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On asymptotic behavior of positive solutions of $x'' = -t^{\alpha\lambda-2}x^{1+\alpha}$ with $\alpha < 0$ and $\lambda = 0, -1$

Ichiro TSUKAMOTO

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Abstract. Given an initial condition $x(T) = A$, $x'(T) = B$ ($' = d/dt$, $0 < T < \infty$, $0 < A < \infty$, $-\infty < B < \infty$) for the differential equation denoted in the title, we shall conclude that if T, A are fixed arbitrarily, then there exists a number B_* such that in every case of $B = B_*$, $B < B_*$, $B > B_*$ we determine analytical expressions of the solution of the initial value problem which shows asymptotic behavior of the solution. That is, these analytical expressions are valid in neighborhoods of ends of the domain of the solution. If $\lambda = -1$, then we shall treat the case $T = 0$, since there exists the solution continuable to $t = 0$.

Key words: Emden and Fowler type differential equations, asymptotic behavior, an initial condition, a first order rational differential equation, a two dimensional autonomous system.

1. Introduction

Let us consider Emden and Fowler type differential equations

$$x'' = -t^{-\alpha-2}x^{1+\alpha} \tag{E_-}$$

$$x'' = -t^{-2}x^{1+\alpha} \tag{E_0}$$

where $' = d/dt$, α is a negative parameter, and t, x are positive variables. The differential equations (E_-) and (E_0) are obtained from putting $\lambda = -1$ and $\lambda = 0$ in

$$x'' = -t^{\alpha\lambda-2}x^{1+\alpha} \tag{E}$$

respectively.

These differential equations are regarded to have several interesting physical applications (cf. [1]). Moreover these are equations of motion in the potential field and so Euler's equation of a variational problem. As for the partial differential equations, these are equations which the positive

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radial solutions fulfill.

Actually many papers treated these in more general form (cf. [5], [7], [8]) and discussed the solutions continuable to ∞. On the other hand, in [15], [21], and [22] we showed asympotic behavior of solutions of (E) under an initial condition

$$x(T) = A, \quad x'(T) = B \tag{I}$$

where

$$0 \le T < \infty, \quad 0 < A < \infty, \quad -\infty < B < \infty.$$

We treated the case $\alpha < 0$ and $\lambda > 0$ in [15], the case $\alpha < 0$ and $\lambda < -1$ or $\lambda > 0$ in [21], and the case $\alpha < 0$ and $-1 < \lambda < 0$ in [22].

However we did not consider (E_-) and (E_0). So in this paper we shall show asymptotic behavior of solutions of (E_-) and (E_0) satisfying (I).

Our discussion will be carried out as follows: In Section 2, we shall state our theorems on (E_-). For preliminaries of the proof, we shall use Sections 3 and 4, and the proof will be completed in Section 5. In Section 6, we shall state theorems on (E_0) which will be obtained directly from applying a transformation written in [9] to (E_-).

2. Theorems on (E_-)

First, suppose $0 < T < \infty$ and fix T and A arbitrarily in (I). Then if $x = x(t)$ denotes a solution of an initial value problem (E_-) and (I), asymptotic behavior of $x = x(t)$ is stated in the following theorems. Notice that ω_+ and ω_- are finite positive numbers in this section and $f(t) \sim g(t)$ as $t \to \tau$ for some τ means $\lim_{t\to\tau} f(t)/g(t) = 1$.

Theorem 1 *There exists a number B_* such that if $B = B_*$, then $x(t)$ is defined for $0 < t < \omega_+$. Moreover $x(t)$ is represented as*

$$x(t) \sim \frac{t}{(\alpha \log t)^{1/\alpha}} \left\{ 1 + \sum_{m+n>0} x_{mn} \left(\frac{1}{\alpha \log t} \log \frac{1}{\alpha \log t} \right)^m \left(\frac{1}{\alpha \log t} \right)^n \right\} \tag{2.1}$$

as $t \to 0$. Here x_{mn} are constants. In the neighborhood of $t = \omega_+$, $x(t)$ is represented as follows:

$$x(t) = K(\omega_+ - t)\Big\{1 + \sum_{j+k+l>0} x_{jkl}(\omega_+ - t)^j \times (\omega_+ - t)^{-(\alpha/2)k}(\omega_+ - t)^{((\alpha+2)/2)l}\Big\} \tag{2.2}$$

where K *and* x_{jkl} *are constants, if* $-2 < \alpha < 0$.

$$x(t) = tU^{1-G(U,C)}e^{CG(U,C)} \tag{2.3}$$

where

$$U \sim -\sqrt{2}\log\frac{t}{\omega_+} \quad \text{as } t \to \omega_+$$

$$G(U,C) = \frac{1}{2}(C - \log U)^{-1}\log(C - \log U) + \sum_{j+k+l\geq 2} d_{jkl}\{U(C - \log U)^2\}^j(C - \log U)^{-k/2} \times \{(C - \log U)^{-1}\log(C - \log U)\}^l$$

d_{jkl} *being constants, if* $\alpha = -2$.

$$x(t) = \Big\{-\frac{2(\alpha+2)\omega_+^{\alpha+2}}{\alpha^2}\Big\}^{1/\alpha}(\omega_+ - t)^{-2/\alpha} \times \Big\{1 + \sum_{m+n>0} x_{mn}(\omega_+ - t)^m(\omega_+ - t)^{(2(\alpha+2)/\alpha)n}\Big\} \tag{2.4}$$

where x_{mn} *are constants, if* $\alpha < -4$, $-4 < \alpha < -2$.

$$x(t) = \sqrt{2\omega_+}\,(\omega_+ - t)^{1/2}\Big\{1 + \sum_{k=1}^{\infty}(\omega_+ - t)^k p_k(\log(\omega_+ - t))\Big\} \tag{2.5}$$

where p_k *are polynomials with* $\deg p_k \leq k$, *if* $\alpha = -4$.

If $B \neq B_*$, then we get the following:

Theorem 2 *If* $B < B_*$, *then* $x(t)$ *is defined for* $0 < t < \omega_+$. *Furthermore* *in the neighborhood of* $t=0$, $x(t)$ *is represented as*

$$x(t) = K\left(1 + \sum_{m+n>0} x_{mn} t^{-\alpha m} t^{n}\right) \tag{2.6}$$

if $-1/\alpha \notin \boldsymbol{N}$, *and as*

$$x(t) = K\left\{1 + \sum_{k=1}^{\infty} t^{-\alpha k} p_k(\log t)\right\} \tag{2.7}$$

if $-1/\alpha \in \boldsymbol{N}$. *Here* K *and* x_{mn} *are constants and* p_k *are polynomials with* $\deg p_k \leq [-\alpha k]$ *where* $[\ \ ]$ *denotes Gaussian symbol. In the neighborhood of* $t=\omega_+$ *we have* (2.2) *through* (2.5).

Theorem 3 *If* $B > B_*$, *then* $x(t)$ *is defined for* $\omega_- < t < \omega_+$. *Moreover in the neighborhood of* $t=\omega_-$, $x(t)$ *is represented as follows*:

$$\begin{aligned} x(t) = K(t-\omega_-)\Big\{1 + \sum_{j+k+l>0} & x_{jkl}(t-\omega_-)^{j} \\ & \times (t-\omega_-)^{-(\alpha/2)k}(t-\omega_-)^{((\alpha+2)/2)l}\Big\} \end{aligned} \tag{2.8}$$

where K *and* x_{jkl} *are constants, if* $-2 < \alpha < 0$.

$$x(t) = tU^{1-G(U,C)}e^{CG(U,C)} \tag{2.9}$$

where

$$U \sim \sqrt{2}\log\frac{t}{\omega_-} \quad \text{as } t \to \omega_-$$

and $G(U,C)$ *is defined in Theorem* 1, *if* $\alpha = -2$.

$$\begin{aligned} x(t) = & \left\{-\frac{2(\alpha+2)\omega_-^{\alpha+2}}{\alpha^2}\right\}^{1/\alpha}(t-\omega_-)^{-2/\alpha} \\ & \times \left\{1 + \sum_{m+n>0} x_{mn}(t-\omega_-)^{m}(t-\omega_-)^{(2(\alpha+2)/\alpha)n}\right\} \end{aligned} \tag{2.10}$$

where x_{mn} are constants, if $\alpha < -4$, $-4 < \alpha < -2$.

$$x(t) = \sqrt{2\omega_-}(t-\omega_-)^{1/2}\left\{1 + \sum_{k=1}^{\infty}(t-\omega_-)^k p_k(\log(t-\omega_-))\right\} \tag{2.11}$$

where p_k are polynomials with $\deg p_k \le k$, if $\alpha = -4$. In the neighborhood of $t = \omega_+$, we get (2.2) through (2.5).

Next, suppose

$$T = 0, \quad 0 < A < \infty, \quad -\infty \le B \le \infty$$

in (I). Then accurately speaking, (I) means

$$\lim_{t\to 0} x(t) = A, \quad \lim_{t\to 0} x'(t) = B.$$

Moreover if $x = x(t)$ denotes a solution of (E_-) and (I) again, we state existence and asymptotic behavior of $x = x(t)$ as follows:

Theorem 4 *Assume $-1/\alpha \notin \boldsymbol{N}$ and $-1 < \alpha < 0$. Then if $B = \infty$, there exist the infinitely many solution represented as*

$$x(t) = A\left\{1 - \frac{A^\alpha}{\alpha(\alpha+1)}t^{-\alpha} + x_{01}t + \sum_{m+n>1} x_{mn}t^{-\alpha m}t^n\right\} \tag{2.12}$$

in the neighborhood of $t = 0$. Here x_{01} is an arbitrary constant. Moreover if $B \neq \infty$, there exist no solution.

Theorem 5 *Suppose $-1/\alpha \notin \boldsymbol{N}$ and $\alpha < -1$. Then if $B \neq \pm\infty$, there exists the unique solution represented as*

$$x(t) = A\left\{1 - \frac{A^\alpha}{\alpha(\alpha+1)}t^{-\alpha} + \frac{B}{A}t + \sum_{m+n>1} x_{mn}t^{-\alpha m}t^n\right\} \tag{2.13}$$

in the neighborhood of $t = 0$ and if $B = \pm\infty$, there exists no solution.

Next we consider the case $-1/\alpha \in \boldsymbol{N}$. If $\alpha = -1$, then (E_-) can be solved explicitly as

$$x(t) = -t\log t + \Gamma t + A$$

where Γ is an arbitrary constant. Obviously this implies that if $B = \infty$, then the infinitely many solutions exist and if $B \neq \infty$, no solution exists.

Theorem 6 *Assume* $-1/\alpha \in \boldsymbol{N}$ *and* $-1 < \alpha < 0$. *Then if* $B = \infty$, *there exists the unique solution represented as*

$$x(t) = A\biggl\{1 - \frac{A^\alpha}{(\alpha+1)\alpha} t^{-\alpha} + \sum_{k=2}^{\infty} t^{-\alpha k} p_k(\log t)\biggr\} \tag{2.14}$$

in the neighborhood of $t = 0$. *Here* p_k *are polynomials with* $\deg p_k \leq [-\alpha k]$. *Furthermore if* $B \neq \infty$, *then there exists no solution.*

Since the solutions (2.12), (2.13), and (2.14) are obtained from (2.6) and (2.7), we immediately get the following from Theorem 2:

Theorem 7 *A solution of* (E_-) *and* (I) *is defined for* $(0, \omega_+)$ *and represented as* (2.2) *through* (2.5) *in the neighborhood of* $t = \omega_+$.

From the proof of Theorems 1, 2, and 3, we shall not get the solution in the cases $B = \pm\infty$. Hence it is not necessary to consider such cases if $0 < T < \infty$.

3. Reduction of (E_-) and investigation of the reduced equation

We use a transformation devised originally in [10]. Now we put

$$y = t^{-\alpha} x^\alpha \quad (\text{namely } x = t y^{1/\alpha}), \quad z = t y' \tag{T}$$

and get a first order rational differential equation

$$\frac{dz}{dy} = \frac{(\alpha-1)z^2 - \alpha y z - \alpha^2 y^3}{\alpha y z}. \tag{R}$$

Using a parameter s, we write this a two dimensional autonomous system

$$\frac{dy}{ds} = \alpha y z, \quad \frac{dz}{ds} = (\alpha-1)z^2 - \alpha y z - \alpha^2 y^3. \tag{S}$$

The critical point of (S) is only $(y, z) = (0, 0)$ and from (T) and $x > 0$ we always have $y > 0$.

Here let us consider (R) in the neighborhood of $y = 0$. For this we put

$$u = y^{-2}z \quad (\text{namely } z = y^2u) \tag{3.1}$$

and get

$$\frac{du}{dy} = -\frac{\alpha^2 + \alpha u + (\alpha+1)yu^2}{\alpha y^2 u}. \tag{3.2}$$

In the righthand side of this, its numerator vanishes as $y = 0$, if and only if

$$u = -\alpha.$$

So, suppose that a solution u of (3.2) accumulates to γ as $y \to 0$. Then we conclude the following:

Lemma 3.1 *γ is a limit point and $\gamma = -\alpha$, $\pm\infty$.*

Proof. It suffices to follow the line of the proof of Lemma 3.1 of [19]. □

Lemma 3.2 *If $\gamma = -\alpha$, then from u we get a unique solution of (R) represented as*

$$z = -\alpha y^2 \left\{ 1 + \sum_{n=1}^{N-1} z_n y^n + O(y^N) \right\} \quad \textit{as } y \to 0 \tag{3.3}$$

where $N \in \boldsymbol{N}$ and z_n are constants. Moreover from (3.3) *we have a solution of (E_-) expressed as* (2.1).

Proof. Following discussion for obtaining (3.5) of [19], we get (3.3) and its unique existence. In addition, from (T) and the proof of Lemma 3.2 of [19] we have (2.1). This completes the proof. □

Since (3.3) exists uniquely, we denote this $z = z_1(y)$.

Here suppose $\gamma = \pm\infty$. Then we put

$$u = 1/v, \quad w = y^{-1}v$$

and get

$$y\frac{dw}{dy} = \frac{1}{\alpha}w + w^2 + \alpha y w^3 \tag{3.4}$$

whose righthand side vanishes if and only if

$$w = 0, \quad -\frac{1}{\alpha},$$

in the case $y = 0$. Now suppose that a solution w of (3.4) accumulates to δ as $y \to 0$. Then the following lemma holds:

Lemma 3.3 *$\delta = -1/\alpha$ and from w we get a solution of (R) represented as*

$$z = -\alpha y\left[1 + \sum_{m+n>0} z_{mn} y^m \{y^{-1/\alpha}(h \log y + C)\}^n\right] \tag{3.5}$$

in the neighborhood of $y = 0$. Here z_{mn}, h, and C are constants and $h = 0$ if $-1/\alpha \notin \boldsymbol{N}$. Moreover from this we have (2.6) and (2.7).

Proof. If $\delta \neq 0$, $-1/\alpha$, $\pm\infty$, then from (3.4) we obtain

$$\frac{dy}{dw} = \frac{y}{(1/\alpha)w + w^2 + \alpha y w^3}.$$

Since the righthand side of this is holomorphic at $(y, w) = (0, \delta)$, we get a contradiction $y \equiv 0$. Hence $\delta = 0$, $-1/\alpha$, $\pm\infty$.

If $\delta = 0$, then we immediately have a contradiction $w \equiv 0$ from Lemma 2.5 of [16]. Moreover if $\delta = \pm\infty$, then we see the proof of Lemma 3.3 of [19] and get the contradiction from the discussion of the case $\delta = \pm\infty$ of this. Hence we obtain $\delta = -1/\alpha$.

So we put $\theta = w + 1/\alpha$ and have

$$y\frac{d\theta}{dy} = -\frac{1}{\alpha}\theta + \theta^2 + \alpha y\left(\theta - \frac{1}{\alpha}\right)^3.$$

Therefore from $-1/\alpha > 0$ we get

$$\theta = \sum_{m+n>0} \theta_{mn} y^m \{y^{-1/\alpha}(h \log y + C)\}^n$$

where θ_{mn}, h, and C are constants, $\theta_{01} = 1$, and $h = 0$ if $-1/\alpha \notin \boldsymbol{N}$. Since we put

$$z = y^2 u, \quad u = 1/v, \quad w = y^{-1} v, \quad \theta = w + 1/\alpha,$$

we eventually get (3.5). Furthermore from (T) we have a differential equation

$$ty' = -\alpha y \left[1 + \sum_{m+n>0} z_{mn} y^m \{ y^{-1/\alpha} (h \log y + C) \}^n \right].$$

Solving this, we obtain

$$y = \Gamma t^{-\alpha} \left[1 + \sum_{m+n>0} y_{mn} t^{-\alpha m} \{ t (\tilde{h} \log t + \tilde{C}) \}^n \right]$$

where Γ and y_{mn} are constants and

$$\tilde{h} = -\alpha h, \quad \tilde{C} = h \log \Gamma + C.$$

Using (T) again, we have

$$x(t) = \Gamma^{1/\alpha} \left[1 + \sum_{m+n>0} x_{mn} t^{-\alpha m} \{ t (\tilde{h} \log t + \tilde{C}) \}^n \right]$$

where x_{mn} are constants. Hence if $-1/\alpha \notin \boldsymbol{N}$, then since $h = \tilde{h} = 0$ we get (2.6) where $K = \Gamma^{1/\alpha}$. Moreover if $1/\alpha \in \boldsymbol{N}$, then we have

$$x(t) = K \left\{ 1 + \sum_{m+n>0} t^{-\alpha(m - n/\alpha)} P_{mn}(\log t) \right\}$$

where P_{mn} are polynomials with $\deg P_{mn} \leq n$. So if we put $k = m - n/\alpha$ and $p_k = P_{mn}$, then we get (2.7). Now the proof is complete. □

4. The investigation of the reduced equation $(\boldsymbol{R})$ in the neighborhood of $\boldsymbol{y = \infty}$

First we show the following:

Lemma 4.1 *If c denotes an arbitrary constant with $0 < c < \infty$, then a solution $z(y)$ of (R) is bounded as $y \to c$.*

Proof. Put $z = 1/\zeta$ in (R). Then we get

$$\frac{d\zeta}{dy} = -\frac{(\alpha - 1 - \alpha yz - \alpha^2 y^3 \zeta^2)\zeta}{\alpha y}. \tag{4.1}$$

If this lemma is false, then there exists a sequence $\{y_n\}$ such that

$$y_n \to c, \quad z(y_n) \to \pm\infty \quad \text{as } n \to \infty.$$

Hence $\zeta(y) = 1/z(y)$ is a solution of (4.1) such that $\zeta(y_n) \to 0$ as $y_n \to c$. Since the righthand side of (4.1) is holomorphic at $(y, \zeta) = (c, 0)$, we therefore get a contradiction $\zeta(y) \equiv 0$. This completes the proof. $\square$

If we apply (T) to a solution $x = x(t)$ of (E_-) and define (y, z), then z is a solution of (R) and (y, z) a solution of (S). Let (ω_-, ω_+) be the domain of $x(t)$. Then we obtain the following:

Lemma 4.2 *As $t \to \omega_\pm$, (y, z) does not converge to a point in a region $0 < y < \infty$, $-\infty < z < \infty$.*

Since this is Lemma 2 of [17] or Lemma 4.1 of [19], the proof is omitted.

Here we consider (R) in the neighborhood of $y = \infty$. For this we put $y = 1/\eta$ in (4.1) and get

$$\frac{d\zeta}{d\eta} = \frac{(\alpha - 1)\eta^3\zeta - \alpha\eta^2\zeta^2 - \alpha^2\zeta^3}{\alpha\eta^4}.$$

Moreover putting $w = \eta^{-3/2}\zeta$ and $\xi = \eta^{1/2}$, we have

$$\xi\frac{dw}{d\xi} = -\frac{\alpha + 2}{\alpha}w - 2\xi w^2 - 2\alpha w^3. \tag{4.2}$$

If the righthand side vanishes in the case $\xi = 0$, then we get

$$w = 0 \quad \text{as } -2 \le \alpha < 0,$$
$$w = 0, \ \pm\rho \quad \text{as } \alpha < -2,$$

where

$$\rho = \frac{1}{\alpha}\sqrt{\frac{\alpha + 2}{-2}}.$$

Suppose that a solution w of (4.2) accumulates to γ as $\xi \to 0$. Then we have the following:

Lemma 4.3 *γ is a limit point and $\gamma = 0$, $\pm\rho$.*

Proof. We get $\gamma = 0$, $\pm\rho$, $\pm\infty$ directly from the reasoning in the proof of Lemma 4.3 of [19]. Hence γ is a limit point. Moreover from the reasoning in the same proof we have a contradiction, if $\gamma = \pm\infty$. Thus $\gamma \neq \pm\infty$ and the proof is complete. □

Here suppose $\gamma = 0$. Then we conclude the following:

Lemma 4.4 *If $-2 < \alpha < 0$, then from w we get a solution z of (R) represented as*

$$z^{-1} = C\xi^{-(\alpha+2)/\alpha+3}\left\{1 + \sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n\right\} \tag{4.3}$$

in the neighborhood of $\xi = 0$. Here C and w_{mn} are constants. Furthermore from z we have a solution of $x = x(t)$ of (E_-) which is represented as (2.2) if $z > 0$, and as (2.8) if $z < 0$.

Proof. From (4.2) we have

$$w = C\xi^{-(\alpha+2)/\alpha}\left\{1 + \sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n\right\},$$

since $-(\alpha+2)/\alpha > 0$ from $-2 < \alpha < 0$ and w devides the righthand side of (4.2). Hence from $w = \eta^{-3/2}\zeta$ and $\zeta = 1/z$ we get (4.3).

On the other hand, we have

$$y' = \left(\frac{1}{\eta}\right)' = -2\xi^{-3}\xi'. \tag{4.4}$$

Therefore applying (T) to (4.3), we obtain

$$C\xi^{-(\alpha+2)/\alpha}\left\{1 + \sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n\right\}\xi' = -\frac{1}{2t}$$

and integrating both sides,

$$-\frac{\alpha}{2}C\xi^{-2/\alpha}\left\{1+\sum_{m+n>0}\hat{w}_{mn}\xi^m(C\xi^{-(\alpha+2)/\alpha})^n\right\}=-\frac{1}{2}\log t+\Gamma \tag{4.5}$$

where $\hat{w}_{mn}$ and Γ are constants. Here suppose that

$$t \to \tau \quad as\ \xi \to 0.$$

Then we get

$$\Gamma = \frac{1}{2}\log\tau.$$

Hence determining ξ from (4.5), we have

$$\begin{aligned}\xi = \left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-\alpha/2}\Bigg[1+\sum_{m+n>0}\xi_{mn}\left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-(\alpha/2)m}\\ \times\left\{C\left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{(\alpha+2)/2}\right\}^n\Bigg]\end{aligned} \tag{4.6}$$

where ξ_{mn} are constants.

Recalling that (ω_-, ω_+) denotes the domain of the solution $x = x(t)$ of (E_-), we get

$$\tau = \omega_+ \text{ if } z > 0, \quad \tau = \omega_- \text{ if } z < 0.$$

Indeed, if $z > 0$ then from (T) we have $y' > 0$ and from (4.4), $\xi' < 0$. Therefore as $\xi \downarrow 0$, y tends to the right end of its domain which is (ω_-, ω_+) from (T). If $z < 0$, then the similar discussion follows.

Consequently expanding $\log t/\tau$ as

$$\log\frac{t}{\omega_-} = \frac{t-\omega_-}{\omega_-} - \frac{1}{2}\left(\frac{t-\omega_-}{\omega_-}\right)^2 + \cdots,$$

$$\log\frac{t}{\omega_+} = -\frac{\omega_+-t}{\omega_+} - \frac{1}{2}\left(\frac{\omega_+-t}{\omega_+}\right)^2 - \cdots,$$

in (4.6), we get (2.2) if $z > 0$, and (2.8) if $z < 0$. Now the proof is complete. $\square$

Lemma 4.5 *If $\alpha = -2$, then from w we have a solution z of (R) represented as*

$$z^{-1} = \pm\frac{\sqrt{2}}{4}\xi^3(C-\log\xi)^{-1/2} \times \left\{1 + \sum_{1\le 2j+k<2(N+1)} w_{jk}\xi^j(C-\log\xi)^{-k/2} + \Omega\right\} \tag{4.7}$$

in the neighborhood of $\xi = 0$. Here C and w_{jk} are constants, $N \in \boldsymbol{N}$, and Ω is a function with

$$|\Omega| \le K|\log\xi|^{-N}$$

where K is a constant. Moreover from z we get a solution $x = x(t)$ of (E_-) which is represented as (2.3)*, if $z > 0$, and as* (2.9)*, if $z < 0$.*

Proof. If $\alpha = -2$, then $-(\alpha+2)/\alpha$ vanishes. So we apply the theory of [3] to (4.2) and have

$$w = \pm\frac{\sqrt{2}}{4}(C-\log\xi)^{-1/2} \times \left\{1 + \sum_{1\le 2j+k<2(N+1)} w_{jk}\xi^j(C-\log\xi)^{-k/2} + \Omega\right\}$$

in the neighborhood of $\xi = 0$. Therefore we get (4.7).

Next applying (T) to (4.7) and discussing as in Section 2 of [18] (or Section 5 of [21]), we have (2.3) and (2.9). As in the proof of Lemma 4.4 we conclude whether we obtain (2.3) or (2.9). Now the proof is complete. □

In the case $-2 \le \alpha < 0$ we got the solutions of (E_-) from a solution w of (4.2). However in the case $\alpha < -2$ we conclude the following:

Lemma 4.6 *If $\alpha < -2$, then there is no solution of (E_-) obtained from w.*

Proof. Since $-(\alpha+2)/\alpha < 0$ in this case, we get $w \equiv 0$ from Lemma 2.5 of [16]. This completes the proof. □

Finally suppose $\alpha < -2$ and $\gamma = \pm\rho$. Then we have the following:

Lemma 4.7 *We get a solution z of (R) represented as*

$$z^{-1} = \xi^3 \left[\gamma + \sum_{m+n>0} u_{mn} \xi^m \{ \xi^{2(\alpha+2)/\alpha} (h \log \xi + C) \}^n \right] \tag{4.8}$$

in the neighborhood of $\xi = 0$. Here u_{mn}, h, and C are constants and $h = 0$ if $\alpha \neq -4$. Furthermore from (4.8) we have a solution of (E_-) which is represented as (2.4) and (2.5), if $\gamma = \rho$, and as (2.10) and (2.11), if $\gamma = -\rho$.

Proof. Putting $u = w - \gamma$, we get

$$z^{-1} = \xi^3(\gamma + u).$$

This is similar to the transformation used in Section 3 of [11]. Hence it suffices to follow the discussion of this. Since in the neighborhood of $\xi = 0$ we have $z > 0$ and $z < 0$ respectively from $\gamma = \rho$ and $\gamma = -\rho$. Therefore as in the proof of Lemma 4.4 we again conclude whether we get (2.4), (2.5) or (2.10), (2.11). Now the proof is complete. □

5. Proofs of our theorems

Recall the conclusions obtained in Sections 3 and 4. Then if a solution $z = z(y)$ of (R) is continuable to $y = 0$, $z = z(y)$ is given only as (3.3) and (3.5). (3.3) (namely $z = z_1(y)$) exists uniquely. It follows from Lemma 4.1 that $z(y)$ is bounded as $y \to c$ $(0 < c < \infty)$. If $z(y)$ can be continued to $y = \infty$, then $z(y)$ is given only as (4.3), (4.7) and (4.8). From these we get

$$z(y) \to \pm\infty \quad \text{as } y \to \infty.$$

Furthermore on the y axis we have

$$\frac{dz}{ds} = -\alpha^2 y^3 < 0$$

from (S). Therefore an orbit of (S) passes the y axis at most once. Now notice that a solution of (R) is an orbit of (S). Then the phase portrait of (S) is as in Figure below. Here the direction of the orbits of (S) are judged from the sign of dy/ds in (S).

Now we consider the case $0 < T < \infty$ in (I).

Proof of Theorem 1. From (T) we get

$$z = -\alpha y\left(1 - \frac{tx'}{x}\right) \tag{5.1}$$

and putting $t = T$ in (5.1),

$$z(y_0) = z_0 \tag{5.2}$$

where

$$y_0 = T^{-\alpha}A^{\alpha}, \quad z_0 = -\alpha y_0\left(1 - \frac{TB}{A}\right).$$

Therefore from the solution of (E_-) and (I) we have a solution of (R) satisfying the initial condition (5.2). Conversely from the solution (R) and (5.2) we get the solution of (E_-) and (I).

Here fix T and A in (I). Then y_0 is fixed and z_0 is a decreasing function of B. Let a point (y_0, z_0) be an intersection of a line $y = y_0$ and an orbit $z = z_1(y)$. Then we suppose that B attains the value B_*. Moreover if we define y, z from applying (T) to a solution $x = x(t)$ of (E_-) and (I), then (y, z) lies on the orbit $z = z_1(y)$ and from Lemma 4.2 we get

$$y \to 0 \quad \text{as } t \to \omega_-, \quad y \to \infty \quad \text{as } t \to \omega_+ \tag{5.3}$$

where (ω_-, ω_+) denotes the domain of $x(t)$ also here. In fact from $z_1(y) = ty' > 0$, y is an increasing function of t. On the other hand, from Lemma 3.2 we have (2.1) and $\omega_- = 0$. In addition from Lemmas 4.4 through 4.7, we get (2.2) through (2.5), and $\omega_+ < \infty$. These complete the proof. □

Proof of Theorem 2. Since z_0 is decreasing in B, if $B < B_*$ then (y_0, z_0) lies above $z = z_1(y)$. Moreover we conclude (5.3) as above. Therefore from Figure below and Lemma 3.3 we get (2.6), (2.7), and $\omega_- = 0$. The rest of the proof is the same as in the proof of Theorem 1. Thus the proof is complete. □

Proof of Theorem 3. If $B > B_*$, then (y_0, z_0) lies under $z = z_1(y)$. Therefore if we define y, z as in the proof of Theorem 1, then (y, z) lies also under $z = z_1(y)$ and from Lemma 4.2, Figure, and the sign of z we have

$$y \to \infty \quad \text{as } t \to \omega_\pm.$$

Here z is not a single-valued function of y. Hence Lemmas 4.4 through 4.7 completes the proof. □

Theorems 4, 5, and 6 are got from the solutions (2.6) and (2.7), since from Theorems 1, 2, and 3 only these are solutions of (E_-) continuable to $t = 0$ and satisfying $x(0) > 0$. In order to start the proofs of Theorems 4, 5, and 6, put $T = 0$ in (I).

Proof of Theorem 4. From $-1/\alpha \notin \boldsymbol{N}$, we require only (2.6). Substituting (2.6) into (E_-), we get

$$\begin{aligned} K &\sum_{m+n>0} (-\alpha m + n)(-\alpha m + n - 1) x_{mn} t^{-\alpha m + n - 2} \\ &= -K^{1+\alpha} t^{-\alpha-2} \left\{ 1 + \sum_{m+n>0} P_{mn}(x_{MN}) t^{-\alpha m + n} \right\} \end{aligned}$$

where $P_{mn}(x_{MN})$ are polynomials of x_{MN} with $M \le m$ and $N \le n$. Therefore we have

$$\begin{aligned} &\sum_{m+n>0} (-\alpha m + n)(-\alpha m + n - 1) x_{mn} t^{-\alpha m + n} \\ &\quad = -K^\alpha t^{-\alpha} - \sum_{m+n>1} Q_{mn}(x_{MN}) t^{-\alpha m + n} \end{aligned}$$

where $Q_{mn} = K^\alpha P_{m-1n}$ and hence Q_{mn} are polynomials of x_{MN} with $M \le m - 1$ and $N \le n$. From this we conclude that

$$x_{10} = -\frac{K^\alpha}{\alpha(\alpha+1)}$$

and x_{01} is an arbitrary constant. Moreover we get

$$\begin{aligned} x_{0n} &= 0 \quad (n \ge 2), \\ x_{mn} &= -\frac{Q_{mn}(x_{MN})}{(\alpha m - n)(\alpha m - n + 1)} \quad (m \ge 1, n \ge 1). \end{aligned}$$

Therefore if we fix x_{01}, then x_{mn} are uniquely determined. Furthermore applying (I) to (2.6) we have (2.12) and

$$x'(t) \sim \frac{A^{\alpha+1}}{\alpha+1} t^{-\alpha-1} \quad \text{as } t \to 0.$$

Hence we get

$$x'(t) \to \infty \quad \text{as } t \to 0$$

which implies nonexistence of the solution, if $B \neq \infty$. This completes the proof. □

Proof of Theorem 5. In the same way, we have (2.12). Therefore from $\alpha < -1$ we get

$$x'(t) \sim Ax_{01} \quad \text{as } t \to 0 \tag{5.4}$$

and from (I),

$$x_{01} = \frac{B}{A}.$$

Thus we get (2.13). Moreover if $B = \pm\infty$, then there exists no solution from (5.4). Now the proof is complete. □

Proof of Theorem 6. Here we require only (2.7), since $-1/\alpha \in \boldsymbol{N}$. Substituting this into (E_-), we have

$$\begin{aligned}\sum_{k=1}^{\infty} & t^{-\alpha k}\{\ddot{p}_k(s) - (2\alpha k + 1)\dot{p}_k(s) + (\alpha k + 1)\alpha k p_k(s)\} \\ &= -K^\alpha t^{-\alpha} - \sum_{k=2}^{\infty} t^{-\alpha k} K^\alpha P_{k-1}(p_K)\end{aligned}$$

where $s = \log t$, $\dot{} = d/ds$, and P_k are polynomials of p_K with $K \leq k$. Therefore if $k = 1$, then we get

$$\ddot{p}_1(s) - (2\alpha + 1)\dot{p}_1(s) + (\alpha + 1)\alpha p_1(s) = -K^\alpha$$

and from $\deg p_1 \leq 1$,

$$p_1(s) = -\frac{K^\alpha}{(\alpha+1)\alpha}.$$

Moreover if $k \geq 2$, then we have

$$\ddot{p}_k(s) - (2\alpha k+1)\dot{p}_k(s) + (\alpha k+1)\alpha k p_k(s) = -K^\alpha P_{k-1}(p_K)$$

and solving this,

$$p_k(s) = K^\alpha \left\{ t^{\alpha k} \int P_{k-1}(p_K) s^{-\alpha k} ds - t^{\alpha k+1} \int P_{k-1}(p_K) s^{-(\alpha k+1)} ds \right\},$$

since $p_k(s)$ are polynomials of s. Thus p_k are uniquely determined. Moreover applying (I) to (2.7), we get (2.14). From (2.14) we have

$$x'(t) \sim \frac{A^{\alpha+1}}{\alpha+1} t^{-\alpha-1} \to \infty \quad \text{as } t \to 0.$$

Hence if $B \neq \infty$, then there exists no solution. This completes the proof. □

Finally notice that the proof of Theorem 7 has been already given in Section 2.

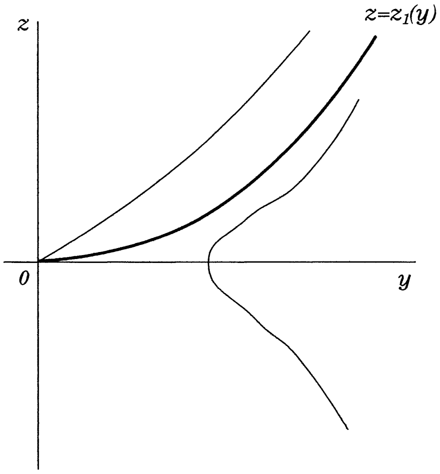

Figure. The phase portrait of (S).

6. On positive solutions of (E_0)

In this section, let us consider (E_0), namely

$$x'' = -t^{-2}x^{1+\alpha}$$

under the initial condition (I). First, suppose $0 < T < \infty$ and fix T and A. Then if $x = x(t)$ is a solution of an initial value problem (E_0) and (I), and $\omega_\pm$ are finite positive numbers with $\omega_- < \omega_+$ again, we conclude the following:

Theorem 8 *There exists a number B_* such that if $B = B_*$, then $x(t)$ is defined for (ω_-, ∞). Moreover $x(t)$ is represented as*

$$x(t) \sim \left(\frac{-1}{\alpha \log t}\right)^{1/\alpha} \left\{1 + \sum_{m+n>0} x_{mn} \left(\frac{\log(-\alpha \log t)}{\alpha \log t}\right)^m \left(\frac{-1}{\alpha \log t}\right)^n\right\} \tag{6.1}$$

as $t \to \infty$. Here x_{mn} are constants. In the neighborhood of $t = \omega_-$, $x(t)$ is represented as follows:

$$x(t) = K(t-\omega_-)\left\{1 + \sum_{j+k+l>0} x_{jkl}(t-\omega_-)^j \right.$$
$$\left. \times (t-\omega_-)^{-(\alpha/2)k}(t-\omega_-)^{((\alpha+2)/2)l}\right\} \tag{6.2}$$

where K and x_{jkl} are constants, if $-2 < \alpha < 0$.

$$x(t) = U^{1-G(U,C)}e^{CG(U,C)} \tag{6.3}$$

where

$$U \sim \sqrt{2}\,\log\frac{t}{\omega_-} \quad \text{as } t \to \omega_-$$

and $G(U,C)$ is defined in Theorem 1, if $\alpha = -2$.

$$x(t) = \left\{ -\frac{2(\alpha+2)\omega_-^2}{\alpha^2} \right\}^{1/\alpha} (t-\omega_-)^{-2/\alpha}$$
$$\times \left\{ 1 + \sum_{m+n>0} x_{mn}(t-\omega_-)^m (t-\omega_-)^{(2(\alpha+2)/\alpha)n} \right\} \tag{6.4}$$

where x_{mn} are constants, if $\alpha < -4$, $-4 < \alpha < -2$.

$$x(t) = \sqrt{\frac{2}{\omega_-}}\,(t-\omega_-)^{1/2} \left\{ 1 + \sum_{k=1}^{\infty} (t-\omega_-)^k p_k(\log(t-\omega_-)) \right\} \tag{6.5}$$

where p_k are polynomials with $\deg p_k \leq k$, if $\alpha = -4$.

Proof. Replace x, t with w, τ respectively in (E_-) and put

$$w = \frac{x}{t}, \quad \tau = \frac{1}{t}.$$

Then it follows from [9] that we get (E_0) from (E_-). Moreover if the initial condition of (E_-) is given as

$$w(\tilde{T}) = \tilde{A}, \quad w'(\tilde{T}) = \tilde{B},$$

then in the initial condition (I) of (E_0) we have

$$T = \frac{1}{\tilde{T}}, \quad A = \frac{\tilde{A}}{\tilde{T}}, \quad B = \tilde{A} - \tilde{B}\tilde{T}$$

since

$$x'(t) = w(\tau) - \tau w'(\tau).$$

Therefore letting B_* of our theorem be $\tilde{A} - B_*\tilde{T}$ where B_* appeared in Theorem 1, we complete the proof from Theorem 1. □

Theorem 9 *If $B < B_*$, then $x(t)$ is defined for $\omega_- < t < \omega_+$. Moreover in the neighborhood of $t = \omega_-$, we get* (6.2) *through* (6.5)*. In the neighborhood of $t = \omega_+$, $x(t)$ is represented as follows*:

$$x(t) = K(\omega_+ - t)\Big\{1 + \sum_{j+k+l>0} x_{jkl}(\omega_+ - t)^j \times (\omega_+ - t)^{-(\alpha/2)k}(\omega_+ - t)^{((\alpha+2)/2)l}\Big\} \tag{6.6}$$

where K *and* x_{jkl} *are constants, if* $-2 < \alpha < 0$.

$$x(t) = U^{1-G(U,C)}e^{CG(U,C)} \tag{6.7}$$

where

$$U \sim -\sqrt{2}\log\frac{t}{\omega_+} \quad \text{as } t \to \omega_+$$

and $G(U,C)$ *is defined in Theorem* 1, *if* $\alpha = -2$.

$$x(t) = \Big\{-\frac{2(\alpha+2)\omega_+^2}{\alpha^2}\Big\}^{1/\alpha}(\omega_+ - t)^{-2/\alpha} \times \Big\{1 + \sum_{m+n>0} x_{mn}(\omega_+ - t)^m(\omega_+ - t)^{(2(\alpha+2)/\alpha)n}\Big\} \tag{6.8}$$

where x_{mn} *are constants, if* $\alpha < -4$, $-4 < \alpha < -2$.

$$x(t) = \sqrt{\frac{2}{\omega_+}}(\omega_+ - t)^{1/2}\Big\{1 + \sum_{k=1}^{\infty}(\omega_+ - t)^k p_k(\log(\omega_+ - t))\Big\} \tag{6.9}$$

where p_k *are polynomials with* $\deg p_k \le k$, *if* $\alpha = -4$.

Proof. Change the variables as in the proof of Theorem 8. Then noticing that B is a decreasing function of $\tilde{B}$ for fixed $\tilde{T}$, $\tilde{A}$, we conclude our theorem from Theorem 3. $\square$

Theorem 10 *If* $B > B_*$, *then* $x(t)$ *is defined for* (ω_-, ∞). *Furthermore in the neighborhood of* $t = \omega_-$, *we get* (6.2) *through* (6.5). *In the neighborhood of* $t = \infty$, $x(t)$ *is represented as*

$$x(t) = Kt\Big(1 + \sum_{m+n>0} x_{mn}t^{\alpha m}t^{-n}\Big) \tag{6.10}$$

if $-1/\alpha \notin \boldsymbol{N}$*, and as*

$$x(t) = Kt\left\{1 + \sum_{k=1}^{\infty} t^{\alpha k} p_k(\log t)\right\} \tag{6.11}$$

if $-1/\alpha \in \boldsymbol{N}$*, where* K *and* x_{mn} *are constants and* p_k *are polynomials with* $\deg p_k \leq [-\alpha k]$.

Proof. As in the proof of Theorem 9, we obtain our theorem directly from Theorem 2. □

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3-10-38 Higashi-kamagaya, Kamagaya-shi
Chiba 273-0104, Japan
E-mail: itkmt@k9.dion.ne.jp