimodule. Let $L, M, N\in {}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{X})_{\mathcal{A}}$ and let $\pi\colon (L, M)\to N$ be a bilinear morphism. We say π is $(\mathcal{A}, \mathcal{A})$-linear if the diagram

$$\begin{tikzcd}
L(X, Y)\otimes M(Y, Z) \arrow[r, "\pi_{X, Y, Z}"] \arrow[d, "A!\otimes A!"'] & N(X, Z) \arrow[d, "A!"] \\
L(AX, AY)\otimes M(AY, AZ) \arrow[r, "\pi_{AX, AY, AZ}"'] & N(AX, AZ)
\end{tikzcd}$$

is commutative and a similar diagram for $!A$ is commutative for all A, X, Y, Z.

Given $L, M\in {}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{X})_{\mathcal{A}}$, the object $L\otimes M\in\mathbf{D}(\mathcal{X}, \mathcal{X})$ naturally admits two-sided action of $\mathcal{A}$ so that $L\otimes M\in {}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{X})_{\mathcal{A}}$ and the universal bilinear morphism $\pi\colon (L, M)\to L\otimes M$ is $(\mathcal{A}, \mathcal{A})$-linear.

With this tensor product the category ${}_{\mathcal{A}}\mathbf{D}(\mathcal{X}, \mathcal{X})_{\mathcal{A}}$ becomes a tensor category.

7. Tensor product in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$

In this section we first review the definition of the tensor product in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ (Day's product [2]) and then examine some isomorphisms of associativity. They are needed later for describing the tensor product in the centralizer $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$.

Let $F, G, H \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$. A *bilinear morphism* $\pi\colon (F, G) \to H$ is a family of linear maps

$$\pi_{X,Y}\colon F(X) \otimes G(Y) \to H(XY)$$

for all $X, Y \in \mathcal{A}$ which are natural in X and Y.

Given $F, G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$, there is a universal bilinear morphism $\pi\colon (F, G) \to F \otimes G$. A construction is given by

$$(F \otimes G)(Z) = \varinjlim F(X) \otimes G(Y),$$

where the limit is taken over morphisms $Z \to XY$ of $\mathcal{A}$.

For $A \in \mathcal{A}$ let $h_A\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ denote the representable functor $X \mapsto \mathrm{Hom}(X, A)$. The bilinear morphism $(h_A, h_B) \to h_{AB}$ given by

$$\mathrm{Hom}(X, A) \otimes \mathrm{Hom}(Y, B) \to \mathrm{Hom}(XY, AB)\colon f \otimes g \mapsto fg$$

yields the isomorphism $h_A \otimes h_B \cong h_{AB}$.

For $F, G, H \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ we similarly define the tensor product $F \otimes G \otimes H$ with universal trilinear morphism $(F, G, H) \to F \otimes G \otimes H$. We have the canonical isomorphisms

$$(F \otimes G) \otimes H \cong F \otimes G \otimes H \cong F \otimes (G \otimes H).$$

The object h_I has the property

$$h_I \otimes F \cong F \cong F \otimes h_I$$

for any $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$.

With this tensor product and the unit object h_I, the category $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ becomes a tensor category.

In Section 3 we defined AF and FA for $A \in \mathcal{A}$ and $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$. We now interpret them in terms of tensor product in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$.

The universal bilinear morphism $(F, G) \to F \otimes G$ and the universal trilinear morphism $(F, G, H) \to F \otimes G \otimes H$ are always denoted by π.

Proposition 7.1 *Let $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ and $A \in \mathcal{A}$. We have an isomorphism*

$$h_A \otimes F \cong AF.$$

This isomorphism takes an element $b \in (AF)(X) = F(A^c X)$ to the element

$$(X \xrightarrow{\eta_A 1} AA^c X)^* \pi_{A,\, A^c X}(1_A \otimes b) \in (h_A \otimes F)(X),$$

and conversely takes an element

$$\pi_{Y,\, Z}((Y \xrightarrow{f} A) \otimes c) \in (h_A \otimes F)(YZ)$$

for $f \in \mathrm{Hom}(Y, A)$ and $c \in F(Z)$ to the element

$$(A^c YZ \xrightarrow{1f1} A^c AZ \xrightarrow{\epsilon_A 1} Z)^*(c) \in (AF)(YZ).$$

Proof. For an object $B \in \mathcal{A}$, let $L_B \colon \mathcal{A} \to \mathcal{A}$ be the functor $X \mapsto BX$. Let $L_B^* \colon \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}) \to \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ be the functor $F \mapsto F \circ L_B$. Since L_{A^c} is a left adjoint of L_A, the functor $L_{A^c}^*$ is a left adjoint of L_A^*. Thus we have a one-to-one correspondence

$$(\text{morphism } \phi \colon F \circ L_{A^c} \to G) \leftrightarrow (\text{morphism } \psi \colon F \to G \circ L_A) \quad (1)$$

in which

$$\phi_Z = (F(A^c Z) \xrightarrow{\psi_{A^c Z}} G(AA^c Z) \xrightarrow{(\eta_A 1)^*} G(Z))$$

and

$$\psi_Z = (F(Z) \xrightarrow{(\epsilon_A 1)^*} F(A^c AZ) \xrightarrow{\phi_{AZ}} G(AZ))$$

for $Z \in \mathcal{A}$.

As in Yoneda's lemma we have also a one-to-one correspondence

$$(\text{bilinear morphism } \theta \colon (h_A, F) \to G) \leftrightarrow (\text{morphism } \psi \colon F \to G \circ L_A) \quad (2)$$

in which

$$\theta_{Y,\, Z}((Y \xrightarrow{f} A) \otimes c) = (YZ \xrightarrow{f1} AZ)^*(\psi_Z(c))$$

for $f \in \mathrm{Hom}(Y, A)$ and $c \in F(Z)$.

Combining (1) and (2), we have a one-to-one correspondence

$$\begin{aligned}&(\text{bilinear morphism } \theta\colon (h_A,\, F) \to G)\\ &\qquad\qquad\leftrightarrow (\text{morphism } \phi\colon F \circ L_{A^c} \to G)\end{aligned}$$

in which

$$\theta_{Y,\,Z}((Y \xrightarrow{f} A) \otimes c) = \phi_{YZ}\big((A^cYZ \xrightarrow{1f1} A^cAZ \xrightarrow{\epsilon_A 1} Z)^*(c)\big)$$

for $f \in \mathrm{Hom}(Y,\, A)$ and $c \in F(Z)$. Also $F \circ L_{A^c} = AF$. When ϕ is the identity, the corresponding $\theta\colon (h_A,\, F) \to AF$ is given by

$$\theta_{Y,\,Z}((Y \xrightarrow{f} A) \otimes c) = (A^cYZ \xrightarrow{1f1} A^cAZ \xrightarrow{\epsilon_A 1} Z)^*(c).$$

This means that we have an isomorphism $h_A \otimes F \cong AF$ taking the element

$$\pi_{Y,\,Z}((Y \xrightarrow{f} A) \otimes c) \in (h_A \otimes F)(YZ)$$

to

$$(A^cYZ \xrightarrow{1f1} A^cAZ \xrightarrow{\epsilon_A 1} Z)^*(c) \in (AF)(YZ).$$

For every $b \in (AF)(X)$ this isomorphism takes the element

$$(X \xrightarrow{\eta_A 1} AA^cX)^*\pi_{A,\,A^cX}(1_A \otimes b) \in (h_A \otimes F)(X)$$

to

$$(A^cX \xrightarrow{1\eta_A 1} A^cAA^cX \xrightarrow{\epsilon_A 11} A^cX)^*(b) = b.$$

This proves the proposition. □

The version for the right action is as follows.

Proposition 7.2 *Let $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\, \mathcal{V})$ and $A \in \mathcal{A}$. We have an isomorphism*

$$F \otimes h_A \cong FA.$$

This isomorphism takes an element $b \in (FA)(X) = F(XA^d)$ to the element

$$(X \xrightarrow{1\eta_A} XA^dA)^*\pi_{XA^d,\,A}(b \otimes 1_A) \in (F \otimes h_A)(X),$$

and conversely takes an element

$$\pi_{Y,Z}(c \otimes (Z \xrightarrow{f} A)) \in (F \otimes h_A)(YZ)$$

for $f \in \mathrm{Hom}(Z, A)$ *and* $c \in F(Y)$ *to the element*

$$(YZA^d \xrightarrow{1f1} YAA^d \xrightarrow{1\epsilon_A} Y)^*(c) \in (FA)(YZ).$$

Our next task is to describe explicitly the natural isomorphisms

$$AF \otimes G \cong A(F \otimes G), \ FA \otimes G \cong F \otimes AG, \ F \otimes GA \cong (F \otimes G)A.$$

Proposition 7.3 *Let* $F, G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ *and* $A \in \mathcal{A}$. *We have an isomorphism*

$$AF \otimes G \cong A(F \otimes G)$$

in which the element

$$(A^c X \xrightarrow{p} YZ)^* \pi_{Y,Z}(a \otimes c) \in (A(F \otimes G))(X)$$

for $p\colon A^c X \to YZ$, $a \in F(Y)$, $c \in G(Z)$ *is mapped to the element*

$$(X \xrightarrow{\eta_A 1} AA^c X \xrightarrow{1p} AYZ)^* \pi_{AY,Z}((A^c AY \xrightarrow{\epsilon_A 1} Y)^*(a) \otimes c) \in (AF \otimes G)(X),$$

and conversely the element

$$(X \xrightarrow{q} YZ)^* \pi_{Y,Z}(b \otimes c) \in (AF \otimes G)(X)$$

for $q\colon X \to YZ$, $b \in (AF)(Y)$, $c \in G(Z)$ *is mapped to the element*

$$(A^c X \xrightarrow{1q} A^c YZ)^* \pi_{A^c Y,Z}(b \otimes c) \in (A(F \otimes G))(X).$$

Proof. The natural isomorphism of associativity

$$(h_A \otimes F) \otimes G \cong h_A \otimes (F \otimes G)$$

and the isomorphism of Proposition 7.1 yield an isomorphism

$$AF \otimes G \cong A(F \otimes G).$$

We examine the correspondence of elements under this isomorphism.

(a) Under the isomorphism of Proposition 7.1 the element

$$(A^c X \xrightarrow{p} YZ)^* \pi_{Y,Z}(a \otimes c) \in (A(F \otimes G))(X)$$

for $a \in F(Y)$, $c \in G(Z)$ corresponds to the element

$$(X \xrightarrow{\eta_A 1} AA^cX)^* \pi_{A,\, A^cX}(1_A \otimes (A^cX \xrightarrow{p} YZ)^* \pi_{Y,\, Z}(a \otimes c)) \\ \in (h_A \otimes (F \otimes G))(X).$$

Under the isomorphism $h_A \otimes (F \otimes G) \cong h_A \otimes F \otimes G$ this element corresponds to the element

$$(X \xrightarrow{\eta_A 1} AA^cX \xrightarrow{1p} AYZ)^* \pi_{A,\, Y,\, Z}(1_A \otimes a \otimes c) \in (h_A \otimes F \otimes G)(X).$$

Thus we have an isomorphism $\alpha\colon A(F \otimes G) \to h_A \otimes F \otimes G$ given by

$$(A^cX \xrightarrow{p} YZ)^* \pi_{Y,\, Z}(a \otimes c) \\ \mapsto (X \xrightarrow{p_\sharp} AYZ)^* \pi_{A,\, Y,\, Z}(1_A \otimes a \otimes c),$$

where $p_\sharp = 1_A p \circ \eta_A 1\colon X \to AYZ$.

(b) Under the isomorphism $AF \otimes G \cong (h_A \otimes F) \otimes G$ the element

$$(X \xrightarrow{q} YZ)^* \pi_{Y,\, Z}(b \otimes c) \in (AF \otimes G)(X)$$

for $b \in (AF)(Y)$ and $c \in G(Z)$ corresponds to the element

$$(X \xrightarrow{q} YZ)^* \pi_{Y,\, Z}([(Y \xrightarrow{\eta_A 1} AA^cY)^* \pi_{A,\, A^cY}(1_A \otimes b)] \otimes c) \\ \in ((h_A \otimes F) \otimes G)(X).$$

Under the isomorphism $(h_A \otimes F) \otimes G \cong h_A \otimes F \otimes G$ this corresponds to the element

$$(X \xrightarrow{\eta_A q} AA^cYZ)^* \pi_{A,\, A^cY,\, Z}(1_A \otimes b \otimes c) \in (h_A \otimes F \otimes G)(X).$$

Thus we have an isomorphism $\beta\colon AF \otimes G \to h_A \otimes F \otimes G$ given by

$$(X \xrightarrow{q} YZ)^* \pi_{Y,\, Z}(b \otimes c) \\ \mapsto (X \xrightarrow{\eta_A q} AA^cYZ)^* \pi_{A,\, A^cY,\, Z}(1_A \otimes b \otimes c).$$

(c) To describe the isomorphism $\beta^{-1} \circ \alpha\colon A(F \otimes G) \to AF \otimes G$, take an element

$$x = (A^cX \xrightarrow{p} YZ)^* \pi_{Y,\, Z}(a \otimes c) \in (A(F \otimes G))(X)$$

for $a \in F(Y)$, $c \in G(Z)$. Put

$$y = (X \xrightarrow{p_\sharp} AYZ)^* \pi_{AY,\, Z}((A^cAY \xrightarrow{\epsilon_A 1} Y)^*(a) \otimes c)) \in (AF \otimes G)(X).$$

Then

$$\begin{aligned}\beta(y) &= (X \xrightarrow{\eta_A p_\sharp} AA^cAYZ)^*\pi_{A,\,A^cAY,\,Z}(1_A \otimes (A^cAY \xrightarrow{\epsilon_A 1} Y)^*(a) \otimes c)\\ &= (X \xrightarrow{p_\sharp} AYZ)^*(AYZ \xrightarrow{\eta_A 111} AA^cAYZ \xrightarrow{1\epsilon_A 11} AYZ)^*\\ &\qquad\qquad\qquad\qquad\qquad\qquad \pi_{A,\,Y,\,Z}(1_A \otimes a \otimes c)\\ &= (X \xrightarrow{p_\sharp} AYZ)^*\pi_{A,\,Y,\,Z}(1_A \otimes a \otimes c)\\ &= \alpha(x).\end{aligned}$$

Hence $\beta^{-1}\alpha(x) = y$.

(d) To describe the isomorphism $\alpha^{-1} \circ \beta \colon AF \otimes G \to A(F \otimes G)$, take an element

$$z = (X \xrightarrow{q} YZ)^*\pi_{Y,\,Z}(b \otimes c) \in (AF \otimes G)(X)$$

for $b \in (AF)(Y)$, $c \in G(Z)$. Put

$$w = (A^cX \xrightarrow{1q} A^cYZ)^*\pi_{A^cY,\,Z}(b \otimes c) \in (A(F \otimes G))(X).$$

Then

$$\begin{aligned}\alpha(w) &= (X \xrightarrow{(1q)_\sharp} AA^cYZ)^*\pi_{A,\,A^cY,\,Z}(1_A \otimes b \otimes c)\\ &= (X \xrightarrow{\eta_A q} AA^cYZ)^*\pi_{A,\,A^cY,\,Z}(1_A \otimes b \otimes c)\\ &= \beta(z).\end{aligned}$$

Hence $\alpha^{-1}\beta(z) = w$.

Thus $\alpha^{-1} \circ \beta$ is the desired isomorphism. □

A version for the right action is analogously obtained.

Proposition 7.4 *We have an isomorphism*

$$F \otimes GA \cong (F \otimes G)A$$

in which the element

$$(XA^d \xrightarrow{p} YZ)^*\pi_{Y,\,Z}(a \otimes c) \in ((F \otimes G)A)(X)$$

for $a \in F(Y)$, $c \in G(Z)$ is mapped to the element

$$(X \xrightarrow{1\eta} XA^dA \xrightarrow{p1} YZA)^*\pi_{Y,\,ZA}(a \otimes (ZAA^d \xrightarrow{1\epsilon} Z)^*(c))$$
$$\in (F \otimes GA)(X),$$

and conversely the element

$$(X \xrightarrow{q} YZ)^*\pi_{Y,\,Z}(a \otimes d) \in (F \otimes GA)(X)$$

for $a \in F(Y)$, $d \in (GA)(Z)$ *is mapped to the element*

$$(XA^d \xrightarrow{q1} YZA^d)^*\pi_{Y,\,ZA^d}(a \otimes d) \in ((F \otimes G)A)(X).$$

Proposition 7.5 *We have an isomorphism*

$$FA \otimes G \cong F \otimes AG$$

in which the element

$$(X \xrightarrow{p} YZ)^*\pi_{Y,\,Z}(b \otimes c) \in (FA \otimes G)(X)$$

for $b \in (FA)(Y)$, $c \in G(Z)$ *is mapped to the element*

$$(X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YA^dAZ)^*\pi_{YA^d,\,AZ}(b \otimes (A^cAZ \xrightarrow{\epsilon 1} Z)^*(c)) \\ \in (F \otimes AG)(X),$$

and conversely the element

$$(X \xrightarrow{p} YZ)^*\pi_{Y,\,Z}(a \otimes d) \in (F \otimes AG)(X)$$

for $a \in F(Y)$, $d \in (AG)(Z)$ *is mapped to the element*

$$(X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YAA^cZ)^*\pi_{YA,\,A^cZ}((YAA^d \xrightarrow{1\epsilon} Y)^*(a) \otimes d) \\ \in (FA \otimes G)(X).$$

Proof. (a) The isomorphism of Proposition 7.2 and the canonical isomorphism yield the isomorphism

$$FA \otimes G \to (F \otimes h_A) \otimes G \to F \otimes h_A \otimes G.$$

Denote this composite by α. It effects as

$$(X \xrightarrow{p} YZ)^*\pi_{Y,\,Z}(b \otimes c)$$
$$\mapsto (X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YA^dAZ)^*\pi_{YA^d,\,A,\,Z}(b \otimes 1_A \otimes c)$$

for $b \in (FA)(Y)$, $c \in G(Z)$.

(b) The isomorphism of Proposition 7.1 and the canonical isomorphism yield the isomorphism

$$F \otimes AG \to F \otimes (h_A \otimes G) \to F \otimes h_A \otimes G.$$

Denote this composite by β. It effects as

$$\begin{aligned}
&(X \xrightarrow{p} YZ)^* \pi_{Y,\,Z}(a \otimes d) \\
&\mapsto (X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YAA^cZ)^* \pi_{Y,\,A,\,A^cZ}(a \otimes 1_A \otimes d)
\end{aligned}$$

for $a \in F(Y)$, $d \in (AG)(Z)$.

(c) To describe $\alpha^{-1} \circ \beta \colon F \otimes AG \to FA \otimes G$, take an element

$$x = (X \xrightarrow{p} YZ)^* \pi_{Y,\,Z}(a \otimes d) \in (F \otimes AG)(X)$$

for $a \in F(Y)$, $d \in (AG)(Z)$. Put

$$\begin{aligned}
y = (X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YAA^cZ)^* \pi_{YA,\,A^cZ}((YAA^d \xrightarrow{1\epsilon} Y)^*(a) \otimes d) \\
\in (FA \otimes G)(X).
\end{aligned}$$

Then

$$\begin{aligned}
\alpha(y) &= (X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YAA^cZ \xrightarrow{11\eta 11} YAA^dAA^cZ)^* \\
&\qquad \pi_{YAA^d,\,A,\,A^cZ}((YAA^d \xrightarrow{1\epsilon} Y)^*(a) \otimes 1_A \otimes d) \\
&= (X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YAA^cZ \xrightarrow{11\eta 11} YAA^dAA^cZ \xrightarrow{1\epsilon 111} YAA^cZ)^* \\
&\qquad \pi_{Y,\,A,\,A^cZ}(a \otimes 1_A \otimes d) \\
&= (X \xrightarrow{p} YZ \xrightarrow{1\eta 1} YAA^cZ)^* \pi_{Y,\,A,\,A^cZ}(a \otimes 1_A \otimes d) \\
&= \beta(x).
\end{aligned}$$

Hence $\alpha^{-1}\beta(x) = y$.

(d) The correspondence in the reverse direction is similarly described. Thus $\alpha^{-1} \circ \beta$ is the desired isomorphism. $\square$

8. Tensor product in $\mathbf{Z}_{\mathcal{A}}(\mathbf{Hom}(\mathcal{A}^{\mathrm{op}},\, \mathcal{V}))$

The purpose of this section is to describe the tensor product in the centralizer $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\, \mathcal{V}))$ induced from the tensor product in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\, \mathcal{V})$.

Proposition 8.1 *We have an isomorphism of categories*

$$\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})) \cong \mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})).$$

Proof. Let $F \in \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ with central structure $\omega_A \colon AF \to FA$ for $A \in \mathcal{A}$. We know the isomorphisms $h_A \otimes F \cong AF$ and $F \otimes h_A \cong FA$ of Propositions 7.1 and 7.2. Since representable functors form generators in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$, the morphisms ω_A for all $A \in \mathcal{A}$ give rise to morphisms $\omega_G \colon G \otimes F \to F \otimes G$ for all $G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$. Namely ω_G are natural in G and $\omega_{h_A} \colon h_A \otimes F \to F \otimes h_A$ corresponds to ω_A through the isomorphisms $h_A \otimes F \cong AF$ and $F \otimes h_A \cong FA$. Then F together with the family $(\omega_G)_G$ is an object of $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$. The correspondence $(\omega_A)_A \mapsto (\omega_G)_G$ of central structures on F gives the desired isomorphism of categories. $\square$

The center $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ is a tensor category (the end of Section 1). Its tensor product is defined as follows. Let F and G be objects of $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ with central structures

$$\omega_H \colon H \otimes F \to F \otimes H, \quad \omega_H \colon H \otimes G \to G \otimes H$$

for $H \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$. Then the tensor product of F and G in $\mathbf{Z}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ has the underlying functor $F \otimes G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ and the central structure

$$\omega_H \colon H \otimes (F \otimes G) \to (F \otimes G) \otimes H$$

given as the composite

$$\begin{aligned} &H \otimes (F \otimes G) \cong (H \otimes F) \otimes G \xrightarrow{\omega_H \otimes 1} (F \otimes H) \otimes G \\ &\cong F \otimes (H \otimes G) \xrightarrow{1 \otimes \omega_H} F \otimes (G \otimes H) \cong (F \otimes G) \otimes H. \end{aligned}$$

The centralizer $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ becomes a tensor category via the isomorphism of Proposition 8.1. Let F and G be objects of $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ with central structures

$$\omega_A \colon AF \to FA, \quad \omega_A \colon AG \to GA$$

for $A \in \mathcal{A}$. Then the tensor product of F and G in $\mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$ has the underlying functor $F \otimes G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$ and the central structure

$$\omega_A \colon A(F \otimes G) \to (F \otimes G)A$$

given as the composite

$$A(F \otimes G) \cong AF \otimes G \xrightarrow{\omega_A \otimes 1} FA \otimes G$$
$$\cong F \otimes AG \xrightarrow{1 \otimes \omega_A} F \otimes GA \cong (F \otimes G)A,$$

where the unlabeled isomorphisms are those of Propositions 7.3–7.5. This is obvious from the definition of those isomorphisms.

Proposition 8.2 *The map $\omega_A\colon (A(F \otimes G))(X) \to ((F \otimes G)A)(X)$ takes the element*

$$p^*\pi_{Y,Z}(a \otimes c)$$

for $p\colon A^cX \to YZ$, $a \in F(Y)$, $c \in G(Z)$ to the element

$$q^*\pi_{AYA^d,\,AZA^d}(b \otimes d),$$

where q is the composite

$$XA^d \xrightarrow{\eta 11} AA^cXA^d \xrightarrow{1p1} AYZA^d \xrightarrow{11\eta 11} AYA^dAZA^d,$$

b is the image of a under the map

$$F(Y) \xrightarrow{(\epsilon 1)^*} F(A^cAY) \xrightarrow{\omega_A} F(AYA^d),$$

and d is the image of c under the map

$$G(Z) \xrightarrow{(\epsilon 1)^*} G(A^cAZ) \xrightarrow{\omega_A} G(AZA^d).$$

Proof. We follow the definition of $\omega_A\colon A(F \otimes G) \to (F \otimes G)A$. The isomorphism $A(F \otimes G) \cong AF \otimes G$ of Proposition 7.3 takes the element

$$x = p^*\pi_{Y,Z}(a \otimes c) \in A(F \otimes G)(X)$$

for $p\colon A^cX \to YZ$, $a \in F(Y)$, $c \in G(Z)$ to the element

$$x_1 = p_\sharp^*\pi_{AY,Z}(a' \otimes c) \in (AF \otimes G)(X),$$

where

$$p_\sharp\colon X \xrightarrow{\eta 1} AA^cX \xrightarrow{1p} AYZ,$$

and a' is the image of a under the map $F(Y) \xrightarrow{(\epsilon 1)^*} F(A^cAY)$. The map

$\omega_A \otimes 1\colon (AF \otimes G)(X) \to (FA \otimes G)(X)$ takes x_1 to the element

$$x_2 = p_\sharp^* \pi_{AY,\,Z}(b \otimes c) \in (FA \otimes G)(X),$$

where $b = \omega_A(a')$. The isomorphism $FA \otimes G \cong F \otimes AG$ of Proposition 7.5 takes x_2 to the element

$$x_3 = r^* \pi_{AYA^d,\,AZ}(b \otimes c') \in (F \otimes AG)(X),$$

where

$$r\colon X \xrightarrow{p_\sharp} AYZ \xrightarrow{11\eta1} AYA^dAZ,$$

and c' is the image of c under the map $G(Z) \xrightarrow{(\epsilon1)^*} G(A^cAZ)$. The map $1 \otimes \omega_A\colon (F \otimes AG)(X) \to (F \otimes GA)(X)$ takes x_3 to the element

$$x_4 = r^* \pi_{AYA^d,\,AZ}(b \otimes d) \in (F \otimes GA)(X),$$

where $d = \omega_A(c')$. Finally the isomorphism $F \otimes GA \cong (F \otimes G)A$ of Proposition 7.4 takes x_4 to the element

$$x_5 = q^* \pi_{AYA^d,\,AZA^d}(b \otimes d) \in ((F \otimes G)A)(X),$$

where

$$q\colon XA^d \xrightarrow{r1} AYA^dAZA^d.$$

Then

$$q\colon XA^d \xrightarrow{\eta11} AA^cXA^d \xrightarrow{1p1} AYZA^d \xrightarrow{11\eta11} AYA^dAZA^d$$

and

$$b = \omega_A(\epsilon1)^*(a), \quad d = \omega_A(\epsilon1)^*(c).$$

Thus the map $\omega_A\colon (A(F \otimes G))(X) \to ((F \otimes G)A)(X)$ takes x to x_5. This proves the proposition. $\square$

By Proposition 4.2 the central structures

$$\omega_A\colon AF \to FA, \quad \omega_A\colon AG \to GA$$

correspond to the conjugate structures

$$\gamma_A\colon F \to F^A, \quad \gamma_A\colon G \to G^A.$$

Define a morphism

$$\gamma_A \colon F \otimes G \to (F \otimes G)^A$$

so that the diagram

$$\begin{tikzcd}
F(Y) \otimes G(Z) \arrow[r, "\gamma_A \otimes \gamma_A"] \arrow[dd, "\pi_{Y,Z}"'] & F(AYA^d) \otimes G(AZA^d) \arrow[d, "\pi_{AYA^d,\, AZA^d}"] \\
 & (F \otimes G)(AYA^dAZA^d) \arrow[d, "(11\eta 11)^*"] \\
(F \otimes G)(YZ) \arrow[r, "\gamma_A"'] & (F \otimes G)(AYZA^d)
\end{tikzcd}$$

commutes for every Y, Z.

Proposition 8.3 *The morphism $\gamma_A \colon F \otimes G \to (F \otimes G)^A$ is the conjugate structure of $F \otimes G$ corresponding to $\omega_A \colon A(F \otimes G) \to (F \otimes G)A$.*

Proof. By the definition of the correspondence between ω and γ ((b) of the proof of Proposition 4.2), it is enough to show that the diagram

$$\begin{tikzcd}
(F \otimes G)(A^cX) \arrow[r, "\gamma_A"] \arrow[dr, "\omega_A"'] & (F \otimes G)(AA^cXA^d) \arrow[d, "(\eta 11)^*"] \\
 & (F \otimes G)(XA^d)
\end{tikzcd}$$

is commutative. Take an element $x = p^*\pi_{Y,Z}(a \otimes c) \in (F \otimes G)(A^cX)$ for $p \colon A^cX \to YZ$, $a \in F(Y)$, $c \in G(Z)$. Then

$$\gamma_A(x) = (AA^cXA^d \xrightarrow{1p1} AYZA^d \xrightarrow{11\eta 11} AYA^dAZA^d)^* \\ \pi_{AYA^d,\, AZA^d}(\gamma_A(a) \otimes \gamma_A(c))$$

by definition, so

$$(\eta 11)^*\gamma_A(x) = q^*\pi_{AYA^d,\, AZA^d}(\gamma_A(a) \otimes \gamma_A(c)),$$

where q is the composite

$$XA^d \xrightarrow{\eta 11} AA^cXA^d \xrightarrow{1p1} AYZA^d \xrightarrow{11\eta 11} AYA^dAZA^d.$$

This coincides with $\omega_A(x)$ by Proposition 8.2. $\square$

9. Tensor equivalence $\mathbf{Z}_{\mathcal{A}}(\mathbf{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})) \simeq {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$

The purpose of this section is to show the equivalence

$$\Delta\colon \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})) \to {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$$

of Theorem 5.1 preserves tensor products. This equivalence is a restriction of the functor

$$\Delta\colon \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}) \to \mathbf{D}(\mathcal{A}, \mathcal{A})$$

given by

$$\Delta F(X, Y) = F(XY^d)$$

for $F \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$. We first construct an isomorphism $\Delta F \otimes \Delta G \cong \Delta(F \otimes G)$ of $\mathbf{D}(\mathcal{A}, \mathcal{A})$ for every $F, G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$, and then show that this is an isomorphism of ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ if $F, G \in \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}))$.

Let $F, G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$. Let

$$\pi_{X,Y}\colon F(X) \otimes G(Y) \to (F \otimes G)(XY)$$

be the universal bilinear morphism. Define the map

$$\mu_{X,Y,Z}\colon \Delta F(X, Y) \otimes \Delta G(Y, Z) \to \Delta(F \otimes G)(X, Z)$$

to be the composite

$$F(XY^d) \otimes G(YZ^d) \xrightarrow{\pi} (F \otimes G)(XY^dYZ^d) \xrightarrow{(1\eta_Y 1)^*} (F \otimes G)(XZ^d).$$

Proposition 9.1 *There exists a unique morphism*

$$\xi\colon \Delta F \otimes \Delta G \to \Delta(F \otimes G)$$

of $\mathbf{D}(\mathcal{A}, \mathcal{A})$ *such that the diagram*

$$\begin{tikzcd}
\Delta F(X, Y) \otimes \Delta G(Y, Z) \arrow[r, "\pi_{X,Y,Z}"] \arrow[dr, "\mu_{X,Y,Z}"'] & (\Delta F \otimes \Delta G)(X, Z) \arrow[d, "\xi_{X,Z}"] \\
& \Delta(F \otimes G)(X, Z)
\end{tikzcd}$$

commutes, where π *is the universal bilinear morphism.*

Proof. It is enough to show that the maps $\mu_{X,Y,Z}$ form a bilinear morphism $(\Delta F, \Delta G) \to \Delta(F \otimes G)$. Let $g\colon Y_1 \to Y_2$ be a morphism. Put $H = F \otimes G$. We have the diagram

$$\begin{tikzcd}
F(XY_1^d) \otimes G(Y_2Z^d) \arrow[r, "(1g^d)^*\otimes 1"] \arrow[d, "1\otimes(g1)^*"'] \arrow[dr, "\pi"'] & F(XY_2^d) \otimes G(Y_2Z^d) \arrow[dr, "\pi"] & \\
F(XY_1^d) \otimes G(Y_1Z^d) \arrow[dr, "\pi"'] & H(XY_1^dY_2Z^d) \arrow[r, "(1g^d11)^*"] \arrow[d, "(11g1)^*"] & H(XY_2^dY_2Z^d) \arrow[d, "(1\eta 1)^*"] \\
 & H(XY_1^dY_1Z^d) \arrow[r, "(1\eta 1)^*"'] & H(XZ^d)
\end{tikzcd}$$

in which the three quadrangles are commutative. Hence the surrounding hexagon is commutative. This means that

$$\begin{tikzcd}
\Delta F(X, Y_1) \otimes \Delta G(Y_2, Z) \arrow[r, "g_*\otimes 1"] \arrow[d, "1\otimes g^*"'] & \Delta F(X, Y_2) \otimes \Delta G(Y_2, Z) \arrow[d, "\mu_{X, Y_2, Z}"] \\
\Delta F(X, Y_1) \otimes \Delta G(Y_1, Z) \arrow[r, "\mu_{X, Y_1, Z}"'] & \Delta H(X, Z)
\end{tikzcd}$$

is commutative. Thus μ is a bilinear morphism. □

Proposition 9.2 *For every $F, G \in \mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$, $\xi\colon \Delta F \otimes \Delta G \to \Delta(F \otimes G)$ is an isomorphism of $\mathbf{D}(\mathcal{A}, \mathcal{A})$.*

Proof. Since $\otimes$ is right exact and representable functors form generators in $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$, it is enough to show that

$$\xi\colon \Delta h_A \otimes \Delta h_B \to \Delta(h_A \otimes h_B)$$

is an isomorphism for every $A, B \in \mathcal{A}$.

For an object $A \in \mathcal{A}$ define $U_A \in \mathbf{D}(\mathcal{A}, \mathcal{A})$ by

$$U_A(X, Y) = \mathrm{Hom}(X, AY).$$

We have an isomorphism $\lambda\colon \Delta h_A \cong U_A$ given by

$$(\Delta h_A)(X, Y) = h_A(XY^d) = \mathrm{Hom}(XY^d, A) \cong \mathrm{Hom}(X, AY).$$

For $X, Y, Z \in \mathcal{A}$ we define a map

$$\nu_{X,Y,Z}\colon U_A(X, Y) \otimes U_B(Y, Z) \to U_{AB}(X, Z)$$

by

$$\nu_{X,Y,Z}((X \xrightarrow{f} AY) \otimes (Y \xrightarrow{g} BZ)) = (X \xrightarrow{f} AY \xrightarrow{1g} ABZ).$$

It is easy to see that the maps $\nu_{X,Y,Z}$ give a bilinear morphism

$$\nu\colon (U_A, U_B) \to U_{AB}.$$

We claim that ν is universal. To prove this, let

$$\pi'\colon (U_A, U_B) \to L$$

be a bilinear morphism. Then

$$\begin{aligned}&\pi'_{X,Y,Z}((X \xrightarrow{f} AY) \otimes (Y \xrightarrow{g} BZ))\\ &= \pi'_{X,BZ,Z}((X \xrightarrow{f} AY \xrightarrow{1g} ABZ) \otimes (BZ \xrightarrow{1} BZ)).\end{aligned}$$

Define $\phi\colon U_{AB} \to L$ by

$$\phi_{X,Z}(X \xrightarrow{h} ABZ) = \pi'_{X,BZ,Z}((X \xrightarrow{h} ABZ) \otimes (BZ \xrightarrow{1} BZ)).$$

Then $\pi'_{X,Y,Z} = \phi_{X,Z} \circ \nu_{X,Y,Z}$. This proves the claim. Therefore ν yields an isomorphism

$$\zeta\colon U_A \otimes U_B \to U_{AB}.$$

So we will know that

$$\xi\colon \Delta h_A \otimes \Delta h_B \to \Delta(h_A \otimes h_B)$$

is an isomorphism once we show that the diagram

$$\begin{tikzcd}
\Delta h_A \otimes \Delta h_B \arrow[r, "\xi"] \arrow[dd, "\lambda\otimes\lambda"'] & \Delta(h_A \otimes h_B) \arrow[d, "\Delta(\theta)"] \\
 & \Delta h_{AB} \arrow[d, "\lambda"] \\
U_A \otimes U_B \arrow[r, "\zeta"'] & U_{AB}
\end{tikzcd}$$

is commutative, where $\theta\colon h_A \otimes h_B \to h_{AB}$ is the canonical isomorphism of Section 7.

In order to show this, it suffices to show that the diagram

$$\begin{tikzcd}
\Delta h_A(X, Y) \otimes \Delta h_B(Y, Z) \arrow[r, "\mu_{X,Y,Z}"] \arrow[dd, "\lambda\otimes\lambda"'] & \Delta(h_A \otimes h_B)(X, Z) \arrow[d, "\Delta(\theta)"] \\
& \Delta h_{AB}(X, Z) \arrow[d, "\lambda"] \\
U_A(X, Y) \otimes U_B(Y, Z) \arrow[r, "\nu_{X,Y,Z}"'] & U_{AB}(X, Z)
\end{tikzcd} \tag{1}$$

is commutative for every $X, Y, Z \in \mathcal{A}$.

By the definition of μ and θ, the composite

$$\Delta h_A(X, Y) \otimes \Delta h_B(Y, Z) \xrightarrow{\mu_{X,Y,Z}} \Delta(h_A \otimes h_B)(X, Z) \xrightarrow{\Delta(\theta)} \Delta h_{AB}(X, Z)$$

is equal to the composite

$$\kappa\colon \operatorname{Hom}(XY^d, A) \otimes \operatorname{Hom}(YZ^d, B) \to \operatorname{Hom}(XY^dYZ^d, AB) \xrightarrow{(1\eta 1)^*} \operatorname{Hom}(XZ^d, AB),$$

where the first arrow is the tensor product of morphisms of $\mathcal{A}$. So it suffices to show the following diagram is commutative.

$$\begin{tikzcd}
\operatorname{Hom}(XY^d, A) \otimes \operatorname{Hom}(YZ^d, B) \arrow[r, "\kappa"] \arrow[d, "\lambda\otimes\lambda"'] & \operatorname{Hom}(XZ^d, AB) \arrow[d, "\lambda"] \\
\operatorname{Hom}(X, AY) \otimes \operatorname{Hom}(Y, BZ) \arrow[r, "\nu_{X,Y,Z}"'] & \operatorname{Hom}(X, ABZ).
\end{tikzcd}$$

Let $f'\colon XY^d \to A$ correspond to $f\colon X \to AY$ under the isomorphism λ, and $g'\colon YZ^d \to B$ correspond to $g\colon Y \to BZ$. We have the diagram

$$\begin{tikzcd}
& XY^dYZ^d \arrow[r, "f'g'"] \arrow[d, "f'11"] & AB \\
XZ^d \arrow[ur, "1\eta 1"] \arrow[r, "f1"'] & AYZ^d \arrow[ur, "1g'"'] \arrow[r, "1g1"'] & ABZZ^d \arrow[u, "11\epsilon"']
\end{tikzcd}$$

in which the three triangles are commutative. Hence the surrounding pentagon is commutative. This means that the map

$$\kappa(f' \otimes g') \colon XZ^d \xrightarrow{1\eta 1} XY^dYZ^d \xrightarrow{f'g'} AB$$

corresponds to the map

$$\nu(f \otimes g) \colon X \xrightarrow{f} AY \xrightarrow{1g} ABZ$$

under λ. This proves the commutativity of (1), and completes the proof. $\square$

Let

$$F,\, G \in \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\, \mathcal{V})).$$

Then

$$F \otimes G \in \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}},\, \mathcal{V}))$$

as defined in Section 8. By Theorem 5.1 the distributors $\Delta(F)$, $\Delta(G)$, and $\Delta(F \otimes G)$ admit two-sided $\mathcal{A}$-action.

Proposition 9.3 *The isomorphism*

$$\xi \colon \Delta(F) \otimes \Delta(G) \to \Delta(F \otimes G)$$

is a morphism of ${}_{\mathcal{A}}\mathbf{D}(\mathcal{A},\, \mathcal{A})_{\mathcal{A}}$.

Proof. We have to show that ξ commutes with the operations $A!$ and $!A$ for every $A \in \mathcal{A}$. For $A!$ we have to show that the diagram

$$\begin{array}{ccc}
(\Delta F \otimes \Delta G)(X,\, Y) & \xrightarrow{A!} & (\Delta F \otimes \Delta G)(AX,\, AY) \\
{\scriptstyle \xi}\big\downarrow & & \big\downarrow{\scriptstyle \xi} \\
\Delta(F \otimes G)(X,\, Y) & \xrightarrow[A!]{} & \Delta(F \otimes G)(AX,\, AY)
\end{array}$$

is commutative. By the definition of ξ it is enough to show that the diagram

$$\begin{array}{ccc}
(\Delta F)(X,\, Z) \otimes (\Delta G)(Z,\, Y) & \xrightarrow{A! \otimes A!} & (\Delta F)(AX,\, AZ) \otimes (\Delta G)(AZ,\, AY) \\
{\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\
\Delta(F \otimes G)(X,\, Y) & \xrightarrow[A!]{} & \Delta(F \otimes G)(AX,\, AY)
\end{array}$$

is commutative. By the definition of μ and the description of $A!$ in terms of γ_A in Theorem 5.1, this diagram reads

$$\begin{tikzcd}
F(XZ^d)\otimes G(ZY^d) \arrow[r, "\gamma_A\otimes\gamma_A"] \arrow[d, "\pi"'] & F(AXZ^dA^d)\otimes G(AZY^dZ^d) \arrow[d, "\pi"] \\
(F\otimes G)(XZ^dZY^d) \arrow[d, "(1\eta_Z 1)^*"'] & (F\otimes G)(AXZ^dA^dAZY^dA^d) \arrow[d, "(11\eta_{AZ}11)^*"] \\
(F\otimes G)(XY^d) \arrow[r, "\gamma_A"'] & (F\otimes G)(AXY^dA^d),
\end{tikzcd}$$

where π is the universal map. This is commutative by the description of γ_A for $F\otimes G$ in Proposition 8.3.

For $!A$ it is enough to show that the diagram

$$\begin{tikzcd}
(\Delta F)(X,\, Z)\otimes(\Delta G)(Z,\, Y) \arrow[r, "!A\otimes !A"] \arrow[d, "\mu"'] & (\Delta F)(XA,\, ZA)\otimes(\Delta G)(ZA,\, YA) \arrow[d, "\mu"] \\
\Delta(F\otimes G)(X,\, Y) \arrow[r, "!A"'] & \Delta(F\otimes G)(XA,\, YA)
\end{tikzcd}$$

is commutative. By the description of $!A$ in Theorem 5.1 this diagram reads

$$\begin{tikzcd}
F(XZ^d)\otimes G(ZY^d) \arrow[r, "(1\epsilon_A 1)^*\otimes(1\epsilon_A 1)^*"] \arrow[d, "\pi"'] & F(XAA^dZ^d)\otimes G(ZAA^dY^d) \arrow[d, "\pi"] \\
(F\otimes G)(XZ^dZY^d) \arrow[d, "(1\eta_Z 1)^*"'] & (F\otimes G)(XAA^dZ^dZAA^dY^d) \arrow[d, "(11\eta_{ZA}11)^*"] \\
(F\otimes G)(XY^d) \arrow[r, "(1\epsilon_A 1)^*"'] & (F\otimes G)(XAA^dY^d).
\end{tikzcd}$$

We are reduced to showing the commutativity of the diagram

$$\begin{tikzcd}
XZ^dZY^d & XAA^dZ^dZAA^dY^d \arrow[l, "1\epsilon_A 11\epsilon_A 1"'] \\
XY^d \arrow[u, "1\eta_Z 1"] & XAA^dY^d. \arrow[l, "1\epsilon_A 1"] \arrow[u, "11\eta_{ZA}11"']
\end{tikzcd}$$

But this follows from the commutative diagram

$$
\begin{tikzcd}
Z^dZ & AA^dZ^dZAA^d \arrow[l, "\epsilon_A 11 \epsilon_A"'] \\
I \arrow[u, "\eta_Z"] & AA^dAA^d \arrow[l, "\epsilon_A\epsilon_A"] \arrow[u, "11\eta_Z 11"'] \\
 & AA^d. \arrow[u, "1\eta_A 1"'] \arrow[ul, "\epsilon_A"]
\end{tikzcd}
$$

The proof is completed. □

From Propositions 9.2 and 9.3 we obtain

Theorem 9.4 *The equivalence* $\Delta\colon \mathbf{Z}_{\mathcal{A}}(\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})) \to {}_{\mathcal{A}}\mathbf{D}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ *preserves tensor products.*

References

[1] Borceux F., *Handbook of Categorical Algebra* 1. Cambridge University Press, Cambridge, 1994.

[2] Day B.J., *On closed categories of functors.* in: MacLane S. (Ed), Reports of the Midwest Category Seminar IV, Lecture Notes in Math. vol. 137, Springer, Berlin, 1970, pp. 1–38.

[3] Joyal A. and Street A., *Tortile Yang-Baxter operators in tensor categories.* Journal of Pure and Applied Algebra **71** (1991), 43–51.

[4] Kassel C., *Quantum Groups.* Springer, New York, 1995.

[5] Majid S., *Representations, duals and quantum doubles of monoidal categories.* Rend. Circ. Math. Palermo (2) Suppl. vol. 26, 1991, pp. 197–206.

[6] Tambara D., *A duality for modules over monoidal categories of representations of semisimple Hopf algebras.* Journal of Algebra **241** (2001), 515–547.

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