Hokkaido Mathematical Journal Vol. 39 (2010) p. 239–259

Limiting absorption principle for the second quantization of self-adjoint operators

Shoji Shimizu

(Received May 7, 2009; Revised February 1, 2010)

Abstract. In this paper we discuss the limiting absorption principle (l.a.p.) of the second quantization of semi-bounded self-adjoint operators. We show that the l.a.p. for a self-adjoint operator on a basic Hilbert space $\mathcal{H}$ is "inherited" to the one for its second quantization on a Fock space $\mathcal{F}(\mathcal{H})$. In order to show such a result, we examine the resolvent of n-body problem and take the limit of the infinite direct sum of those operators in a suitable subspace of $\mathcal{F}(\mathcal{H})$.

Key words: limiting absorption principle, resolvent estimates.

1. Introduction

We consider the limiting absorption principle for the second quantization $d\Gamma(T)$ of a semi-bounded self-adjoint operator T in a Hilbert space $\mathcal{H}$. The second quantization $d\Gamma(T)$ is defined as follows. Let $\otimes_n\mathcal{H}$ be a n-fold tensor product of $\mathcal{H}$ and define an operator

$$T^{(n)} := \sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \overbrace{T}^{j\text{-th}} \otimes I \otimes \cdots \otimes I \tag{1.1}$$

on $\otimes_n\mathcal{H}$. Then we define the second quantization of T by

$$d\Gamma(T) := \oplus_{n=0}^{\infty} T^{(n)}$$

on the Fock space $\mathcal{F}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_n \mathcal{H}$. It is known that $d\Gamma(T)$ is a self-adjoint operator (see e.g. Reed and Simon [8], [9]). Roughly speaking, our purpose is to show that the limiting absorption principle for T in $\mathcal{H}$ is "inherited" to one for its second quantization $d\Gamma(T)$ in a dense subset of the Fock space $\mathcal{F}(\mathcal{H})$.

In the following, we denote by $\sigma(T)$ the spectrum of a self-adjoint

2000 Mathematics Subject Classification : 47B15, 48B25.

operator T in $\mathcal{H}$ and by $R[T](z) \equiv (T-z)^{-1}$ the resolvent of T for $z \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T)$. It is well known that $\sigma(T)$ is contained in $\mathbb{R}$ and that $R[T](z)$ is an analytic function in $\rho(T)$, if T is a self-adjoint operator. Therefore, it is of special interest to compute the boundary values of the resolvent on the real axis. However, if $\mu \in \sigma(T)$, then $R[T](\mu \pm i\epsilon)$ diverges as $\epsilon \to 0+$ in the norm topology of $B(\mathcal{H})$, the space of bounded linear operators from $\mathcal{H}$ into itself. This is the reason why we need to consider a slightly different topology. To be more precise, we introduce the following notion.

Definition 1.1 Let $\mathcal{H}$ be a Hilbert space, and $X \subset \mathcal{H}$ be a dense, continuously embedded Banach space. We say that the self-adjoint operator T on $\mathcal{H}$ satisfies the limiting absorption principle (l.a.p. for short) in $K \subseteq \sigma(T)$ with respect to the norm topology of $B(X, X^*)$, if the limits

$$R^{\pm}[T](\mu) := \lim_{\epsilon \to 0+} R[T](\mu \pm i\epsilon)$$

exist in the norm topology of $B(X, X^*)$, uniformly in $\mu \in K$. Here $B(X, X^*)$ denotes the space of bounded linear operators of X into its dual X^*, and $i = \sqrt{-1}$.

Remark 1.2 When T satisfies the l.a.p., $R^{\pm}[T](\mu)$ are operator-valued continuous functions in μ w.r.t. the norm topology of $B(X, X^*)$, since $R[T]$ is continuous function in upper or lower half plane w.r.t. the norm topology of $B(X, X^*)$ and converges to uniformly $R^{\pm}[T](\mu)$.

As a typical example of $\mathcal{H}$ and X, we have in mind $\mathcal{H} \equiv L^2(\mathbb{R}^d)$, the space of square integrable functions, and $X \equiv L^{2,s}(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} \mid \int_{\mathbb{R}^d} |f(x)|^2 (1+|x|^2)^s dx < \infty\}$ with $s > 0$.

Now we state our main result of this paper:

Theorem 1.3 *Let $\mathcal{H}$ and X be as in Definition* 1.1. *Suppose that a self-adjoint operator T on $\mathcal{H}$ satisfies the following three conditions*:

(a) *There exists a constant $a < 0$ such that $T \geq a$ holds.*
(b) *T satisfies the l.a.p. in every compact interval in $(0, \infty)$ w.r.t. the norm topology of $B(X, X^*)$.*
(c) *For any $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(T)X \subseteq X$ and $\|\varphi(T)\|_{B(X)} \leq \sup_{x \in \mathbb{R}} |\varphi(x)|$ hold.*

Then, for any positive numbers λ and θ with $\lambda < \theta$, there exists a continuously embedded Hilbert space $\mathcal{F}_{\lambda,\theta}(X)$ in $\mathcal{F}(\mathcal{H})$ such that $d\Gamma(T)$ satisfies the l.a.p. in every compact interval in (λ, θ) w.r.t. the norm topology of $B(\mathcal{F}_{\lambda,\theta}(X), \mathcal{F}_{\lambda,\theta}(X)^*)$. Furthermore, for any $\mu \in (\lambda, \theta)$ we have

$$\|R^{\pm}[d\Gamma(T)](\mu)\|_{B(\mathcal{F}_{\lambda,\theta}(X),\mathcal{F}_{\lambda,\theta}(X)^*)} \leq 1. \tag{1.2}$$

We can get similar results by replacing $\mathcal{F}(\mathcal{H})$ by the bosonic Fock space $\mathcal{F}_b(\mathcal{H}) \equiv \oplus_{n=0}^{\infty} S_n(\otimes_n \mathcal{H})$ or the fermionic Fock space $\mathcal{F}_f(\mathcal{H}) \equiv \oplus_{n=0}^{\infty} A_n(\otimes_n \mathcal{H})$, and $d\Gamma(T)$ by its reduced part on $\mathcal{F}_b(\mathcal{H})$ or $\mathcal{F}_f(\mathcal{H})$, respectively, where S_n (resp. A_n) is the n-th symmetrization (resp. anti-symmetrization) operator.

Corollary 1.4 *Assume that T satisfies the assumptions of Theorem* 1.3 *and that*

$$S_n(\otimes_n X) \subset \otimes_n X, \quad A_n(\otimes_n X) \subset \otimes_n X \tag{1.3}$$

for any $n \in \mathbb{N}$. Let $d\Gamma_\sharp(T)$ be the reduced part of $d\Gamma(T)$ on $\mathcal{F}_\sharp(\mathcal{H})$ with $\sharp = b$ or f. Then, for any positive numbers λ and θ with $\lambda < \theta$, there exist continuously embedded Hilbert spaces $\mathcal{F}^{\sharp}_{\lambda,\theta}(X) \subset \mathcal{F}_\sharp(\mathcal{H})$ such that $d\Gamma_\sharp(T)$ satisfy the l.a.p. in every compact interval in (λ, θ) w.r.t. the norm topology of $B(\mathcal{F}^{\sharp}_{\lambda,\theta}(X), \mathcal{F}^{\sharp}_{\lambda,\theta}(X)^*)$ with $\sharp = b$ or f.

Since the proof of this assertion is similar to that of Theorem 1.3, we omit it. We remark that (1.3) is satisfied if we choose $(\mathcal{H}, X)$ as in Remark 1.2 with $s > 1/2$.

In order to prove Theorem 1.3, we shall apply the following result, which is an extension of Theorem 4.1 in Ben-Artzi and Devinatz [2] where the case of two particles was studied.

Theorem 1.5 *Assume T satisfies the assumptions of Theorem* 1.3*. Then $T^{(n)}$, defined by* (1.1)*, satisfies the l.a.p. in every compact interval in* $(0, \infty)$ *w.r.t. the norm topology of $B(\otimes_n X, (\otimes_n X)^*)$. Moreover, for any* $0 < \lambda < \theta$*, we have*

$$\left\|R^{\pm}[T^{(n)}](\mu)\right\|_{B(\otimes_n X,(\otimes_n X)^*)} \leq C_n, \quad \mu \in [\lambda, \theta], \tag{1.4}$$

where $C_n := n \sup_{\frac{\lambda}{4^{n-1}} \leq \tau \leq \theta-(n-1)a} \|R^{\pm}[T](\tau)\|_{B(X,X^*)} + (6 \cdot 4^{n-2}/\lambda)$ and n

is a positive integer.

Let us mention the sketch of proofs of these results. We prove Theorem 1.5 by following the argument of [2]. Since $T^{(n)}$ can be written as $T \otimes I^{(n-1)} + I \otimes T^{(n-1)}$ as an operator on $\otimes_n \mathcal{H}$, our proof will be reduced to the two particle case, where I and $I^{(n-1)}$ are the identity operators on $\mathcal{H}$ and $\otimes_{n-1}\mathcal{H}$, respectively. As for Theorem 1.3, we shall introduce a "space of weighted square summable sequences" (for the notion, we refer to e.g. Obata [6]), and apply Theorem 1.5.

The plan of this paper is as follows. In the next section we prepare notations and preliminaries. In Section 3 we prove Theorem 1.5. Finally, Theorem 1.3 is shown in Section 4.

2. Preliminaries

First we prepare some notations.

- We denote by $\langle \cdot, \cdot \rangle$ an inner product on a Hilbert space, which is antilinear with respect to the left variable.
- $\| \cdot \|_V$ stands for the norm of a Banach space V. As long as it does not cause any confusion, we sometimes omit the index.
- For a bounded sesqui-linear form q on Hilbert space $\mathcal{H}$, we define $\|q\| \equiv \sup_{\|f\|_{\mathcal{H}}, \|g\|_{\mathcal{H}}=1} |q(f, g)|$.
- We denote by E and E_n the spectral measures of T and $T^{(n)}$, respectively. We denote by R_n the resolvent of $T^{(n)}$, i.e., $R_n(z) = R[T^{(n)}](z)$.
- We denote by $\mathcal{H}\hat{\otimes}\mathcal{K}$ the algebraic tensor product of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.
- We identify $(\mathcal{H}\otimes\mathcal{K})^*$ with $\mathcal{H}^*\otimes\mathcal{K}^*$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces.

Next we prepare three lemmas which will be used in the subsequent sections. The first one is an elementary fact.

Lemma 2.1 *Let $\mathcal{H}$ be a Hilbert space and $q(\cdot, \cdot)$ be a sesqui-linear form defined on a dense subspace D of $\mathcal{H}$. Suppose there exists some constant C such that $|q(f, g)| \leq C\|f\|\|g\|$ for any $f, g \in D$. Then $q(\cdot, \cdot)$ can be uniquely extended to a bounded sesqui-linear form $\tilde{q}(\cdot, \cdot)$ defined on whole $\mathcal{H}$. Moreover, it holds that $|\tilde{q}(f, g)| \leq C\|f\|\|g\|$ for any $f, g \in \mathcal{H}$.*

The second one is a representation of R_n, the resolvent of $T^{(n)}$. It will play an important role in this paper.

Lemma 2.2 *Let $\mathcal{H}$ be a Hilbert space, and let T be a self-adjoint operator (not necessarily semi-bounded) on $\mathcal{H}$. Then for any $\mu \in \mathbb{R}$, $\epsilon > 0$, we have*

$$\begin{aligned}\langle R_n(\mu \pm i\varepsilon)f, g\rangle &= \int_{\mathbb{R}} \langle R_1(\mu \pm i\varepsilon - \omega)f_1, g_1\rangle d\langle f_{n-1}, E_{n-1}(\omega)g_{n-1}\rangle \\ &= \int_{\mathbb{R}} \langle R_{n-1}(\mu \pm i\varepsilon - \omega)f_{n-1}, g_{n-1}\rangle d\langle f_1, E(\omega)g_1\rangle, \qquad (2.1)\end{aligned}$$

where $f = f_1 \otimes f_{n-1}$ and $g = g_1 \otimes g_{n-1}$ with f_1, $g_1 \in \mathcal{H}$ and f_{n-1}, $g_{n-1} \in \otimes_{n-1}\mathcal{H}$.

For the proof, see Berezanskii [3, Ch.VI, Section 4, pp. 462–469].

The last one concerns with the relation among tensor products of different Hilbert spaces.

Lemma 2.3 *Let $\mathcal{H}$ and X be Hilbert spaces such that $X \subset \mathcal{H}$ and $\|f\|_{\mathcal{H}} \leq \|f\|_X$, for any $f \in X$. Then for any positive integer n and any pair of integers $(i_1, \ldots, i_m)$ satisfying $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, the Hilbert space $X^n_{i_1,\ldots,i_m} := \underbrace{\mathcal{H} \otimes \cdots \otimes \overbrace{X}^{i_1\text{-}th} \otimes \cdots \otimes \overbrace{X}^{i_2\text{-}th} \otimes \cdots \otimes \overbrace{X}^{i_m\text{-}th} \otimes \cdots \otimes \mathcal{H}}_{n}$ is a subspace of $\otimes_n\mathcal{H}$, and $\|f\|_{\otimes_n\mathcal{H}} \leq \|f\|_{X^n_{i_1,\ldots,i_m}}$ for any $f \in X^n_{i_1,\ldots,i_m}$.*

Proof. We shall only treat the case of $n = 2$ and $i_1 = 1$ (that is, $X^2_1 \equiv X \otimes \mathcal{H}$). Let us define a sesqui-linear form $q(f, g) := \langle f, g\rangle_X$, $f, g \in X$ on $\mathcal{H}$ whose domain is equal to X. By the assumption, it is easy to show that $q(\cdot, \cdot)$ is a non-negative closed symmetric form. Then there exists a self-adjoint operator $A \geq 0$ on $\mathcal{H}$ such that $q(f, g) = \langle Af, Ag\rangle_{\mathcal{H}}$ for $f, g \in X$. Then we can find immediately that the domain of A is equal to X and $\|Af\|_{\mathcal{H}} = \|f\|_X \geq \|f\|_{\mathcal{H}}$. We recall that $X \otimes \mathcal{H}$ is completion of $X \hat{\otimes} \mathcal{H}$ with its norm $\|f \otimes g\|_{X\otimes\mathcal{H}} := \|f\|_X \|g\|_{\mathcal{H}}$. Since $\|f \otimes g\|_{X\otimes\mathcal{H}} = \|(A \otimes I)f \otimes g\|_{\mathcal{H}\otimes\mathcal{H}}$ for $f \in X, g \in \mathcal{H}$, we conclude that $X \otimes \mathcal{H}$ is equal to the domain of $A \otimes I$ and $\|\psi\|_{X\otimes\mathcal{H}} = \|(A \otimes I)\psi\|_{\mathcal{H}\otimes\mathcal{H}} \geq \|\psi\|_{\mathcal{H}\otimes\mathcal{H}}$ for $\psi \in X \otimes \mathcal{H}$. Thus the assertion is proved. $\square$

3. Proof of Theorem 1.5

As is mentioned at the end of Section 1, our proof of Theorem 1.5 is based on the work of Ben-Artzi and Devinatz [2]. The first step is to decompose the resolvent $R_n(z)$. Let λ be any positive number and fix it. Suppose μ is an arbitrary fixed number such that $\mu \geq \lambda$. We put $\delta = \lambda/4$. Let φ, ψ and χ be a partition of unity over $[a-1, \infty)$. More explicitly, they are functions in $C^\infty(\mathbb{R})$ satisfying $\varphi(x) + \psi(x) + \chi(x) \equiv 1$ $(x \in [a-1, \infty))$ and

$$\begin{aligned} &\operatorname{supp} \varphi \subseteq (-\infty, \mu - \delta), \\ &\operatorname{supp} \psi \subseteq (\mu - 2\delta, \mu - (n-1)a + 2\delta), \\ &\operatorname{supp} \chi \subseteq (\mu - (n-1)a + \delta, \infty). \end{aligned}$$

If $z = \mu \pm i\epsilon$ with $\epsilon > 0$, then we have

$$\begin{aligned} R_n(z) = R_n(z)\{\varphi(T) \otimes I^{(n-1)}\} &+ R_n(z)\big\{\psi(T) \otimes E_{n-1}((-\infty, \mu - \delta])\big\} \\ &+ R_n(z)\big\{\chi(T) \otimes E_{n-1}((-\infty, \mu - \delta])\big\} \\ &+ R_n(z)\big\{(I - \varphi(T)) \otimes E_{n-1}((\mu - \delta, \infty))\big\} \end{aligned} \tag{3.1}$$

as an operator on $\otimes_n \mathcal{H}$. In fact, since operators on the right-hand side are all bounded, and $E_{n-1}(\omega)$ is a resolution of identity, one can get (3.1) by an algebraic tensorial computation. Therefore, if we set

$$\begin{aligned} q_1(\mu \pm i\epsilon)(f, g) &:= \big\langle R_n(\mu \pm i\epsilon)\{\varphi(T) \otimes I^{(n-1)}\}f, g\big\rangle, \\ q_2(\mu \pm i\epsilon)(f, g) &:= \big\langle R_n(\mu \pm i\epsilon)\{\psi(T) \otimes E_{n-1}((-\infty, \mu - \delta])\}f, g\big\rangle, \\ q_3(\mu \pm i\epsilon)(f, g) &:= \big\langle R_n(\mu \pm i\epsilon)\{\chi(T) \otimes E_{n-1}((-\infty, \mu - \delta])\}f, g\big\rangle, \\ q_4(\mu \pm i\epsilon)(f, g) &:= \big\langle R_n(\mu \pm i\epsilon)\{(I - \varphi(T)) \otimes E_{n-1}((\mu - \delta, \infty))\}f, g\big\rangle \end{aligned} \tag{3.2}$$

for $f, g \in \otimes_n \mathcal{H}$ and $\epsilon > 0$, then we have

$$\langle R_n(\mu \pm i\epsilon)f, g\rangle = \sum_{j=1}^{4} q_j(\mu \pm i\epsilon)(f, g).$$

We remark that the meaning of those sesqui-linear forms are rather clear than that in [2] where the computation is carried out based on the framework of Berezanskii [3].

The next step is to derive estimates for $q_j(\mu \pm i\epsilon)(f, g)$ for $j = 1, 2, 3, 4$ and take their limits, by assuming that the assumptions (a), (b) in Theorem 1.3 hold and that $T^{(k)}$ satisfies the l.a.p. in every compact interval in $(0, \infty)$, w.r.t. the norm topology of $B(X_k, X_k^*)$ for $k = 1, 2, \ldots, n-1$. We denote by $R_k^{\pm}(\mu)$ the limit of $R_k(\mu \pm i\epsilon)$ in $B(\otimes_k X, (\otimes_k X)^*)$ for $k = 1, 2, \ldots, n-1$. In the following, we shall denote $\otimes_k \mathcal{H}, \otimes_k X$ by $\mathcal{H}_k, X_k$ for $k \in \mathbb{N}$, respectively, for the sake of simplicity.

First we consider $q_1(\mu \pm i\epsilon)$.

Lemma 3.1 *For any $\epsilon > 0$, it holds that*

$$q_1(\mu \pm i\epsilon)(f, g) = \lim_{\Pi_N \to 0} \sum_{j=1}^{N} \big\langle \varphi(\xi_j)\{I \otimes R_{n-1}(\mu \pm i\epsilon - \xi_j)\} f, \\ \{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g \big\rangle \tag{3.3}$$

for any $f, g \in \mathcal{H} \hat{\otimes} X_{n-1}$. Here Π_N is a partition of $[a, \mu - \delta]$;

$$a =: \omega_0 < \omega_1 < \cdots < \omega_{n-1} < \omega_n := \mu - \delta,$$

$\xi_j \in (\omega_{j-1}, \omega_j]$ for $j = 1, \ldots, N$, and we mean by $\Pi_N \to 0$ that $\max_{1 \le j \le N}$ $(\omega_j - \omega_{j-1})$ converges to zero.

Moreover, we have

$$|q_1(\mu \pm i\epsilon)(f, g)| \le \sup_{a \le \omega \le \mu - \delta} \|R_{n-1}(\mu \pm i\epsilon - \omega)\|_{B(X_{n-1}, X_{n-1}^*)} \\ \cdot \|f\|_{\mathcal{H} \otimes X_{n-1}} \|g\|_{\mathcal{H} \otimes X_{n-1}} \tag{3.4}$$

for any $f, g \in \mathcal{H} \otimes X_{n-1}$.

Proof. First we shall prove (3.3). It is enough to show (3.3) for the case where $f = f_1 \otimes f_{n-1}$, $g = g_1 \otimes g_{n-1}$ being $f_1, g_1 \in \mathcal{H}$ and $f_{n-1}, g_{n-1} \in X_{n-1}$, because of the linearity of both hand sides w.r.t. f and g. Then by the continuity of $\langle R_{n-1}(\mu \pm i\epsilon - \omega) f_{n-1}, g_{n-1} \rangle$ in ω, the right-hand side of (3.3) can be written as

$$\int_{[a,\mu-\delta]} \varphi(\omega)\langle R_{n-1}(\mu \pm i\epsilon - \omega) f_{n-1}, g_{n-1}\rangle d\langle E(\omega) f_1, g_1\rangle \tag{3.5}$$

(recall the assumption (a)). By the functional calculus and the variable transformation concerning the Lebesgue-Stieltjes integral, we see that the above integral is equal to

$$\int_{\mathbb{R}} \langle R_{n-1}(\mu \pm i\epsilon - \omega) f_{n-1}, g_{n-1}\rangle d\langle E(\omega)\varphi(T) f_1, g_1\rangle. \tag{3.6}$$

This expression is further equal to $q_1(\mu \pm i\epsilon)(f, g)$, by (2.1) and (3.2). We thus get (3.3).

Next we shall prove (3.4). To begin with, we remark that

$$\begin{aligned} &\big|\langle\{I \otimes R_{n-1}(\mu \pm i\epsilon - \xi_j)\} f, \{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g\rangle\big| \\ &\quad \leq C_{n-1} \big\|\{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} f\big\|_{\mathcal{H}\otimes X_{n-1}} \\ &\qquad \cdot \big\|\{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g\big\|_{\mathcal{H}\otimes X_{n-1}} \end{aligned} \tag{3.7}$$

holds for any $f, g \in \mathcal{H}\hat{\otimes} X_{n-1}$, where we have set

$$C_{n-1} = \sup_{a \leq \omega \leq \mu-\delta} \|R_{n-1}(\mu \pm i\epsilon - \omega)\|_{B(X_{n-1}, X^*_{n-1})}.$$

In fact, since $E(\cdot)$ is an orthogonal projection, we have

$$\begin{aligned} &\big\langle\{I \otimes R_{n-1}(\mu \pm i\epsilon - \xi_j)\} f, \{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g\big\rangle \\ &\quad = \big\langle\{(I \otimes R_{n-1}(\mu \pm i\epsilon - \xi_j))(E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)})\} f, \\ &\qquad\qquad \{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g\big\rangle. \end{aligned} \tag{3.8}$$

Since

$$\|I \otimes R_{n-1}(\mu \pm i\epsilon - \xi_j)\|_{B(\mathcal{H}\otimes X_{n-1}, \mathcal{H}^*\otimes X^*_{n-1})} \leq C_{n-1}$$

for $a \leq \xi_j \leq \mu - \delta$, we obtain (3.7) from (3.8). Now, by (3.7) we have

$$
\begin{aligned}
&\Big|\sum_{j=1}^{N}\varphi(\xi_j)\big\langle\{I\otimes R_{n-1}(\mu\pm i\epsilon-\omega)\}f,\{E((\omega_{j-1},\omega_j])\otimes I^{(n-1)}\}g\big\rangle\Big| \\
&\le C_{n-1}\sum_{j=1}^{N}|\varphi(\xi_j)|\big\|\{E((\omega_{j-1},\omega_j])\otimes I^{(n-1)}\}f\big\|\big\|\{E((\omega_{j-1},\omega_j])\otimes I^{(n-1)}\}g\big\| \\
&\le C_{n-1}\sqrt{\sum_{j=1}^{N}\|\{E((\omega_{j-1},\omega_j])\otimes I^{(n-1)}\}f\|^2} \\
&\qquad\qquad\cdot\sqrt{\sum_{j=1}^{N}\|\{E((\omega_{j-1},\omega_j])\otimes I^{(n-1)}\}g\|^2} \\
&= C_{n-1}\big\|\{E([a,\mu-\delta])\otimes I^{(n-1)}\}f\big\|\big\|\{E([a,\mu-\delta])\otimes I^{(n-1)}\}g\big\| \\
&\le C_{n-1}\|f\|\|g\| \qquad (3.9)
\end{aligned}
$$

for $f,g\in\mathcal{H}\hat{\otimes}X_{n-1}$. Here the norms are taken in $\mathcal{H}\otimes X_{n-1}$. Hence $q_1(\mu\pm i\epsilon)(\cdot,\cdot)$ is a bounded sesqui-linear form on $\mathcal{H}\hat{\otimes}X_{n-1}$. From (3.3), (3.9) and Lemma 2.1, we find (3.4). This completes the proof. □

Similarly to the proof of Lemma 3.1, we can prove the following lemma with the help of the assumption (b).

Lemma 3.2 *For any $\epsilon>0$, it holds that*

$$
\begin{aligned}
q_2(\mu\pm i\epsilon)(f,g)=\lim_{\Lambda_N\to 0}\sum_{j=1}^{N}\big\langle&\{R_1(\mu\pm i\epsilon-\xi_j)\psi(T)\otimes I^{(n-1)}\}f, \\
&\{I\otimes E_{n-1}((\omega_{j-1},\omega_j])\}g\big\rangle \qquad (3.10)
\end{aligned}
$$

for any $f,g\in X\hat{\otimes}\mathcal{H}_{n-1}$. Here Λ_N is a partition of $[(n-1)a,\mu-\delta]$;

$$(n-1)a=:\omega_0<\omega_1<\cdots<\omega_{n-1}<\omega_n:=\mu-\delta.$$

Moreover, we have

$$
\begin{aligned}
|q_2(\mu\pm i\epsilon)(f,g)|\le\ &\sup_{(n-1)a\le\omega\le\mu-\delta}\|R_1(\mu\pm i\epsilon-\omega)\|_{B(X,X^*)} \\
&\cdot\|f\|_{X\otimes\mathcal{H}_{n-1}}\|g\|_{X\otimes\mathcal{H}_{n-1}} \qquad (3.11)
\end{aligned}
$$

for any $f, g \in X \otimes \mathcal{H}_{n-1}$.

Next we consider the limits of the sesqui-linear forms $q_1(\mu \pm i\epsilon)$ and $q_2(\mu \pm i\epsilon)$ as $\epsilon \to 0+$. We define a sesqui-linear form on $\mathcal{H}\hat{\otimes}X_{n-1}$ by

$$q_1^{\pm}(\mu)(f,g) = \lim_{\Pi_N \to 0} \sum_{j=1}^{N} \varphi(\xi_j) \big\langle \{I \otimes R_{n-1}^{\pm}(\mu - \xi_j)\} f, \\ \{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g \big\rangle \quad (3.12)$$

for any $f, g \in \mathcal{H}\hat{\otimes}X_{n-1}$. Here we have used the same notation as in (3.3), $R_{n-1}^{\pm}(\mu)$ is the limit of $R_{n-1}(\mu \pm i\epsilon)$ in $B(X_{n-1}, X_{n-1}^*)$, and $\langle \{I \otimes R_{n-1}^{\pm}(\mu - \xi_j)\} f, \{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g\rangle$ means $\{I \otimes R_{n-1}^{\pm}(\mu - \xi_j)\} f$ acts on $\{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g$ as a linear functional on $\mathcal{H} \otimes X_{n-1}$. Note that $q_1^{\pm}(\mu)$ is well defined. In fact, if $f = f_1 \otimes f_{n-1}$, $g = g_1 \otimes g_{n-1}$, then $q_1^{\pm}(\mu)(f,g)$ is equal to $\int_{[a,\mu-\delta]} \varphi(\omega) \langle R_{n-1}^{\pm}(\mu - \omega) f_{n-1}, g_{n-1}\rangle d\langle E(\omega) f_1, g_1\rangle$. By the assumption that l.a.p. $T^{(n-1)}$ satisfies the l.a.p. $\langle R_{n-1}^{\pm}(\mu - \omega) f_{n-1}, g_{n-1}\rangle$ is continuous in ω, and hence the integral is finite. For general elements in $\mathcal{H}\hat{\otimes}X_{n-1}$, it can be linearly extended. Similarly to (3.9), we have

$$\big|q_1^{\pm}(\mu)(f,g)\big| \le \sup_{a \le \omega \le \mu-\delta} \big\| R_{n-1}^{\pm}(\mu - \omega) \big\|_{B(X_{n-1}, X_{n-1}^*)} \\ \cdot \|f\|_{\mathcal{H}\otimes X_{n-1}} \|g\|_{\mathcal{H}\otimes X_{n-1}}$$

for any $f, g \in \mathcal{H}\hat{\otimes}X_{n-1}$. By using Lemma 2.1, it is hence extended to a bounded sesqui-linear form on $\mathcal{H} \otimes X_{n-1}$.

For any $f, g \in \mathcal{H} \otimes X_{n-1}$, let $\{f_k\}, \{g_k\}$ be sequences in $\mathcal{H}\hat{\otimes}X_{n-1}$ such that $f_k \to f$ and $g_k \to g$ as $k \to \infty$. Then it follows from (3.3) and (3.12) that

$$\big|q_1(\mu \pm i\epsilon)(f,g) - q_1^{\pm}(\mu)(f,g)\big| \\ = \lim_{k \to \infty} \bigg| \lim_{\Pi_N \to 0} \sum_{j=1}^{N} \varphi(\xi_j) \big\langle \{I \otimes (R_{n-1}(\mu \pm i\epsilon - \xi_j) - R_{n-1}^{\pm}(\mu - \xi_j))\} f_k, \\ \{E((\omega_{j-1}, \omega_j]) \otimes I^{(n-1)}\} g_k \big\rangle \bigg|$$

$$\leq \lim_{k\to\infty} \sup_{a\leq\omega\leq\mu-\delta}\big\|R_{n-1}(\mu\pm i\epsilon-\omega)-R^{\pm}_{n-1}(\mu-\omega)\big\|_{B(X_{n-1},X^*_{n-1})} \cdot \|f_k\|\|g_k\|$$
$$= \sup_{a\leq\omega\leq\mu-\delta}\big\|R_{n-1}(\mu\pm i\epsilon-\omega)-R^{\pm}_{n-1}(\mu-\omega)\big\|_{B(X_{n-1},X^*_{n-1})}\|f\|\|g\|, \tag{3.13}$$

where the norms are taken in $\mathcal{H}\otimes X_{n-1}$. By Lemma 2.3, we have $\|f\|_{\mathcal{H}\otimes X_{n-1}} \leq \|f\|_{X_n}$ for any $f\in X_n$. Therefore we conclude that

Lemma 3.3 *The sesqui-linear form $q_1(\mu\pm i\epsilon)(f,g)$ converges to $q_1^{\pm}(\mu)(f,g)$ uniformly in $f,g\in X_n$, as $\epsilon\to 0+$. Moreover, we have*

$$\big\|q_1^{\pm}(\mu)\big\| \leq \sup_{a\leq\omega\leq\mu-\delta}\big\|R^{\pm}_{n-1}(\mu-\omega)\big\|_{B(X_{n-1},X^*_{n-1})}. \tag{3.14}$$

Similarly, we have the following lemma.

Lemma 3.4 *Let $q_2^{\pm}(\mu)(\cdot,\cdot)$ be the linear extension over $X\otimes\mathcal{H}_{n-1}$ of a sesqui-linear form defined by*

$$q_2^{\pm}(\mu)(f,g) = \lim_{\Lambda_N\to 0}\sum_{j=1}^{N}\big\langle\{R_1^{\pm}(\mu-\xi_j)\otimes I^{(n-1)}\}\{\psi(T)\otimes I^{(n-1)}\}f, \{I\otimes E_{n-1}((\omega_{j-1},\omega_j])\}g\big\rangle \tag{3.15}$$

for any $f,g\in X\hat{\otimes}\mathcal{H}_{n-1}$. Then $q_2(\mu\pm i\epsilon)(f,g)$ converges to $q_2^{\pm}(\mu)(f,g)$ uniformly in $f,g\in X_n$, as $\epsilon\to 0+$. Moreover, we have

$$\big\|q_2^{\pm}(\mu)\big\| \leq \sup_{(n-1)a\leq\omega\leq\mu-\delta}\big\|R^{\pm}(\mu-\omega)\big\|_{B(X,X^*)}. \tag{3.16}$$

Next we deal with $q_3(\mu\pm i\epsilon)$.

Lemma 3.5 *For any $\epsilon>0$, it holds that*

$$q_3(\mu\pm i\epsilon)(f,g) = \lim_{\Lambda_N\to 0}\sum_{j=1}^{N}\big\langle[\{\chi(T)R_1(\mu\pm i\epsilon-\xi_j)\}\otimes I^{(n-1)}]f, \{I\otimes E_{n-1}((\omega_{j-1},\omega_j])\}g\big\rangle. \tag{3.17}$$

for any $f,g\in\mathcal{H}\hat{\otimes}\mathcal{H}_{n-1}$. Here Λ_N and ξ_j are as in Lemma 3.2.

Moreover, we have $|q_3(\mu \pm i\epsilon)(f,g)| \leq \delta^{-1}\|f\|_{\mathcal{H}_n}\|g\|_{\mathcal{H}_n}$ *for any* $f, g \in \mathcal{H}_n$.

Proof. The equation (3.17) can be shown similarly to the proof of (3.3). In order to prove the inequality, we first remark that

$$\sup_{(n-1)a\leq\omega\leq\mu-\delta}\|\chi(T)R_1(\mu \pm i\epsilon-\omega)\|_{B(\mathcal{H})} \leq \delta^{-1}.$$

In fact, by the functional calculus, we have

$$\|\chi(T)R_1(\mu \pm i\epsilon - \omega)f\|^2_{\mathcal{H}} = \int_{(\mu-(n-1)a+\delta,\infty)} \left|\frac{\chi(\rho)}{\rho-(\mu \pm i\epsilon - \omega)}\right|^2 d\|E(\rho)f\|^2$$

for any $f \in \mathcal{H}$ and $\omega \in [(n-1)a, \mu-\delta]$. Since $\rho - (\mu - \omega) \geq \delta$ for any $\rho \in (\mu-(n-1)a+\delta, \infty)$ and $\omega \in [(n-1)a, \mu-\delta]$, the last quantity is bounded by $\delta^{-2}\|f\|^2$. If we let $f, g \in \mathcal{H}\hat{\otimes}\mathcal{H}_{n-1}$, then we have

$$\begin{aligned}
&|q_3(\mu \pm i\epsilon)(f,g)| \\
&\leq \lim_{\Lambda_N\to 0}\left|\sum_{j=1}^{N}\left\langle[\{\chi(T)R_1(\mu \pm i\epsilon-\xi_j)\}\otimes I^{(n-1)}]f, \{I\otimes E_{n-1}((\omega_{j-1},\omega_j])\}g\right\rangle\right| \\
&= \lim_{\Lambda_N\to 0}\left|\sum_{j=1}^{N}\left\langle\{\chi(T)R_1(\mu \pm i\epsilon-\xi_j)\otimes I^{(n-1)}\}\{I\otimes E_{n-1}((\omega_{j-1},\omega_j]\}f,\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.\left.\{I\otimes E_{n-1}((\omega_{j-1},\omega_j])\}g\right\rangle\right| \\
&\leq \delta^{-1}\|f\|_{\mathcal{H}_n}\|g\|_{\mathcal{H}_n}.
\end{aligned}$$

This completes the proof. □

Proceeding as in the proof of Lemmas 3.3 and 3.4, we have the following. We remark that $\chi(T)/\{T-(\mu-\xi_j)\}$ is well defined thanks to the cut-off function χ.

Lemma 3.6 *Let* $q_3^{\pm}(\mu)(\cdot,\cdot)$ *be a sesqui-linear form on* $\mathcal{H}\otimes\mathcal{H}_{n-1}$ *such that*

$$
\begin{aligned}
&q_3^{\pm}(\mu)(f,g)\\
&\quad= \lim_{\Lambda_N \to 0} \sum_{j=1}^{N} \left\langle \left\{ \left(\frac{\chi(T)}{T-(\mu-\xi_j)} \right) \otimes I^{(n-1)} \right\} f, \{ I \otimes E_{n-1}((\omega_{j-1},\omega_j])g \right\rangle
\end{aligned}
\tag{3.18}
$$

for any $f, g \in \mathcal{H}\hat{\otimes}\mathcal{H}_{n-1}$. Then $q_3(\mu \pm i\epsilon)(f,g)$ converges to $q_3^{\pm}(\mu)(f,g)$ uniformly in $f, g \in X_n$, as $\epsilon \to 0+$. Moreover, we have $\|q_3^{\pm}(\mu)\| \leq \delta^{-1}$.

Next we consider $q_4(\mu \pm i\epsilon)$.

Lemma 3.7 *For any $\epsilon > 0$, it holds that*

$$
\begin{aligned}
&q_4(\mu \pm i\epsilon)(f,g)\\
&\quad= \int_{(\mu-\delta,\infty)} \langle \{I-\varphi(T)\}\{R_1(\mu \pm i\epsilon - \omega)\} f_1, g_1 \rangle d\langle f_{n-1}, E_{n-1}(\omega) g_{n-1} \rangle
\end{aligned}
\tag{3.19}
$$

for any $f = f_1 \otimes f_{n-1}$, $g = g_1 \otimes g_{n-1}$, where $f_1, g_1 \in \mathcal{H}$, $f_{n-1}, g_{n-1} \in \mathcal{H}_{n-1}$. Moreover, we have

$$
|q_4(\mu \pm i\epsilon)(f,g)| \leq \frac{2}{\lambda} \|f\|_{\mathcal{H}_n} \|g\|_{\mathcal{H}_n}, \quad f, g \in \mathcal{H} \otimes \mathcal{H}_{n-1}. \tag{3.20}
$$

Proof. The equation (3.19) can be shown similarly to (3.3). We shall next prove (3.20). We remark that $\|\{I-\varphi(T)\}\{R_1(\mu \pm i\epsilon - \omega)\}\|_{B(\mathcal{H})} < 2/\lambda$. In fact, by the functional calculus, we have

$$
\begin{aligned}
&\|\{I-\varphi(T)\}\{R_1(\mu \pm i\epsilon - \omega)\} f\|_{\mathcal{H}}^2\\
&\quad= \int_{[a,\infty)} \left| \frac{1-\varphi(\rho)}{\rho - (\mu \pm i\epsilon - \omega)} \right|^2 d\|E(\rho) f\|^2
\end{aligned}
\tag{3.21}
$$

for any $\omega \in (\mu-\delta, \infty)$. Considering the support of $1-\varphi(\rho)$ under the domain of the integration (3.19), we have $\rho - \mu + \omega > \mu - 2\delta > \lambda/2$. Therefore the last quantity is bounded by $(2/\lambda)^2 \|f\|_{\mathcal{H}}^2$.

Let $f = f_1 \otimes f_{n-1}$, $g = g_1 \otimes g_{n-1}$, and denote by $q_{4,R}(\mu \pm i\epsilon)(f,g)$ the following integral

$$\int_{(\mu-\delta,R]} \left\langle \frac{I-\varphi(T)}{T-(\mu\pm i\epsilon-\omega)} f_1, g_1 \right\rangle d\langle f_{n-1}, E_{n-1}(\omega) g_{n-1}\rangle,$$

$$R > \mu - \delta. \quad (3.22)$$

Then for any $c > 0$, there exists $R_0 > \mu - \delta$ such that

$$\left| q_4(\mu \pm i\epsilon)(f,g) - q_{4,R_0}(\mu \pm i\epsilon)(f,g) \right| < c.$$

Hence we have

$$|q_4(\mu \pm i\epsilon)(f,g)| < c + \left| q_{4,R_0}(\mu \pm i\epsilon)(f,g) \right|. \quad (3.23)$$

In order to establish an estimate of $q_{4,R_0}(\mu \pm i\epsilon)(f,g)$, it is sufficient to evaluate the partial sum;

$$\sum_{l=1}^{N} \left\langle \frac{I-\varphi(T)}{T-(\mu\pm i\epsilon-\omega_l)} f_1, g_1 \right\rangle \langle f_{n-1}, E_{n-1}(\Delta_l) g_{n-1}\rangle$$
$$= \sum_{l=1}^{N} \left\langle \left\{ \frac{I-\varphi(T)}{T-(\mu\pm i\epsilon-\omega_l)} \otimes I^{(n-1)} \right\} f, \{I \otimes E_{n-1}(\Delta_l)\} g \right\rangle$$

where $(\mu-\delta, R_0] = \sum_{l=1}^{N} \Delta_l$ is a partition and $\omega_l \in \Delta_l$. Using (3.21), we have

$$\left| \sum_{l=1}^{N} \left\langle \left\{ \frac{I-\varphi(T)}{T-(\mu\pm i\epsilon-\omega_l)} \otimes I^{(n-1)} \right\} f, \{I \otimes E_{n-1}(\Delta_l)\} g \right\rangle \right|$$
$$\leq \sum_{l=1}^{N} \left\| \frac{I-\varphi(T)}{T-(\mu\pm i\epsilon-\omega_l)} \otimes I^{(n-1)} \right\|_{B(\mathcal{H}_n)}$$
$$\cdot \|\{I \otimes E_{n-1}(\Delta_l)\} f\| \|\{I \otimes E_{n-1}(\Delta_l)\} g\|$$
$$\leq (2/\lambda) \|f\| \|g\|.$$

We thus get

$$\left| q_{4,R_0}(\mu \pm i\epsilon)(f,g) \right| < (2/\lambda) \|f\| \|g\|. \quad (3.24)$$

Hence by (3.23) and (3.24), (3.20) holds when $f = f_1 \otimes f_{n-1}$, $g = g_1 \otimes g_{n-1}$. For general elements of $\mathcal{H}\hat{\otimes}\mathcal{H}_{n-1}$, we can show (3.20) by the linear extension. Therefore, by Lemma 2.1, the assertion is proved. □

Finally, we handle the limit of $q_4(\mu \pm i\epsilon)$.

Lemma 3.8 *Let $q_4^{\pm}(\mu)$ be a sesqui-linear form on $\mathcal{H}\hat{\otimes}\mathcal{H}_{n-1}$ such that*

$$q_4^{\pm}(\mu)(f,g) = \int_{(\mu-\delta,\infty)} \left\langle \frac{I-\varphi(T)}{T-(\mu-\omega)} f_1, g_1 \right\rangle d\langle f_{n-1}, E_{n-1}(\omega) g_{n-1}\rangle \quad (3.25)$$

for any $f = f_1 \otimes f_{n-1}$, $g = g_1 \otimes g_{n-1}$ where $f_1, g_1 \in \mathcal{H}$, $f_{n-1}, g_{n-1} \in \mathcal{H}_{n-1}$ (for general elements of $\mathcal{H}\hat{\otimes}\mathcal{H}_{n-1}$, we linearly extend it). Then $q_4(\mu \pm i\epsilon)(f,g)$ converges to $q_4^{\pm}(\mu)(f,g)$ uniformly in $f, g \in X_n$. Moreover, we have $\|q_4^{\pm}(\mu)\| \le 2/\lambda$.

Proof. By the functional calculus, we have $\sup_{\omega>\mu-\delta}\|\{I-\varphi(T)\}/\{T-(\mu \pm i\epsilon-\omega)\}^{-1} - \{I-\varphi(T)\}/\{T-(\mu-\omega)\}\|_{B(\mathcal{H})} < \epsilon$. Since the rest of the proof is similar to that of Lemma 3.3, we omit it. □

End of the proof of Theorem 1.5. For $\mu \ge \lambda$, we set $q^{\pm}(\mu) := \sum_{i=1}^{4} q_i^{\pm}(\mu)$. We see from Lemmas 3.3, 3.4, 3.6 and 3.8 that it is a bounded sesqui-linear form, and hence it can be identified with an element of $B(X_n, X_n^*)$ by setting $\langle R_n^{\pm}(\mu) f, g\rangle = q^{\pm}(\mu)(f,g)$ for $f, g \in X_n$. Then Lemmas 3.3, 3.4, 3.6 and 3.8 show that $R_n^{\pm}(\mu \pm i\epsilon)$ converge to $R_n^{\pm}(\mu)$ in $B(X_n, X_n^*)$, as $\epsilon \to 0+$.

Next we prove the resolvent estimate (1.4). We first consider the case $n = 2$. Fix any $\theta > \lambda > 0$. Then for any $\mu \in [\lambda, \theta]$, we have from Lemmas 3.1, 3.2, 3.5 and 3.7

$$\|R_2(\mu \pm i\epsilon)\|_{B(X_2,X_2^*)} \le 2\sup_{\delta\le\rho\le\theta-a}\|R_1(\rho \pm i\epsilon)\|_{B(X,X^*)} + 6/\lambda, \quad (3.26)$$

because $\delta = 4/\lambda$.

Next we treat the general case. Suppose the following inequality holds for $n = k-1$;

$$\begin{aligned}&\|R_n(\mu \pm i\epsilon)\|_{B(X_n,X_n^*)} \\ &\quad \le n\sup_{\frac{\lambda}{4^{n-1}}\le\tau\le\theta-(n-1)a}\|R_1(\tau \pm i\epsilon)\|_{B(X,X^*)} + (6\cdot 4^{n-2}/\lambda). \quad (3.27)\end{aligned}$$

Similarly to (3.26), for $\mu \in [\lambda, \theta]$, we have

$$
\begin{aligned}
&\|R_k(\mu \pm i\epsilon)\|_{B(X_k,X_k^*)} \\
&\quad \leq \sup_{\delta\leq\rho\leq\theta-a}\|R_{k-1}(\rho \pm i\epsilon)\|_{B(X_{k-1},X_{k-1}^*)} \\
&\qquad + \sup_{\delta\leq\rho\leq\theta-(k-1)a}\|R_1(\rho \pm i\epsilon)\|_{B(X,X^*)} + 6/\lambda.
\end{aligned} \tag{3.28}
$$

On the other hand, replacing λ by δ in (3.27), we have

$$
\begin{aligned}
&\sup_{\delta\leq\rho\leq\theta}\|R_{k-1}(\rho \pm i\epsilon)\|_{B(X_{k-1},X_{k-1}^*)} \\
&\quad \leq (k-1)\sup_{\frac{\delta}{4^{k-1}}\leq\tau\leq\theta-(k-1)a}\|R_1(\tau \pm i\epsilon)\|_{B(X,X^*)} + 6(k-1)4^{k-3}/\delta \\
&\quad = (k-1)\sup_{\frac{\lambda}{4^{k}}\leq\tau\leq\theta-(k-1)a}\|R_1(\tau \pm i\epsilon)\|_{B(X,X^*)} + 6(k-1)4^{k-2}/\lambda.
\end{aligned} \tag{3.29}
$$

Combining this estimate with (3.28), we conclude that (3.27) holds for $n = k$. Letting $\epsilon \to 0+$, we obtain (1.4). This completes the proof of Theorem 1.5.

4. Proof of Theorem 1.3

Let us now consider the l.a.p. for $d\Gamma(T)$ in a Fock space, as an application of Theorem 1.5. First of all, observe that (1.4) and (3.27) with $n = k$ yield the following:

Lemma 4.1 *We fix any two positive numbers λ, θ with $\lambda < \theta$. For any $n \in \mathbb{N}$, we set $\delta_{n-1} := \lambda/4^{n-1}$ and*

$$
\begin{aligned}
\tilde{C_n} := n\max\big\{ &\sup_{z\in K_n}\|R(z)\|_{B(X,X^*)}, \\
&\sup_{\delta_{n-1}\leq\mu\leq\theta-(n-1)a}\|R^{\pm}(\mu)\|_{B(X,X^*)}, 1\big\} + 6\cdot 4^{n-2}/\lambda.
\end{aligned} \tag{4.1}
$$

Here K_n is defined by

$$K_n := \{z = x+iy | \delta_{n-1} \leq x \leq \theta-(n-1)a,\ -1 < y < 1, y \neq 0\}$$

Then we have

$$\|R_n(\mu \pm i\epsilon)\|_{B(\otimes_n X,\otimes_n X^*)} \leq \tilde{C_n} \tag{4.2}$$

$$\|R_n^{\pm}(\mu)\|_{B(\otimes_n X,\otimes_n X^*)} \leq \tilde{C_n} \tag{4.3}$$

for any $\mu \in [\delta_{n-1}, \theta - (n-1)a]$, $0 < \epsilon < 1$ *and* $n \in \mathbb{N}$.

Now we shall introduce a suitable subspace $\mathcal{F}_{\lambda,\theta}(X)$ of $\mathcal{F}(X)$, following the standard argument in treating Fock space (we refer to, for instance, Obata [6]). Let $\tilde{C_n}$ the number from Lemma 4.1. We define

$$\mathcal{F}_{\lambda,\theta}(X) := \Big\{ \phi = \{\phi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}(\mathcal{H}) |\ \phi^{(n)} \in \otimes_n X, \\ \sum_{n=0}^{\infty} 2^n \tilde{C_n} \|\phi^{(n)}\|^2_{\otimes_n X} < \infty \Big\} \tag{4.4}$$

with the inner product

$$\begin{gathered} \langle \phi, \psi \rangle_{F_{\lambda,\theta}(X)} := \sum_{n=0}^{\infty} 2^n \tilde{C_n} \big\langle \phi^{(n)}, \psi^{(n)} \big\rangle_{\otimes_n X}, \\ \phi = \{\phi^{(n)}\}, \quad \psi = \{\psi^{(n)}\} \in \mathcal{F}_{\lambda,\theta}(X). \end{gathered} \tag{4.5}$$

It is not difficult to see that $\mathcal{F}_{\lambda,\theta}(X)$ is a Hilbert space. By the definition of $\mathcal{F}_{\lambda,\theta}(X)$, it is continuously embedded into $\mathcal{F}(X)$. By Lemma 2.3, $\mathcal{F}(X)$ is embedded into $\mathcal{F}(\mathcal{H})$, so that

$$\mathcal{F}_{\lambda,\theta}(X) \hookrightarrow \mathcal{F}(X) \hookrightarrow \mathcal{F}(\mathcal{H}).$$

Next we define $R^{\pm}(\mu)$ for $\mu \in \mathbb{R}$ by

$$\begin{gathered} \langle R^{\pm}(\mu) f, g \rangle := \sum_{n=0}^{\infty} \big\langle R_n^{\pm}(\mu) f^{(n)}, g^{(n)} \big\rangle, \\ f = \{f^{(n)}\}, \quad g = \{g^{(n)}\} \in \mathcal{F}_{\lambda,\theta}(X). \end{gathered} \tag{4.6}$$

We recall that $R_n^{\pm}(\mu) f^{(n)}$ is an element of $(\otimes_n X)^*$. It follows from Theorem 1.5 that $R^{\pm}(\mu) \in B(\mathcal{F}_{\lambda,\theta}(X), \mathcal{F}_{\lambda,\theta}(X)^*)$. In fact, by (4.3) we have

$$|\langle R^{\pm}(\mu) f, g \rangle| \le \sum_{n=0}^{\infty} 2^n \tilde{C_n} \|f^{(n)}\|_{\otimes_n X} \|g^{(n)}\|_{\otimes_n X}$$

$$\leq \sqrt{\sum_{n=0}^{\infty} 2^n \tilde{C}_n \|f^{(n)}\|^2_{\otimes_n X}} \sqrt{\sum_{n=0}^{\infty} 2^n \tilde{C}_n \|g^{(n)}\|^2_{\otimes_n X}}$$
$$= \|f\|_{\mathcal{F}_{\lambda,\theta}(X)} \|g\|_{\mathcal{F}_{\lambda,\theta}(X)}. \tag{4.7}$$

We are ready to show $d\Gamma(T)$ satisfies the l.a.p. in $B(\mathcal{F}_{\lambda,\theta}(X), \mathcal{F}_{\lambda,\theta}(X)^*)$. Note that if $f \in \mathcal{F}_{\lambda,\theta}(X)$ satisfies $\|f\|_{\mathcal{F}_{\lambda,\theta}(X)} = 1$, then

$$\|f^{(n)}\|_{\otimes_n X} \leq 1/\sqrt{2^n \tilde{C}_n} \quad \text{for any} \quad n \in \mathbb{N}, \tag{4.8}$$

by (4.5). Therefore we have

$$\sum_{n=0}^{\infty} \left\|R_n(\mu \pm i\epsilon) - R_n^{\pm}(\mu)\right\|_{B(\otimes_n X, \otimes_n X^*)} \|f^{(n)}\|_{\otimes_n X} \|g^{(n)}\|_{\otimes_n X} \leq \sum_{n=0}^{\infty} 1/2^{n-1} \tag{4.9}$$

by (4.2) and (4.3). Take any compact interval $K \subseteq [\lambda, \theta]$ and $\eta > 0$. Then there exists a large integer N such that

$$\sum_{n=N+1}^{\infty} \left\|R_n(\mu \pm i\epsilon) - R_n^{\pm}(\mu)\right\|_{B(\otimes_n X, \otimes_n X^*)} \|f^{(n)}\|_{\otimes_n X} \|g^{(n)}\|_{\otimes_n X} < \eta \tag{4.10}$$

for any $\mu \in K, f$ and $g \in \mathcal{F}_{\lambda,\theta}(X)$ such that $\|f\|_{\mathcal{F}_{\lambda,\theta}(X)} = \|g\|_{\mathcal{F}_{\lambda,\theta}(X)} = 1$ by (4.9). Therefore, we have

$$\left|\langle R(\mu \pm i\epsilon) f, g\rangle - \langle R^{\pm}(\mu) f, g\rangle\right|$$
$$\leq \sum_{n=0}^{N} \left\|R_n(\mu \pm i\epsilon) - R_n^{\pm}(\mu)\right\|_{B(\otimes_n X, \otimes_n X^*)} \|f^{(n)}\|_{\otimes_n X} \|g^{(n)}\|_{\otimes_n X}$$
$$+ \sum_{n=N+1}^{\infty} \left\|R_n(\mu \pm i\epsilon) - R_n^{\pm}(\mu)\right\|_{B(\otimes_n X, \otimes_n X^*)} \|f^{(n)}\|_{\otimes_n X} \|g^{(n)}\|_{\otimes_n X}$$
$$\leq \sum_{n=0}^{N} \sup_{\mu \in K} \left\|R_n(\mu \pm i\epsilon) - R_n^{\pm}(\mu)\right\|_{B(\otimes_n X, \otimes_n X^*)} + \eta. \tag{4.11}$$

By the uniformly compact convergence of $R_n(\mu \pm i\epsilon)$ $(n = 0, \dots N)$, as $\epsilon \to 0+$, the above inequality shows the desired conclusion. This completes the proof.

5. Application to Schrödinger operators

In this section, we shall consider an application of Theorem 1.3 and 1.5 to the so called Schrödinger operators of the form $-\Delta + V(x)$, where Δ is the generalized Laplacian and $V(x)$ is a potential regarded as a multiplication operator acting in $L^2(\mathbb{R}^d)$. It is not simple to specify the class of potentials for which the assumptions in Theorem 1.3 are fulfilled. However if $V(x)$, for example, is the Coulomb potential we can verify the assumption.

If we consider some specific potential, we can improve the assertion of Theorems 1.3 and 1.5. we shall see this improvement in the following paragraph. Assume that T satisfies the assumptions (a) and (b) of Theorem 1.3 and has the following two conditions;

(c') For any $\varphi \in \{f \in C^\infty(\mathbb{R}) | \sup_{x\in\mathbb{R}}|f(x)| < \infty\}$, $\varphi(T)X \subseteq X$ and $\|\varphi(T)\|_{B(X)} \leq \sup_{x\in\mathbb{R}}|\varphi(x)|$.

(d) There exists a constant C_0 such that

$$\sup_{a\geq 0, b\neq 0}\|R[T](a+ib)\|_{B(X,X^*)} \leq C_0,$$
$$\sup_{a>0}\|R^{\pm}[T](a)\|_{B(X,X^*)} \leq C_0.$$

Then we can choose $\mathcal{F}_{\lambda,\theta}(X)$ independent of λ and θ, and the estimate (1.4) becomes

$$\left\|R^{\pm}[T^{(n)}](\mu)\right\|_{B(\otimes_n X, \otimes_n X^*)} \leq (3n-2)C_0 \tag{5.1}$$

for any $\mu \in (0, \infty)$.

Proof. We shall first claim that

$$\sup_{\mu\in(0,\infty)}\left\|q_j^{\pm}(\mu)\right\| \leq C_0 \tag{5.2}$$

for $j = 2, 3, 4$. (5.2) can be proved as follows. If $j = 2$, it follows from immediately (3.16) and the assumption (d). For the case $j = 3$, we remark that

$$q_3^{\pm}(\mu)(f,g) = \lim_{\Lambda_N \to 0} \sum_{j=1}^{N} \big\langle \{(R^{\pm}(\mu - \xi_j)\chi(T)) \otimes I^{(n-1)}\} f, \\ \{I \otimes E_{n-1}((\omega_{j-1}, \omega_j])g\big\rangle \quad (5.3)$$

for any $f, g \in X \hat{\otimes} \mathcal{H}_{n-1}$. Here we have used the same notations as in (3.18). Then we have

$$\begin{aligned} \big|q_3^{\pm}(\mu)(f,g)\big| &\leq \lim_{\Lambda_N \to 0} \sum_{j=1}^{N} C_0 \|\chi(T)\|_{B(X)} \big\|\{I \otimes E_{n-1}((\omega_{j-1}, \omega_j])\} f\big\| \\ &\quad \cdot \big\|\{I \otimes E_{n-1}((\omega_{j-1}, \omega_j])\} g\big\| \\ &\leq C_0 \|f\| \|g\| \end{aligned} \quad (5.4)$$

for any $f, g \in X \otimes \mathcal{H}_{n-1}$ (see also the proof of Lemma 3.5). Here the norms are taken in $X \otimes \mathcal{H}_{n-1}$, and we use the assumptions (c') and (d) in (5.3). From (5.4), Lemma 2.1 and Lemma 2.3, we conclude that $\sup_{\mu \in (0,\infty)} \|q_3^{\pm}(\mu)\| \leq C_0$. We can prove (5.2) for the case $j = 4$ in the same way as the case $j = 3$. Thus we have the following estimate instead of (3.28);

$$\begin{aligned} &\|R_k(\mu \pm i\epsilon)\|_{B(X_k, X_k^*)} \\ &\qquad \leq \sup_{\delta \leq \rho \leq \theta - a} \|R_{k-1}(\rho \pm i\epsilon)\|_{B(X_{k-1}, X_{k-1}^*)} + 3C_0, \end{aligned} \quad (5.5)$$

from which we get

$$\sup_{\mu \in (0,\infty)} \big\|R[T^{(n)}](\mu \pm i\epsilon)\big\|_{B(\otimes_n X, (\otimes_n X)^*)} \leq (3n-2)C_0 \quad (5.6)$$

for any $n \in \mathbb{N}$. Letting $\epsilon \to 0+$, we find (5.1). □

The conditions (c') and (d) are deduced, if $V(x)$ does not have any zero eigenvalue nor any zero resonance and $X = H^{-1,s} \equiv \{f : \mathbb{R}^d \to \mathbb{C} \mid \|(1+|x|^2)^{s/2}(1-\Delta)^{-1/2} f\|_{L^2(\mathbb{R}^d)} < \infty\}$ with the space dimension $d = 3$ and $s > 5/2$ (see Jensen and Kato [4]).

Acknowledgements First of all, the author is grateful to Professor H. Kubo for leading him to the problem and constant encouragement. He would like to express his gratitude to Professor A. Arai who kindly gave a chance to discuss with him. He is also thanks to Dr. K. Doi and Dr.

Y. Matuzawa who pointed out essential mistakes. He is grateful to the referee for careful reading and useful comments.

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Division of Mathematics
Graduate School of Information Science
Tohoku University
Aoba-ku, Sendai, 980-8579, Japan
E-mail: shimizu@ims.is.tohoku.ac.jp