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# Weak solution of a singular semilinear elliptic problem

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**Abstract.** We study the singular semilinear elliptic equation $\Delta u + f(.,u) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$, $N \geq 3$. $f\colon \mathbb{R}^N \times (0,\infty) \to [0,\infty)$ is such that $f(.,u) \in L^1(\mathbb{R}^N)$ for $u > 0$ and $u \to f(x,u)$ is continuous and nonincreasing for a.e. $x$ in $\mathbb{R}^N$. We assume that there exists a subset $\Omega \subset \mathbb{R}^N$ with positive measure such that $f(x,u) > 0$ for $x \in \Omega$ and $u > 0$ and that $\int_{\mathbb{R}^N} f(x, c|x|^{2-N})dx < \infty$ for some $c > 0$. Then we show that there exists a unique solution $u$ in the Marcinkiewicz space $M^{N/(N-2)}(\mathbb{R}^N)$ such that $\Delta u \in L^1(\mathbb{R}^N)$, $u > 0$ a.e. in $\mathbb{R}^N$.

*Key words*: singular elliptic equation, weak solution.

## 1. Introduction

We study the semilinear elliptic equation

$$\Delta u + f(.,u) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 3$ and $f$ satisfies the following assumptions:

(H1) $f\colon \mathbb{R}^N \times (0,\infty) \to [0,\infty)$. For all $u > 0$, $x \to f(x,u)$ is in $L^1(\mathbb{R}^N)$, and $u \to f(x,u)$ is continuous and nonincreasing for a.e. $x$ in $\mathbb{R}^N$;

(H2) There exists $\Omega \subset \mathbb{R}^N$ with positive measure such that $f(x,u) > 0$ for $x \in \Omega$ and $u > 0$;

(H3) There exists $c > 0$ such that

$$\int_{\mathbb{R}^N} f(x, c|x|^{2-N})dx < +\infty.$$

**Definition 1** $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ is a solution of (1.1) if $u > 0$ a.e. in $\mathbb{R}^N$, $\Delta u \in L^1(\mathbb{R}^N)$ (in the sense of distributions) and

$$\Delta u(x) + f(x, u(x)) = 0 \quad \text{a.e. in } \mathbb{R}^N.$$

The aim of this paper is to give a general existence and uniqueness result under sufficiently weak conditions.

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The particular case

$$f(x,u) = p(x)u^{-\lambda} \quad x \in \mathbb{R}^N,\ u > 0, \tag{1.2}$$

where $\lambda > 0$ has been considered by several authors ([3, 4, 6, 7, 8, 9] and their references). More precisely Kusano and Swanson [7] treated the case $\lambda \in (0,1)$ when $p \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ and $p > 0$ in $\mathbb{R}^N \setminus 0$. They established the existence of a classical solution $u \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ under the following conditions:

(H4) There exists a constant $C > 0$ such that $C\phi(|x|) \le p(x)$ for $x \in \mathbb{R}^N$, where $\phi(t) = \max_{|x|=t} p(x)$, $t \ge 0$;

(H5) $\displaystyle\int_0^{+\infty} t^{N-1+\lambda(N-2)}\phi(t)\,dt < \infty.$

Moreover they showed that

$$m \le |x|^{N-2}u(x) \le M \quad |x| \ge R, \tag{1.3}$$

for some constants $R$, $M \ge m > 0$.

This result was first generalized to all $\lambda > 0$ in [3]. In both cases the upper and lower solution method was used.

Before going further we need a second definition.

**Definition 2** Let $1 < p < \infty$ and $1/p + 1/p' = 1$. The Marcinkiewicz space $M^p(\mathbb{R}^N)$ is the space of measurable functions $u$ on $\mathbb{R}^N$ such that $\|u\|_{M^p} < \infty$, where

$$\|u\|_{M^p} = \min\Big\{C \in [0,\infty];\ \int_K |u(x)|\,dx \le C|K|^{1/p'} \quad \forall K \subset \mathbb{R}^N \quad \text{measurable}\Big\}.$$

($|K|$ is the Lebesgue measure of $K$).

It is easy to verify that $M^p(\mathbb{R}^N)$ equipped with the $\|.\|_{M^p}$ norm is a Banach space. Moreover $M^p(\mathbb{R}^N)$ is continuously imbedded in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ (see [1]).

The regularity assumption on $p$ in (1.2) was weakened in [4] to

(H6) $p \in C(\mathbb{R}^N)$, $p(x) > 0$ for $x \in \mathbb{R}^N \setminus 0$.

If moreover

(H7) $\displaystyle\int_{\mathbb{R}^N} |x|^{\lambda(N-2)}p(x)\,dx < \infty,$

we established the existence of a unique solution $u$ in the Marcinkiewicz space $M^{N/(N-2)}(\mathbb{R}^N)$ satisfying $\Delta u \in L^1(\mathbb{R}^N)$, via the upper and lower solution method. Assumption (H4) was not required.

The positivity assumption on $p$ and the decay condition (H5) were relaxed in [8]. The authors proved the existence and uniqueness of a classical solution vanishing at infinity under the following hypotheses:

(H8) $p \in C^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ and wherever $p(x_0) = 0$ there exists $r > 0$ such that $p(x) > 0$ for $x \in \partial B(x_0, r)$ where $B(x_0, r)$ is the ball of radius $r$ centered at $x_0$;

(H9) $\displaystyle\int_0^{+\infty} t\phi(t)\,dt < \infty.$

It may be noted again that the above result does not require (H4).

Then Lair and Shaker [9] treated the term

$$f(x,u) = p(x)g(u),$$

where $p \in C(\mathbb{R}^N)$ is nontrivial and nonnegative and $g$ is such that $g'(s) \le 0$ and $g(s) > 0$ for $s > 0$. Assuming (H9) they established the existence of a unique positive solution $u \in D(\Delta)$ decaying to zero at infinity ($D(\Delta)$ denotes the domain of the Laplace operator $\Delta$, such that the images of its elements are in $C(\mathbb{R}^N)$).

Jin [6] considered the more general case $f(x,u)$ under some smoothness assumptions. Moreover the results obtained are complementary to the cases already mentioned. Finally Mâagli and Zribi [10] studied the case where $f(x,u)$ satisfies weaker regularity conditions. However their hypotheses, which are different from ours, lead to the existence and uniqueness of a continuous positive solution decaying to zero at infinity.

Now we can state our result.

**Theorem 1** *Let* $f\colon \mathbb{R}^N \times (0,\infty) \to [0,\infty)$*,* $N \ge 3$*, satisfy* (H1)–(H3)*. Then problem* (1.1) *has a unique solution* $u \in M^{N/(N-2)}(\mathbb{R}^N)$ *such that* $f(.,u) \in L^1(\mathbb{R}^N)$*,* $u > 0$ *a.e. in* $\mathbb{R}^N$*.*

In Section 2 we give a preliminary result. Theorem 1 is proved in Section 3.

## 2. A preliminary result

We recall a variant of Kato's inequality (see [2] Lemma A1 in the Appendix).

**Lemma 1** *Let $\Omega \subset \mathbb{R}^N$ be any open set. Let $v \in L^1_{\text{loc}}(\Omega)$ and $f \in L^1_{\text{loc}}(\Omega)$ be such that*

$$\Delta v \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

*Then*

$$\Delta v^+ \geq f \operatorname{sign}^+ v \quad \text{in } \mathcal{D}'(\Omega).$$

Now we shall prove the following lemma which is a slight extension of LEMMA A8 in [1].

**Lemma 2** *Let $v \in L^1_{\text{loc}}(\mathbb{R}^N)$ be such that $\Delta v \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$. If $v$ satisfies*

$$\lim_{n\to\infty} n^{-N} \int_{n\leq|y|\leq 2n} |v(x+y)|\, dy = 0 \tag{2.1}$$

*for all $x \in \mathbb{R}^N$, then $v \leq 0$ a.e. in $\mathbb{R}^N$.*

*Proof.* By Lemma 1 we have

$$\Delta v^+ \geq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

i.e. $v^+$ is subharmonic. If $w$ is a function defined on $\mathbb{R}^N$ and $a \in \mathbb{R}^N$, $\tau_a w$ denotes the translate of $w$ ($\tau_a w(x) = w(x-a)$). If $w$ is integrable on the sphere $S_R = \{x \in \mathbb{R}^N; |x| = R\}$ we will denote the average of $w$ over $S_R$ by $w_R$. Of course $v^+$ also satisfies (2.1). Let $x \in \mathbb{R}^N$ be fixed. Since the average of $v^+(x+y)$ over $n \leq |y| \leq 2n$ may be expressed as a weighted average of $(\tau_{-x}v^+)_r$ over $n \leq r \leq 2n$, (2.1) implies that there is a sequence $r_n \to \infty$ such that $(\tau_{-x}v^+)_{r_n} \to 0$. Since $v^+$ is subharmonic on $\mathbb{R}^N$, we deduce that

$$v^+(x) \leq (\tau_{-x}v^+)_{r_n} \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Letting $n \to \infty$ in the above inequality we get $v^+(x) = 0$ for a.e. $x \in \mathbb{R}^N$. The proof of the Lemma is complete.

**Remark 1** Notice that Lemma 2 is also valid for $N = 1$ or 2. If $v \in L^1(\mathbb{R}^N)$ or $v \in M^p(\mathbb{R}^N)$ for $1 < p < \infty$, then $v$ satisfies (2.1).

## 3. Proof of Theorem 1

1) Uniqueness. The proof is the same as in [4]. For completeness we provide the details. We shall need the following Lemma ([1, LEMMA A10]).

**Lemma 3** *Let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a nondecreasing function satisfying $p(0) = 0$. For $u \in M^{N/(N-2)}(\mathbb{R}^N)$ such that $\Delta u \in L^1(\mathbb{R}^N)$ we have*

$$\sqrt{p'(u)}\,|\mathrm{grad}\,u| \in L^2(\mathbb{R}^N),$$

*and*

$$\int p'(u)|\mathrm{grad}\,u|^2 + \int \Delta u.p(u) \leq 0.$$

Let $u_1,\ u_2 \in M^{N/(N-2)}(\mathbb{R}^N)$ be two solutions of problem (1.1) such that $\Delta u_j \in L^1(\mathbb{R}^N)$ for $j = 1, 2$. Let $u = u_1 - u_2$ and $v = \Delta u$. Then $u \in M^{N/(N-2)}(\mathbb{R}^N)$, $v \in L^1(\mathbb{R}^N)$ and $uv \geq 0$ a.e. in $\mathbb{R}^N$ by (H1). Now let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a strictly increasing function satisfying $p(0) = 0$. Then $p(u)v \geq 0$ a.e. in $\mathbb{R}^N$ and Lemma 3 implies that $|\mathrm{grad}\,u| = 0$. We deduce that $u$ is a constant function in $M^{N/(N-2)}(\mathbb{R}^N)$, hence $u = 0$.

2) Existence. We begin with the following Lemma.

**Lemma 4** *Let $j \in \mathbb{N}^\star$. There exists a unique $u_j \in M^{N/(N-2)}(\mathbb{R}^N)$ such that $f(.,u_j+1/j) \in L^1(\mathbb{R}^N)$, $u_j \geq 0$ a.e. in $\mathbb{R}^N$ and $\Delta u_j + f(.,u_j+1/j) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$.*

*Proof.* Define

$$\beta_j(x,u) = f\Big(x, \frac{1}{j}\Big) - f\Big(x, u + \frac{1}{j}\Big), \quad x \in \mathbb{R}^N,\ u \geq 0,$$

and

$$\beta_j(x,u) = 0, \quad x \in \mathbb{R}^N,\ u \leq 0.$$

Then we have:

- For all $u \in \mathbb{R}$, $x \to \beta_j(x,u)$ is in $L^1(\mathbb{R}^N)$;
- $\mathbb{R} \ni u \to \beta_j(x,u)$ is continuous and nondecreasing for a.e. $x$ in $\mathbb{R}^N$;
- $\beta_j(x,0) = 0$ for a.e. $x$ in $\mathbb{R}^N$.

Since $f(.,1/j) \in L^1(\mathbb{R}^N)$ Theorem 1 in [5] implies the existence of a unique $u_j \in M^{N/(N-2)}(\mathbb{R}^N)$ satisfying $\beta_j(.,u_j) \in L^1(\mathbb{R}^N)$ and

$$-\Delta u_j + \beta_j(.,u_j) = f\Big(.,\frac{1}{j}\Big) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).$$

Since $\Delta u_j \leq 0$ in $\mathcal{D}'(\mathbb{R}^N)$, Remark 1 and Lemma 2 imply that $u_j \geq 0$ a.e. in $\mathbb{R}^N$. Therefore we have $f(.,u_j + 1/j) \in L^1(\mathbb{R}^N)$ and

$$\Delta u_j + f\Big(.,u_j + \frac{1}{j}\Big) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N),$$

and the Lemma is proved.

Now let $E_N$ be defined by

$$E_N(x) = \frac{1}{(N-2)\Omega_N |x|^{N-2}}\,,$$

where $\Omega_N$ is the volume of the unit $N$-ball. We recall (see the appendix in [1]) that $E_N \in M^{N/(N-2)}(\mathbb{R}^N)$ and that for any $g \in L^1(\mathbb{R}^N)$, $v = E_N \star g \in M^{N/(N-2)}(\mathbb{R}^N)$ is the unique function in $M^{N/(N-2)}(\mathbb{R}^N)$ satisfying $-\Delta v = g$. We have the following Lemma.

**Lemma 5** *Let $g \in L^1(\mathbb{R}^N)$ with compact support. Then*

$$\lim_{|x|\to\infty} \frac{E_N \star g(x)}{E_N(x)} = \int_{\mathbb{R}^N} g(y)\,dy.$$

*Proof.* We have

$$\frac{E_N \star g(x)}{E_N(x)} = \int_{\mathbb{R}^N} \frac{|x|^{N-2}}{|x-y|^{N-2}}\, g(y)\,dy,$$

and the result easily follows from the Lebesgue dominated convergence theorem.

Now we define

$$c_j = \int_{\mathbb{R}^N} f\Big(x, u_j(x) + \frac{1}{j}\Big)\,dx, \quad j \in \mathbb{N}^\star.$$

**Lemma 6** *Let $j \in \mathbb{N}^\star$. Assume that $0 < \gamma < c_j$. Then there exist $R_j > 0$ and $a_j > 0$ such that*

$$u_j(x) \geq \frac{\gamma}{(N-2)\Omega_N |x|^{N-2}} \quad \textit{a.e. in } \{x \in \mathbb{R}^N; |x| \geq R_j\},$$

*and*

$$\operatorname{ess}\inf_{\{x;|x|\le R_j\}} u_j(x) > a_j.$$

*Proof.* Clearly (H2) implies that there exist $A_j > 0$ and $M_j > 0$ such that

$$\int_{|x|\le A_j} \min\Big(f\Big(x, u_j(x) + \frac{1}{j}\Big), M_j\Big)\, dx > \gamma. \tag{3.1}$$

Now define

$$\tilde f_j(x) = \min\Big(f\Big(x, u_j(x) + \frac{1}{j}\Big), M_j\Big)\, \mathbf{1}_{\{x;|x|\le A_j\}}(x), \quad x \in \mathbb{R}^N.$$

Since $-\Delta u_j \ge \tilde f_j$ a.e. in $\mathbb{R}^N$ we obtain $u_j \ge E_N \star \tilde f_j$ a.e. in $\mathbb{R}^N$. We have $E_N \star \tilde f_j \in C^1(\mathbb{R}^N)$ and $E_N \star \tilde f_j > 0$ on $\mathbb{R}^N$. By (3.1) and Lemma 5 there exists $R_j > 0$ such that

$$E_N \star \tilde f_j \ge \gamma E_N \text{ on } \{x \in \mathbb{R}^N; |x| \ge R_j\},$$

and the Lemma follows.

**Lemma 7** *For every $j \in \mathbb{N}^\star$ we have $u_j + 1/j \ge u_{j+1} + 1/(j+1)$ a.e. in $\mathbb{R}^N$.*

*Proof.* Let $u = \big(u_{j+1} + 1/(j+1)\big) - \big(u_j + 1/j\big)$. From Lemma 1 using (H1) we deduce that

$$\Delta u^+ \ge 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Now define $\Omega = \{x \in \mathbb{R}^N; u(x) > 0\}$. Since $u \le u_{j+1} - u_j$ a.e. in $\mathbb{R}^N$, we obtain that $u^+ \le \mathbf{1}_\Omega(u_{j+1} - u_j)$ a.e. in $\mathbb{R}^N$, hence $u^+ \in M^{N/(N-2)}(\mathbb{R}^N)$. Therefore Remark 1 and Lemma 2 imply that $u \le 0$ a.e. in $\mathbb{R}^N$ and the Lemma is proved. □

**Lemma 8** *For every $j \in \mathbb{N}^\star$ we have $u_j \le u_{j+1}$ a.e. in $\mathbb{R}^N$.*

*Proof.* Using (H1) and Lemma 7 we get

$$u_j - u_{j+1} = E_N \star \Big(f\Big(., u_j + \frac{1}{j}\Big) - f\Big(., u_{j+1} + \frac{1}{j+1}\Big)\Big) \le 0$$

a.e. in $\mathbb{R}^N$.

Now we claim that

$$\sup_{j\in\mathbb{N}^\star} c_j < \infty. \tag{3.2}$$

Indeed, assume the contrary. Then there exists $j_0 \in \mathbb{N}^\star$ such that $c_{j_0} > (N-2)\Omega_N c$ where $c$ is given in (H3). By Lemma 6 there exist $R_{j_0} > 0$ and $a_{j_0} > 0$ such that

$$u_{j_0}(x) \geq \frac{c}{|x|^{N-2}} \quad \text{for a.e. } x \in \{x \in \mathbb{R}^N; |x| \geq R_{j_0}\}, \tag{3.3}$$

and

$$\operatorname{ess}\inf_{\{x;|x|\leq R_{j_0}\}} u_{j_0}(x) > a_{j_0}. \tag{3.4}$$

Let $j \geq j_0$. Using (3.3), (3.4) and Lemma 8 we deduce that

$$c_j \leq \int_{|x|\leq R_{j_0}} f(x, a_{j_0})\, dx + \int_{|x|\geq R_{j_0}} f(x, c|x|^{2-N})\, dx,$$

and (H3) gives a contradiction.

Now we can prove the existence. By (H1) and Lemma 7 $j \to f(., u_j + 1/j)$ is nondecreasing. (3.2) and the Beppo Levi theorem for monotonic sequences imply that there exists $g \in L^1(\mathbb{R}^N)$ such that

$$f\Big(., u_j + \frac{1}{j}\Big) \to g \quad \text{in } L^1(\mathbb{R}^N) \quad \text{when } j \to \infty.$$

Therefore

$$u_j = E_N \star f\Big(., u_j + \frac{1}{j}\Big) \to E_N \star g = u \qquad \text{in } M^{N/(N-2)}(\mathbb{R}^N) \quad \text{when } j \to \infty$$

(see [1] Lemma A4) and

$$-\Delta u = g \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

By Lemma 8 and the Fischer-Riesz theorem $u_j \to u$ a.e. in $\mathbb{R}^N$. Lemma 6 and Lemma 8 imply that $u > 0$ a.e. in $\mathbb{R}^N$. Clearly we have $g = f(., u)$. The proof is complete. □

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