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Transference of Density

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1. Introduction and Notation

Let $H = \{(x, y) : y \geq 0\}$ denote the upper half-plane. This paper concerns various linear densities of a set $E \subset H$ at points of $\mathbb{R}$, which we identify with the boundary of H.

We shall denote by $L(x, \theta)$ the ray $\{(x + t\cos\theta, t\sin\theta) : t \geq 0\}$ for every $x \in \mathbb{R}$ and $\theta \in (0, \pi)$. The segment $\{(x + t\cos\theta, t\sin\theta) : 0 \leq t \leq r\}$ will be denoted by $L(x, \theta, r)$. The density of E along the ray $L(x, \theta)$ is defined by

$$d(E, x, \theta) = \lim_{r\to 0+} \frac{\lambda(E \cap L(x, \theta, r))}{r}, \tag{1}$$

where λ denotes the linear measure (one-dimensional Hausdorff measure) in $\mathbb{R}^2$. Replacing the limit in (1) by lim sup and lim inf, we obtain the respective upper and lower densities $\bar{d}(E, x, \theta)$ and $\underline{d}(E, x, \theta)$. Should E be non-Borel, there are several additional possibilities defined by replacing λ in (1) with either the linear outer measure λ^* or the linear inner measure λ_* and again replacing the limit by either lim sup and lim inf. So, for example, the upper inner density of E along the ray $L(x, \theta)$ is defined as

$$\bar{d}_*(E, x, \theta) = \limsup_{r\to 0+} \frac{\lambda_*(E \cap L(x, \theta, r))}{r}.$$

If $d\#$ denotes any of these density operators, then the set E is said to have positive density relative to $d\#$ at a point $x \in \mathbb{R}$ if $d\#(x) > 0$.

In this paper we are interested in whether linear densities in one sense or another are transferable. For example, if we know that a set E has one of these linear densities in a given direction, can we infer that there are points at which E has a linear density of the same or different variety in another direction? The strongest hypothesis for linear densities would be that a set E has full linear density in a given direction and at every $x \in \mathbb{R}$, and the weakest conclusion is that there is a point $x_0 \in \mathbb{R}$ and a direction θ_0 at which $\bar{d}^*(E, x_0, \theta_0) > 0$.

If $E \subset H$ is Borel, then we denote by $D(E, x)$ the two-dimensional density of E at the point $(x, 0)$ relative to H. That is,

$$D(E, x) = \lim_{h\to 0+} \frac{\lambda_2(E \cap B(x, h))}{\lambda_2(H \cap B(x, h))},$$

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where λ_2 is the two-dimensional Lebesgue measure and $B(x,h)$ is the ball with center $(x,0)$ and radius h. The vertical and horizontal sections of the set $E \subset \mathbb{R}^2$ will be denoted, respectively, by

$$E_x = \{y : (x,y) \in E\} \quad \text{and} \quad E^y = \{x : (x,y) \in E\}$$

for every $x, y \in \mathbb{R}$.

2. The Terrain

In this section we examine what sort of density transfer might be expected, first in the case of Borel subsets of H and then for arbitrary subsets of H. Our starting point is an example due to Goffman and Sledd [3, Exm. 3].

THEOREM 1 (Goffman and Sledd). *There exists an open* (*or closed*) *set* $E \subset H$ *such that*:

(i) $\bar{d}(E,x,\pi/2) = 1$ *for every* $x \in \mathbb{R}$; *and*
(ii) $d(E,x,\theta) = 0$ *for every* $x \in \mathbb{R}$ *and* $\theta \neq \pi/2$.

This theorem shows that one can expect no transfer whatsoever of upper density. Also, by considering the complement of the set constructed in Theorem 1 and then applying an affine transformation, one obtains the following corollary.

COROLLARY 2. *Let* $\theta \neq \pi/2$ *be fixed. Then there exists an open* (*or closed*) *set* $E \subset H$ *such that*:

(i) $d(E,x,\pi/2) = 1$ *for every* $x \in \mathbb{R}$; *and*
(ii) $\underline{d}(E,x,\theta) = 0$ *for every* $x \in \mathbb{R}$.

Expanding on the techniques employed by Goffman and Sledd in their proof, we extend this latter example to show that one cannot expect to transfer full density even for Borel sets under quite stringent assumptions.

THEOREM 3. *There exists an open set* $G \subset H$ *such that*:

(i) $d(G,x,\pi/2) = 1$ *for every* $x \in \mathbb{R}$;
(ii) $\underline{d}(G,x,\theta) = 0$ *for every* $x \in \mathbb{R}$ *and every* $\theta \neq \pi/2$; *and*
(iii) $D(G,x) = 1$ *for every* $x \in \mathbb{R}$.

The proofs of this theorem, and of the next several examples, are given in Section 7.

As a consequence of the preceding examples, the strongest result we might expect when E is Borel would entail the transfer of full density in a given direction, say $\pi/2$, to full upper density in some other directions. Yet we cannot expect to transfer to *every* direction $\theta \neq \pi/2$, as the following example shows. This example makes use of a Besicovitch-type set discovered by Kinney [5, Exm. 1]: a two-dimensional null set contained in H that contains a ray emanating from each point of the x-axis.

THEOREM 4. *There exists an F_σ (or a G_δ) set $E \subset \mathbb{R}^2$ such that*:

(i) *E_x is of full linear measure for every $x \in \mathbb{R}$; and*
(ii) *for every $x \in \mathbb{R}$ there is an angle $\theta \in (0,\pi)\backslash\{\pi/2\}$ such that $L(x,\theta,1) \cap E$ is null.*

Moreover, we cannot expect to transfer density to *every* $x \in \mathbb{R}$, as the following simple example shows. Let C denote the Cantor ternary set, and take

$$E = \{(x,y) : x \in [0,1],\ 0 \le y \le \operatorname{dist}(x,C)^2\} \cup (C \times \mathbb{R}^+) \cup ((\mathbb{R}\backslash[0,1]) \times \mathbb{R}^+).$$

Then we have $d(E,x,\pi/2) = 1$ for every x but $d(E,x,\theta) = 0$ for every $x \in C$ and $\theta \neq \pi/2$.

In light of these examples, we have the following two candidate theorems.

THEOREM 5. *Suppose $E \subset H$ is Borel and $d(E,x,\pi/2) = 1$ for every $x \in \mathbb{R}$. Let $\theta \neq \pi/2$ be fixed. Then, for almost every $x \in \mathbb{R}$, $\bar{d}(E,x,\theta) = 1$.*

THEOREM 6. *Suppose $E \subset H$ is Borel and $d(E,x,\pi/2) = 1$ for every $x \in \mathbb{R}$. Then, for almost every $x \in \mathbb{R}$, $\bar{d}(E,x,\theta) = 1$ for almost every $\theta \in (0,\pi)$.*

Theorem 5 follows immediately from [3, Lemma 1], and Theorem 6 is an easy consequence of Theorem 5. Indeed, suppose $E \subset H$ is Borel and $d(E,x,\pi/2) = 1$ for every $x \in \mathbb{R}$. Let C be the set of pairs $(x,\theta) \in \mathbb{R} \times (0,\pi)$ such that $\bar{d}(E,x,\theta) = 1$. One can check that C is a Borel subset of $\mathbb{R}^2$. For every fixed θ, the section C^θ is of full measure by Theorem 5. Hence, by Fubini's theorem, C is of full measure in $\mathbb{R} \times [0,\pi]$. We can then apply Fubini's theorem again to find that, for a.e. x, the section C_x is of full measure in $(0,\pi)$; this is the statement of the theorem. (In Section 6 we give another proof.)

Our main objective is to investigate the transfer of density for arbitrary sets. However, in the nonmeasurable case we immediately stumble upon the following obstacles.

THEOREM 7. *There exists a set $E \subset H$ such that*:

(i) *$d^*(E,x,\pi/2) = 1$ for every $x \in \mathbb{R}$; and*
(ii) *$d^*(E,x,\theta) = 0$ for every $x \in \mathbb{R}$ and $\theta \neq \pi/2$.*

THEOREM 8. *There exists a set $E \subset H$ such that*:

(i) *$d_*(E,x,\pi/2) = 1$ for every $x \in \mathbb{R}$; and*
(ii) *$d_*(E,x,\theta) = 0$ for every $x \in \mathbb{R}$ and $\theta \neq \pi/2$.*

In fact, the set E in Theorem 7 can have full outer measure on every vertical line even as $|E \cap L(x,\theta)| \le 2$ for every $x \in \mathbb{R}$ and every $\theta \neq \pi/2$. The set E in Theorem 8 can be the complement of a singleton on every vertical line, yet the inner linear measure is 0 on every nonvertical line. These theorems are proved in Section 7.3.

This leaves only the possibility of transferring inner density to upper outer density. However, even this cannot be proved in Zermelo–Fraenkel set theory with the axiom of choice (ZFC), as the following example shows.

THEOREM 9. *Assuming the continuum hypothesis* (*CH*), *there exists a set* $E \subset H$ *such that*:

(i) $d_*(E, x, \pi/2) = 1$ *for every* $x \in \mathbb{R}$; *and*
(ii) $d^*(E, x, \theta) = 0$ *for every* $x \in \mathbb{R}$ *and* $\theta \neq \pi/2$.

This example is a special case of the following theorem of Erdős [1]: If we decompose the set of all lines in the plane into two arbitrary disjoint sets L_1 and L_2, then there exists a decomposition of the plane into two sets S_1 and S_2 such that each line of L_i intersects S_i $(i = 1, 2)$ in a set of power less than $2^{\aleph_0}$. Assuming CH, this means that each line of L_i intersects S_i $(i = 1, 2)$ in a countable set. If L_1 and L_2 consist of the vertical lines and the nonvertical lines, respectively, then the set $E = S_2$ is co-countable in every vertical line and countable in every nonvertical line. We can see that, in order to obtain Theorem 9, we may assume $\operatorname{non}\mathcal{N} = 2^{\aleph_0}$ instead of CH. (Here $\operatorname{non}\mathcal{N}$ is the smallest cardinal of subsets of $\mathbb{R}$ having positive outer measure.) Additional or higher-dimensional examples of this sort can be formulated using the ideas and techniques found in [2].

3. Transference of Density for Arbitrary Sets

According to Section 2, the strongest results we can expect are as follows.

THEOREM 10. *The following statement is consistent with ZFC. Suppose* $E \subset H$ *is such that* $d_*(E, x, \pi/2) = 1$ *for every* $x \in \mathbb{R}$; *then, for every fixed* $\theta \neq \pi/2$ *and almost every* $x \in \mathbb{R}$, $\bar{d}^*(E, x, \theta) = 1$.

THEOREM 11. *The following statement is consistent with ZFC. Suppose* $E \subset H$ *is such that* $d_*(E, x, \pi/2) = 1$ *for every* $x \in \mathbb{R}$; *then, for almost every* $x \in \mathbb{R}$, $\bar{d}^*(E, x, \theta) = 1$ *for almost every* θ.

Note that, since these statements are concerned with nonmeasurable sets, neither implies the other. Our aim is to prove them both. In fact, we rank the "consistency strength" of these statements in terms of the nonexistence of certain sets with paradoxical properties. We denote $I = [0, 1]$. A set $H \subset I \times I$ is a *0-1 set* if $\lambda(H_x) = 1$ for every $x \in I$ and $\lambda(H^y) = 0$ for every $y \in I$. Such sets were introduced by Sierpiński in [7].

We will be mainly interested in the following weaker version of this property. A set $H \subset I \times I$ is a *weak 0-1 set* if $\lambda(H_x) = 1$ for every $x \in I$ and

$$\lambda^*(\{y \in I : \lambda(H^y) = 0\}) > 0.$$

It is known that the existence of weak 0-1 sets is independent of ZFC. Indeed, CH or Martin's axiom implies the existence of weak 0-1 sets. In the random real model, however, there are no weak 0-1 sets (see [6, p. 673]). Our main results are expressed in the next two theorems.

THEOREM 12. *The following statements are equivalent.*

(i) *There is no weak 0-1 set.*

(ii) *If $E \subset H$ is such that $d_*(E, x, \pi/2) = 1$ for every $x \in \mathbb{R}$ then, for every fixed $\theta \neq \pi/2$ and almost every $x \in \mathbb{R}$, $\bar{d}^*(E, x, \theta) = 1$.*

THEOREM 13. *If there is no weak 0-1 set, then the following statement holds.*

(iii) *If $E \subset H$ is such that $d_*(E, x, \pi/2) = 1$ for every $x \in \mathbb{R}$ then, for almost every $x \in \mathbb{R}$, $\bar{d}^*(E, x, \theta) = 1$ for almost every θ.*

Because the existence of a weak 0-1 set is independent of ZFC, Theorem 12 implies Theorem 10; similarly, Theorem 13 implies Theorem 11. However:

> *We do not know whether the negation of* (iii) *is actually equivalent to the existence of a weak 0-1 set.*

Statement (iii) is certainly false under various set-theoretic hypotheses—for example, CH or non$\mathcal{N} = 2^{\aleph_0}$. Another condition, which does not involve the value of the continuum, is that add$\mathcal{N} =$ cof$\mathcal{N}$. Yet another sufficient condition for (iii) to be false is if $\mathbb{R}$ can be linearly ordered in such a way that each initial segment has (1/2)-dimensional Hausdorff measure 0.

Now we turn to the proofs. Theorem 12 will be proved in the next section. Section 5 will be devoted to the preliminaries of the proof of Theorem 13, which will be proved in Section 6. The constructions of the examples (Theorems 3, 4, 7, and 8) are given in the Section 7.

4. No Weak 0-1 Set Is Equivalent to Fixed Directional Transfer of Density a.e.

First we prove the implication (ii) $\Rightarrow$ (i) of Theorem 12. Suppose S is a weak 0-1 set, and let $\bar{S} = \bigcup_{n,m\in\mathbb{Z}}(S + (n, m))$. Then $\bar{S}_x$ is of full measure in $\mathbb{R}$ for every $x \in \mathbb{R}$, and there is a set Y such that $\lambda^*(Y) > 0$ and $\lambda(\bar{S}^y) = 0$ for every $y \in Y$. Rotating $\bar{S}$ about the origin with angle $\pi/4$, we obtain the set T. Then we have $d(T, x, 3\pi/4) = 1$ for every x and $\bar{d}^*(T, x, \pi/4) = 0$ for every $x \in -\sqrt{2} \cdot Y$.

It is easy to see that, for every $0 < \theta < \pi/2$, the affine transformation $L(x, y) = (x + y, 2y \tan\theta)$ maps the directions $\pi/4$ and $3\pi/4$ to the directions θ and $\pi/2$. Then the set $E = L(T)$ has the property that $d(E, x, \pi/2) = 1$ for every x and $\bar{d}^*(E, x, \theta) = 0$ for every $x \in -\sqrt{2} \cdot Y$. Therefore, Theorem 12(ii) is false for every $0 < \theta < \pi/2$. (By reflecting the set E about the y-axis, we can see that Theorem 12 is false for every $\pi/2 < \theta < \pi$ as well.)

In order to prove (i) $\Rightarrow$ (ii) we first need to establish some notation and to prove a preliminary lemma.

If $(X, \mathcal{A}, \mu)$ is a measure space, then we shall denote by μ^* the outer measure generated by μ. That is, for each $H \subset X$, $\mu^*(H) = \inf\{\mu(A) : H \subset A,\ A \in \mathcal{A}\}$.

Let $A \subset X$ be measurable and let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the extended real line. If $f\colon A \to \bar{\mathbb{R}}$ then we shall define the lower integral of f, $\underline{\int}_A f\, d\mu(x)$, as the supremum of the integrals $\int_A g\, d\mu(x)$; here $g\colon A \to \bar{\mathbb{R}}$ is an arbitrary summable function such that $g \le f$ everywhere on A. If there is no such g then we put $\underline{\int}_A f\, d\mu(x) = -\infty$. The upper integral $\overline{\int}_A f\, d\mu(x)$ is defined analogously. When integrating over $I = [0, 1]$, we shall omit the subscript I; that is, $\underline{\int}$, $\overline{\int}$, and $\int$ denote (respectively) $\underline{\int}_I$, $\overline{\int}_I$, and $\int_I$.

The next three lemmas were proved in [4] and are restated here for completeness.

LEMMA 14. *If $\underline{\int}_A f\, d\mu(x)$ is finite, then there is a summable function $g\colon A \to \bar{\mathbb{R}}$ such that: $g \le f$ on A; $\int_A g\, d\mu(x) = \underline{\int}_A f\, d\mu(x)$; and, for every $\varepsilon > 0$, $\mu^*(\{x \in A : f(x) < g(x) + \varepsilon\}) = \mu(A)$.*

LEMMA 15. *For every $f\colon A \to \bar{\mathbb{R}}$ and $g\colon A \to \bar{\mathbb{R}}$, we have*
(i) $\underline{\int}_A (f+g)\, d\mu(x) \ge \underline{\int}_A f\, d\mu(x) + \underline{\int}_A g\, d\mu(x)$ *and*
(ii) $\overline{\int}_A (f+g)\, d\mu(x) \le \overline{\int}_A f\, d\mu(x) + \overline{\int}_A g\, d\mu(x)$
whenever the right-hand sides make sense.

LEMMA 16. *For every sequence of nonnegative functions $f_n\colon A \to [0, \infty]$, we have*

$$\overline{\int_A} \liminf_n f_n\, d\mu(x) \le \liminf_n \overline{\int_A} f_n\, d\mu(x).$$

We shall also need the following change-of-variable formula for lower and upper integrals.

LEMMA 17. *If ϕ is a strictly monotonic C^1 map from the interval $[a, b]$ onto the interval $[c, d]$ with nonvanishing derivative, then $\underline{\int}_c^d f\, d\lambda = \underline{\int}_a^b (f \circ \phi) \cdot |\phi'|\, d\lambda$ for every $f : [c, d] \to \bar{\mathbb{R}}$. A similar statement holds for the upper integral.*

Proof. Let ψ denote the inverse of ϕ. The statement involving lower integrals follows from the observations that if g is summable and $g \le f$ on $[c, d]$ then $(g \circ \phi) \cdot |\phi'| \le (f \circ \phi) \cdot |\phi'|$ on $[a, b]$ and if h is summable and $h \le (f \circ \phi) \cdot |\phi'|$ on $[a, b]$ then $(h \circ \psi) \cdot |\psi'| \le f$ on $[c, d]$.

The formula involving upper integrals is proved similarly. □

Our next lemma is a generalization of [4, Thm. 6].

LEMMA 18. *The following statements are equivalent.*
(i) *There is no weak 0-1 set.*
(ii) *Whenever $f_n\colon (I \times I) \to [0, \infty]$ is a sequence of nonnegative functions such that $\sup_{y \in I} \sup_n \overline{\int} f_n(x, y)\, dx < \infty$, we have*

$$\underline{\int} \left(\liminf_n \underline{\int} f_n(x, y)\, dy \right) dx \le \underline{\int} \left(\limsup_n \overline{\int} f_n(x, y)\, dx \right) dy. \quad (2)$$

Proof. Suppose S is a weak 0-1 set. Let $Y = \{y \in I : \lambda(S^y) = 0\}$, so $\lambda^*(Y) > 0$. Let $f_n = \chi_S$ for every n. Then $\overline{\int} f_n(x, y)\, dx \leq 1$ for every $y \in I$ and every n. Also, the left-hand side of (2) equals 1 while the right-hand side is at most $1 - \lambda^*(Y) < 1$, and thus (ii) fails.

Now suppose that there is no weak 0-1 set. Let f_n be a sequence of functions as in (ii), and suppose that (2) is false. Then, multiplying the functions f_n by a suitable positive constant, we may assume that

$$\underline{\int}\left(\limsup_n \overline{\int} f_n(x, y)\, dx\right) dy < 1 - 3\varepsilon$$
$$< 1 < \underline{\int}\left(\liminf_n \underline{\int} f_n(x, y)\, dy\right) dx \tag{3}$$

for some $\varepsilon > 0$. The last inequality of (3) implies that there is a nonnegative summable function ϕ on I such that $\int \phi\, dx \geq 1$ and

$$\liminf_n \underline{\int} f_n(x, y)\, dy > \phi(x) \quad \text{for every } x \in I.$$

Then, for every $x \in I$, there exists an index $n_0(x)$ such that, for $n \geq n_0(x)$,

$$\underline{\int} f_n(x, y)\, dy > \phi(x). \tag{4}$$

Set $K = \sup_{y \in I} \sup_n \overline{\int} f_n(x, y)\, dx$ and put $g_n(y) = \overline{\int} f_n(x, y)\, dx$ $(y \in I)$. Then $K < \infty$ by assumption, and $g_n \leq K$ on I for every n. Let $g(y) = \limsup_n g_n(y)$ $(y \in I)$. Then $\underline{\int} g\, dy < 1 - 3\varepsilon$ by (3) and so, by Lemma 14, there is a nonnegative summable function h such that $\int h\, dy < 1 - 3\varepsilon$ and $\lambda^*(B) = 1$, where $B = \{y \in I : g(y) < h(y) + \varepsilon\}$. Let $B_k = \{y \in B : g_n(y) < h(y) + \varepsilon \ (n \geq k)\}$. Since $g = \limsup_n g_n$, it follows that $B_1 \subset B_2 \subset \cdots$ and $\bigcup_k B_k = B$. Hence $\lim_{k \to \infty} \lambda^*(B_k) = \lambda^*(B) = 1$ and we can therefore find a k such that $\lambda^*(B_k) > 1 - (\varepsilon/K)$. Fix such a k and select a measurable set $C \subset I \setminus B_k$ such that $\lambda(C) < \varepsilon/K$ and $\lambda^*(C \cup B_k) = 1$. Define

$$h_1(y) = \begin{cases} h(y) + \varepsilon & \text{if } y \in I \setminus C, \\ K & \text{if } y \in C; \end{cases}$$

note that h_1 is summable and that $\int h_1\, dy < 1 - \varepsilon$. Put $D = C \cup B_k$. Then $\lambda^*(D) = 1$, and if $n \geq k$ then

$$\overline{\int} f_n(x, y)\, dx = g_n(y) \leq h_1(y) \quad \text{for every } y \in D. \tag{5}$$

In the rest of this proof we take advantage of the preceding information to show the existence of a weak 0-1 set. To facilitate this demonstration we alter our venue somewhat. Let Ω denote the measure space $I^{\mathbb{N}} = I \times I \times \cdots$ with the product measure ν. The generic element of Ω will be denoted by $\omega = (\omega_1, \omega_2, \ldots)$, where each ω_i belongs to I. Note that the measure space (Ω, ν) is isomorphic to (I, λ). The outer measure generated by ν is denoted by ν^*. We define

$$F_n(x,\omega) = \liminf_m \frac{1}{m} \cdot \sum_{j=1}^m f_n(x,\omega_j) \quad (x \in I,\ \omega \in \Omega)$$

and put

$$H_n = \{(x,\omega) : F_n(x,\omega) \geq \phi(x)\}.$$

Since ϕ is summable, there is an $\eta > 0$ such that $\int_H \phi\,dx < \varepsilon$ for every measurable set $H \subset I$ with $\lambda(H) < \eta$. We claim that the sets H_n have the following properties:

$$\text{for every } x \in I \text{ we have } \nu(\Omega \setminus (H_n)_x) = 0 \text{ if } n \geq n_0(x) \tag{6}$$

and there is a set $E \subset \Omega$ such that $\nu^*(E) = 1$; and

$$\lambda^*((H_n)^\omega) < 1 - \eta \quad \text{for every } n \geq k \text{ and } \omega \in E. \tag{7}$$

Indeed, if $x \in I$ and $n \geq n_0(x)$ then, by (4) and the strong law of large numbers,

$$F_n(x,\omega) = \liminf_m \frac{1}{m} \cdot \sum_{j=1}^m f_n(x,\omega_j) \geq \underline{\int} f_n(x,y)\,dy \geq \phi(x)$$

for ν-almost every ω; that is, (6) holds. Applying the strong law of large numbers again, we find that the set

$$U = \left\{\omega \in \Omega : \lim_m \frac{1}{m} \cdot \sum_{j=1}^m h_1(\omega_j) = \int h_1\,dy < 1 - \varepsilon\right\}$$

is a measurable set of full measure in Ω. Let $D^{\mathbb{N}} = D \times D \times \cdots$. Then $D^{\mathbb{N}}$ is of full outer measure in Ω, and the same is true for $E = U \cap D^{\mathbb{N}}$. If $\omega \in E$ and $n \geq k$ are fixed then, by (5) and Lemmas 15 and 16, we have

$$\begin{aligned}
\overline{\int} F_n(x,\omega)\,dx &= \overline{\int}\left(\liminf_m \frac{1}{m} \cdot \sum_{j=1}^m f_n(x,\omega_j)\right) dx \\
&\leq \liminf_m \overline{\int}\left(\frac{1}{m} \cdot \sum_{j=1}^m f_n(x,\omega_j)\right) dx \\
&\leq \liminf_m \frac{1}{m} \cdot \sum_{j=1}^m \overline{\int} f_n(x,\omega_j)\,dx \\
&\leq \liminf_m \frac{1}{m} \cdot \sum_{j=1}^m h_1(\omega_j) < 1 - \varepsilon.
\end{aligned} \tag{8}$$

Let $A \subset I$ be a measurable hull of $(H_n)^\omega$. Since $F_n(x,\omega) \geq \phi(x)$ for every $x \in (H_n)^\omega$, it follows that $\int_A \phi\,dx \leq \overline{\int} F_n(x,\omega)\,dx$ and thus $\int_A \phi\,dx < 1 - \varepsilon$ by (8). Therefore, $\int_{I\setminus A} \phi\,dx > \varepsilon$ and hence $\lambda(I \setminus A) > \eta$ by the choice of η. We obtain $\lambda^*((H_n)^\omega) = \lambda(A) < 1 - \eta$, proving (7).

It is an easy matter to see that (6) and (7) imply the existence of a weak 0-1 set. Since Ω is isomorphic to I, we can find sets $T_n \subset I \times I$ such that

$$\text{for every } x \in I \text{ we have } \lambda(I \setminus (T_n)_x) = 0 \text{ if } n \geq n_0(x) \tag{9}$$

and there is a set $M \subset I$ such that $\lambda^*(M) = 1$; and

$$\lambda^*((T_n)^y) < 1 - \eta \quad \text{for every } n \geq k \text{ and } y \in M. \tag{10}$$

Now define $T^m = \bigcap_{n=m}^{\infty} T_n$ and $T = \bigcup_{m=1}^{\infty} T^m$. It is clear from (9) that $\lambda(I \setminus T_x) = 0$ for every $x \in I$. Let $y \in M$. Then, by (10), $\lambda^*((T^m)^y) < 1 - \eta$ for every m. Since $T^1 \subset T^2 \subset \cdots$, it follows that $\lambda^*(T^y) \leq 1 - \eta$ for every $y \in M$.

We define $S = \{(x, y) : (x + r, y) \in T \text{ for every } r \in [-x, 1 - x] \cap \mathbb{Q}\}$. Then $\lambda(I \setminus S_x) = 0$ for every $x \in I$ and $\lambda(S^y) = 0$ for every $y \in A$ (since $\lambda^*(T^y) < 1$). Thus S is a weak 0-1 set. However, this contradicts our assumption, which completes the proof of (2). $\square$

Remark. The statement of Lemma 18 is not true without the assumption that

$$\sup_{y \in I} \sup_n \overline{\int} f_n(x, y)\, dx < \infty.$$

Indeed, let $f_n(x, y) = 1/(ny)$ if $y > 0$ and let $f_n(x, y) = 0$ if $y = 0$. Then the left-hand side of (2) is infinity while the right-hand side is zero.

Now we resume our proof of (i) $\Rightarrow$ (ii) for Theorem 12. Suppose there is no weak 0-1 set and suppose $d_*(E, x, \pi/2) = 1$ for every $x \in \mathbb{R}$. (See Section 1 for the definitions of these density operators.) Again we transform the set E by rotating and applying an affine transformation to obtain a set A such that $d_*(A_x, x) = \lim_{h \to 0^+} \lambda_*(A_x \cap [x - h, x + h])/(2h) = 1$ for every $x \in \mathbb{R}$. We will show that $\bar{d}^*(A^y, y) = \limsup_{r \to 0+} \lambda^*(A^y \cap [y - h, y + h]))/(2h) = 1$ for almost every $y \in \mathbb{R}$. It is enough to prove this for a.e. $y \in I$.

Put $A_n = A \cap \{(x, y) : |y - x| \leq 1/n\}$ and $f_n(x, y) = (n/2) \cdot \chi_{A_n}$ for every $n = 1, 2, \ldots$. We can apply Lemma 18(ii) with our sequence f_n. If $x \in (0, 1)$, then x is a density point of A_x and so $\lim_n \underline{\int} f_n(x, y)\, dy = 1$. Hence the left-hand side of (2) equals 1, in which case the right-hand side must be at least 1. Since $\overline{\int} f_n(x, y)\, dx \leq 1$ for every y, it follows that $\limsup_n \overline{\int} f_n(x, y)\, dx = 1$ for a.e. y. Given

$$\overline{\int} f_n(x, y)\, dx = \left(\frac{n}{2}\right) \cdot \lambda^*\left(A^y \cap \left[y - \left(\frac{1}{n}\right), y + \left(\frac{1}{n}\right)\right]\right),$$

we conclude that $\bar{d}^*(A^y, y) = 1$ for a.e. y. This completes the proof of Theorem 12.

5. The Sierpiński Coefficient of Measure Spaces

Let $(X, \mathcal{A}, \mu)$ be a finite measure space. By the *Sierpiński coefficient* of X we mean the supremum of the numbers

$$\underline{\int}_X \underline{\int} g(x, y)\, dy\, d\mu(x) - \underline{\int} \overline{\int}_X g(x, y)\, d\mu(x)\, dy, \tag{11}$$

where g runs through all functions mapping $X \times I$ into I. In other words, the Sierpiński coefficient of X is the smallest real number s such that

$$\underline{\int}_X \underline{\int} g(x,y)\,dy\,d\mu(x) \le \underline{\int} \overline{\int}_X g(x,y)\,d\mu(x)\,dy + s \tag{12}$$

for every $g\colon (X \times I) \to I$. We denote the Sierpiński coefficient of X by $s(X)$. Clearly, $0 \le s(X) \le \mu(X)$ for every finite measure space X, and $s(X) = 0$ if and only if

$$\underline{\int}_X \underline{\int} g(x,y)\,dy\,d\mu(x) \le \underline{\int} \overline{\int}_X g(x,y)\,d\mu(x)\,dy$$

for every bounded function $g\colon (X \times I) \to \mathbb{R}$.

LEMMA 19. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let b and M be positive numbers. Then, for every function $f\colon (X \times [0,b]) \to [0,M]$,*

$$\underline{\int}_X \underline{\int}_0^b f(x,y)\,dy\,d\mu(x) \le \underline{\int}_0^b \overline{\int}_X f(x,y)\,d\mu(x)\,dy + Mb \cdot s(X). \tag{13}$$

Proof. Let $g(x,y) = (1/M)f(x,by)$ for $(x,y) \in X \times I$. The values of g are in I and so (12) follows. Applying the substitution $z = by$ (and Lemma 17) and then multiplying by Mb, we obtain (13). $\square$

We shall now establish the basic properties of the Sierpiński coefficient. These results will be used in Section 6 to prove Theorem 13.

We say that a subset $S \subset X \times I$ is a *weak 0-1 set* in $X \times I$ if S_x is of full measure for every $x \in X$ and if $\lambda^*(\{y \in I : \mu(H^y) = 0\}) > 0$. In Corollary 21 we prove that $s(X) = \mu(X)$ if and only if there is a weak 0-1 set in $X \times I$. This implies that, for some measure spaces (including the standard measure space $(I, \mathcal{L}, \lambda)$), the value of $s(X)$ cannot be determined in ZFC. We begin by characterizing the Sierpiński coefficient.

THEOREM 20. *For every finite measure space X, the Sierpiński coefficient $s(X)$ is the largest real number s for which* (a) *there exists a set $U \subset X \times I$ such that U_x is of full measure for every $x \in X$ and* (b) *$\mu^*(U^y) \le \mu(X) - s$ for a set of points y that is of full outer measure.*

Proof. Let s be a real number and let U be a set as in the statement of the theorem. If $f = \chi_U$ then the value of the difference of integrals in (11) is at least $\mu(X) - (\mu(X) - s) = s$, from which it follows that $s(X) \ge s$.

In order to prove $s(X) \le s$, we must construct a set $U \subset X \times I$ such that U_x is of full measure for every $x \in X$ and $\mu^*(U^y) \le \mu(X) - s(X)$ for a set of y's of full outer measure.

Let $\varepsilon > 0$ be arbitrary, and fix a function $f\colon (X \times I) \to I$ such that $L - R > s(X) - \varepsilon$; here $L = \underline{\int}_X \underline{\int} f(x,y)\,dy\,d\mu(x)$ and $R = \underline{\int} \overline{\int}_X f(x,y)\,d\mu(x)\,dy$. Then there is a measurable function $h\colon X \to I$ such that (a) $h(x) \le \underline{\int} f(x,y)\,dy$ for every $x \in X$ and (b) $\int_X h\,d\mu(x) = L$. Similarly, there is a measurable function

$k\colon I \to [0,\mu(X)]$ such that (a) $k(y) \le \overline{\int}_X f(x,y)\,d\mu(x)$ for every $y \in I$ and (b) $\int k\,dy = R$. Furthermore, the set $D = \left\{y \in I : \overline{\int}_X f(x,y)\,d\mu(x) < k(y)+\varepsilon\right\}$ is of full outer measure.

We define $F\colon (X \times \Omega) \to I$ by

$$F(x,\omega) = \liminf_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} f(x,\omega_i)$$

and put

$$U = \{(x,\omega) \in X \times \Omega : F(x,\omega) \ge h(x)\}.$$

For every $x \in X$ we have $\int f(x,y)\,dy \ge h(x)$ and thus, by the strong law of large numbers, $F(x,\omega) \ge h(x)$ for a.e. $\omega \in \Omega$. Therefore, U_x is of full measure in Ω for every $x \in X$.

Applying the strong law of large numbers again, we find that the set

$$E = \left\{\omega \in \Omega : \lim_n \frac{1}{n} \cdot \sum_{i=1}^{n} k(\omega_i) = \int k\,dy < R+\varepsilon\right\}$$

is a set of full measure in Ω. Since D is of full outer measure in I, it follows that $D^{\mathbb{N}} = D \times D \times \cdots$ is of full outer measure in Ω, so the same is true for $E \cap D^{\mathbb{N}}$. If $\omega \in E \cap D^{\mathbb{N}}$ then, by Lemmas 15 and 16,

$$\begin{aligned}
\overline{\int}_X F(x,\omega)\,d\mu(x) &= \overline{\int}_X \left(\liminf_n \frac{1}{n} \cdot \sum_{i=1}^{n} f(x,\omega_i)\right) d\mu(x) \\
&\le \liminf_n \overline{\int}_X \left(\frac{1}{n} \cdot \sum_{i=1}^{n} f(x,\omega_i)\right) d\mu(x) \\
&\le \liminf_n \frac{1}{n} \cdot \sum_{i=1}^{n} \overline{\int}_X f(x,\omega_i)\,d\mu(x) \\
&\le \liminf_n \frac{1}{n} \cdot \sum_{i=1}^{n} (k(\omega_i)+\varepsilon) \\
&< R + 2\varepsilon. \qquad (14)
\end{aligned}$$

Let $\omega \in E \cap D^{\mathbb{N}}$ be fixed, and let A be a measurable hull of U^ω. (That is, let $A \in \mathcal{A}$ be such that $U^\omega \subset A$ and $\mu^*(U^\omega) = \mu(A)$.) Then, by (14),

$$\begin{aligned}
R + 2\varepsilon &> \overline{\int}_X F(x,\omega)\,d\mu(x) \ge \overline{\int}_{U^\omega} F(x,\omega)\,d\mu(x) \ge \overline{\int}_{U^\omega} h(x)\,d\mu(x) \\
&= \int_A h(x)\,d\mu(x) = \int_X h\,d\mu(x) - \int_{X\setminus A} h\,d\mu(x) \\
&\ge L - \mu(X \setminus A) = L - \mu(X) + \mu(A) \\
&= L - \mu(X) + \mu^*(U^\omega);
\end{aligned}$$

we therefore have

$$\mu^*(U^\omega) \le \mu(X) + R - L + 2\varepsilon < \mu(X) - s(X) + 3\varepsilon.$$

In sum: the set $U \subset X \times \Omega$ has the properties that U_x is of full measure for every $x \in X$ and that $\mu^*(U^\omega) < \mu(X) - s(X) + 3\varepsilon$ for a set of ω's of full outer measure. Because Ω is isomorphic to I, there is a subset of $X \times I$ with the same properties. In particular, for every positive integer n, there exists a set $U_n \subset X \times I$ such that $(U_n)_x$ is of full measure for every $x \in X$ and such that $\mu^*(U_n^y) < \mu(X) - s(X) + (1/n)$ for every $y \in Y_n$, where $\lambda^*(Y_n) = 1$.

We now construct sets U_n^* that have similar properties and that satisfy the extra condition $Y_1 = Y_2 = \cdots$. Let $Y = Y_1 \times Y_2 \times \cdots \subset \Omega$; then Y is of full outer measure in Ω. We define

$$U_n^* = \{(x, \omega) \in X \times \Omega : (x, \omega_n) \in U_n\}.$$

It is easy to check that $(U_n^*)_x$ is of full measure for every $x \in X$ and that $\mu^*((U_n^*)^\omega) = \mu^*(U_n^{\omega_n}) < \mu(X) - s(X) + (1/n)$ for every $\omega \in Y$. That is, the $U_n^* \subset X_n \times \Omega$ $(n = 1, 2, \ldots)$ have the desired properties.

Put $U^* = \bigcap_{n=1}^\infty U_n^*$. Then U_x^* is of full measure for every $x \in X$ and $\mu^*((U^*)^\omega) \le \mu(X) - s(X)$ for every $\omega \in Y$. Given that Ω is isomorphic to I, there exists a set $U \subset X \times I$ such that U_x is of full measure for every $x \in X$ and such that $\mu^*(U^y) \le \mu(X) - s(X)$ for a set of y's of full outer measure. $\square$

COROLLARY 21. *The equality $s(X) = \mu(X)$ holds if and only if there is a weak 0-1 set in $X \times I$.*

Proof. Let $S \subset X \times I$ be a weak 0-1 set. Then the set

$$U = \{(x, y) \in X \times I : (x, y + r) \in S \cup (X \times (\mathbb{R} \setminus I)) \text{ for every } r \in \mathbb{Q}\}$$

has the properties that U_x is of full measure for every $x \in X$ and that U^y is null for a set of y's of full outer measure. By Theorem 20, $s(X) = \mu(X)$. The reverse implication is an immediate consequence of Theorem 20. $\square$

If $E \subset X$, then by the *measure space E* we mean $(E, \mathcal{A}|_E, \mu_E)$; here $\mathcal{A}|_E$ is the σ-algebra $\{E \cap A : A \in \mathcal{A}\}$, and $\mu_E(B) = \mu^*(B)$ for every $B \in \mathcal{A}|_E$.

LEMMA 22. *Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space, and suppose the subsets $A \subset X$ and $B \subset X$ can be separated in the sense that there is a measurable set $C \subset \mathcal{A}$ with $A \subset C$ and $B \cap C = \emptyset$. Then $s(A \cup B) = s(A) + s(B)$.*

Proof. First note that, for every bounded function $g \colon (A \cup B) \to I$,

$$\underline{\int}_{A \cup B} g\,dx = \underline{\int}_A g\,dx + \underline{\int}_B g\,dx \quad \text{and} \quad \overline{\int}_{A \cup B} g\,dx = \overline{\int}_A g\,dx + \overline{\int}_B g\,dx.$$

Therefore, if $f \colon ((A \cup B) \times I) \to I$ is arbitrary then

$$\underline{\int}_{A \cup B} \underline{\int} f\,dy\,dx = \underline{\int}_A \underline{\int} f\,dy\,dx + \underline{\int}_B \underline{\int} f\,dy\,dx. \tag{15}$$

Also, using Lemma 15 yields

$$\begin{aligned}\underline{\int}\overline{\int}_{A\cup B} f\,dx\,dy &= \underline{\int}\Big(\overline{\int}_A f\,dx + \overline{\int}_B f\,dx\Big)\,dy\\ &\geq \underline{\int}\overline{\int}_A f\,dx\,dy + \underline{\int}\overline{\int}_B f\,dx\,dy. \end{aligned}\tag{16}$$

Now combining (15) and (16), we obtain

$$\begin{aligned}&\underline{\int}_{A\cup B}\underline{\int} f\,dy\,dx - \underline{\int}_{A\cup B}\overline{\int} f\,dx\,dy\\ &\quad\leq \Big(\underline{\int}_A\underline{\int} f\,dy\,dx - \underline{\int}_A\overline{\int} f\,dx\,dy\Big) + \Big(\underline{\int}_B\underline{\int} f\,dy\,dx - \underline{\int}_B\overline{\int} f\,dx\,dy\Big)\\ &\quad\leq s(A)+s(B);\end{aligned}$$

therefore, $s(A\cup B)\leq s(A)+s(B)$.

For the reverse inequality, it follows from Theorem 20 that there exists a set $S\subset A\times I$ such that, for every $x\in A$, S_x is of full measure and $\mu^*(S^y)\leq\mu^*(A)-s(A)$ for every $y\in Y_1$, where $\bar{\lambda}(Y_1)=1$. Similarly, there is a set $T\subset B\times I$ with T_x of full measure for every $x\in B$ and $\mu^*(T^y)\leq\mu^*(B)-s(B)$ for every $y\in Y_2$, where $\lambda^*(Y_2)=1$. Define

$$\begin{aligned}\bar{S}&=\{(x,y,z)\in A\times I^2 : (x,y)\in S\},\\ \bar{T}&=\{(x,y,z)\in B\times I^2 : (x,z)\in T\}.\end{aligned}$$

Then $\bar{S}_x$ is of full measure in I^2 for every $x\in A$ and $\bar{T}_x$ is of full measure in I^2 for every $x\in B$. Now if $(y_1,y_2)\in Y_1\times Y_2$, then

$$\begin{aligned}\mu^*(\bar{S}^{(y_1,y_2)})&=\mu^*(S^{y_1})\leq\mu^*(A)-s(A),\\ \mu^*(\bar{T}^{(y_1,y_2)})&=\mu^*(T^{y_2})\leq\mu^*(B)-s(B).\end{aligned}$$

Hence, by using that I and I^2 are isomorphic as measure spaces, we can find sets $\tilde{S}\subset A\times I$, $\tilde{T}\subset B\times I$, and $Z\subset I$ such that

(i) $\tilde{S}_x$ is full measure for every $x\in A$,
(ii) $\tilde{T}_x$ is full measure for every $x\in B$,
(iii) $\lambda^*(Z)=1$,
(iv) $\mu^*(\tilde{S}^z)\leq\mu^*(A)-s(A)$ for every $z\in Z$, and
(v) $\mu^*(\tilde{T}^z)\leq\mu^*(B)-s(B)$ for every $z\in Z$.

The set $W=\tilde{S}\cup\tilde{T}\subset(A\cup B)\times I$ has the following properties: W_x is of full measure for every $x\in A\cup B$ and, for every $y\in Z$,

$$\mu^*(W^y)\leq\mu^*(A)-s(A)+\mu^*(B)-s(B)=\mu^*(A\cup B)-(s(A)+s(B)).$$

By Theorem 20, this implies that $s(A\cup B)\geq s(A)+s(B)$. □

THEOREM 23. *Let X be a finite measure space. If $H_1\subset H_2\subset\cdots$ and $\bigcup_{n=1}^\infty H_n=X$, then $s(H_n)\to s(X)$.*

Proof. By Theorem 20, there is a set $U\subset X\times I$ such that U_x is of full measure for every $x\in X$ and $\mu^*(U^y)\leq\mu(X)-s(X)$ for every $y\in Y$, where $Y\subset I$ is a set of full outer measure. Let $H \subset X$ be arbitrary. If $V = U \cap (H \times I)$, then V_x is of full measure for every $x \in H$ and

$$\mu_H^*(V^y) \le \mu^*(U^y) \le \mu(X) - s(X) = \mu_H(H) - (s(X) + \mu_H(H) - \mu(X))$$

for every $y \in Y$. Again by Theorem 20, this inequality implies that

$$s(H) \ge s(X) + \mu^*(H) - \mu(X). \tag{17}$$

If $H_1 \subset H_s \subset \cdots$ and $\bigcup_{n=1}^{\infty} H_n = X$, then $\mu^*(H_n) \to \mu(X)$; therefore, by (17), $\liminf_{n\to\infty} s(H_n) \ge s(X)$.

Now we prove that $\limsup_{n\to\infty} s(H_n) \le s(X)$. Suppose this is not true. Then, by passing to a subsequence if necessary, we may assume that $s(H_n) > s(X) + \varepsilon$ for every n, where $\varepsilon > 0$. According to Theorem 20, there exist sets $U_n \subset H_n \times I$ and $Y_n \subset I$ such that $(U_n)_x$ is of full measure for every $x \in H_n$ and $\mu^*(Y_n) = 1$ and such that $\mu^*((U_n)^y) \le \mu^*(H_n) - s(H_n)$ for every $y \in Y_n$.

Let $Y = Y_1 \times Y_2 \times \cdots \subset \Omega$; then Y is of full outer measure in Ω. We define

$$U_n^* = \{(x,\omega) \in H_n \times \Omega : (x,\omega_n) \in U_n\}.$$

Then $(U_n^*)_x$ is of full measure for every $x \in H_n$, and

$$\mu^*((U_n^*)^{\omega}) = \mu^*(U_n^{\omega_n}) \le \mu^*(H_n) - s(H_n) < \mu(X) - s(X) - \varepsilon$$

for every $\omega \in Y$.

Now we put $U^* = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} U_n^*$. It is easy to check that $(U^*)_x$ is of full measure for every $x \in X$ and that $\mu^*((U^*)^{\omega}) \le \mu(X) - s(X) - \varepsilon$ for every $\omega \in Y$.

Since Ω is isomorphic to I, there exists a set $U \subset X \times I$ such that (a) U_x is of full measure for every $x \in X$ and (b) $\mu^*(U^y) \le \mu(X) - s(X) - \varepsilon$ for a set of y's of full outer measure. If $f = \chi_U$, then the value of (11) is at least $s(X) + \varepsilon$. However, its value cannot be greater than $s(X)$—a contradiction. Thus we have proved that $\limsup_{n\to\infty} s(H_n) \le s(X)$. □

LEMMA 24. *For every $H \subset I$ and $\varepsilon > 0$ we have $\lambda^*(\bar{H}) \le s(H)/\varepsilon$, where*

$$\bar{H} = \bar{H}(\varepsilon) = \left\{x \in H : \limsup_{h\to 0+} \frac{s(H \cap [x, x+h])}{h} > \varepsilon\right\}.$$

Proof. For each $x \in \bar{H}$ there exist arbitrarily small numbers $h > 0$ with $h < s(H \cap [x, x+h])/\varepsilon$. We can now apply the Vitali covering theorem to find a sequence of disjoint intervals $[x_i, x_i + h_i]$, $i = 1, 2, \ldots$, that cover almost every point of $\bar{H}$. Therefore,

$$\lambda^*(\bar{H}) \le \sum_{i=1}^{\infty} h_i \le \frac{1}{\varepsilon} \sum_{i=1}^{\infty} s(H \cap [x_i, x_i + h_i]).$$

For each $n \in \mathbb{N}$, by Lemma 22 we have

$$s(H) = \sum_{i=1}^{n} s(H \cap [x_i, x_i + h_i]) + s(H \cap J);$$

here $J = I \setminus \bigcup_{i=1}^{n} [x_i, x_i + h_i]$. Hence $\sum_{i=1}^{n} s(H \cap [x_i, x_i + h_i]) \le s(H)$ for each $n \in \mathbb{N}$ and so $\lambda^*(\bar{H}) \le s(H)/\varepsilon$. □

We conclude by proving some supplementary results that may be of interest in their own right.

PROPOSITION 25. *For every $c \in [0,1]$, there exists a probability space X such that $s(X) = c$.*

Proof. If $c = 0$, then we may take the one-element probability space. Suppose $0 < c \le 1$. Let $\kappa = \mathrm{non}\mathcal{N}$; that is, let κ be the smallest cardinality of subsets of $\mathbb{R}$ having positive outer measure. Then there clearly exists a set $A \subset [0,1]$ of cardinality κ such that $\lambda^*(A) = c$. Let $\bar{A}$ be a measurable hull of A, and let p be a point not in A. We define $X = A \cup \{p\}$. The elements of the σ-algebra $\mathcal{A}$ will be sets of the form $E \cap A$ or $(E \cap A) \cup \{p\}$, where $E \subset [0,1]$ is Lebesgue measurable. Then the measure μ is defined by $\mu(E \cap A) = \lambda(E \cap \bar{A})$ and $\mu((E \cap A) \cup \{p\}) = \lambda(E \cap \bar{A}) + 1 - c$ for every Lebesgue measurable $E \subset [0,1]$. It is clear that $(X, \mathcal{A}, \mu)$ is a probability space.

Let $B \subset I$ be a set of cardinality κ with full outer measure, and let $A = \{a_\alpha : \alpha < \kappa\}$ and $B = \{b_\alpha : \alpha < \kappa\}$ be well-orderings of A and B (respectively) that are of type κ. The set

$$S = \{(a_\alpha, b_\beta) : \alpha < \beta < \kappa\} \cup (A \times (I \setminus B)) \cup (\{p\} \times I)$$

has the property that S_x is of full measure in I for every $x \in X$ (because the cardinality of $I \setminus S_x$ is less than κ), and $\mu(S^y) = 1 - c$ for every $y \in B$. By Theorem 20, these statements imply that $s(X) \ge c$. Yet if $f : (X \times I) \to [0,1]$ then the value of the difference of integrals in (11) is clearly no more than c, so $s(X) = c$. □

Note that, in contrast with Proposition 25, there are some measure spaces, including the standard measure space $(I, \mathcal{L}, \lambda)$, for which the value of $s(X)$ must be either 0 or $\mu(X)$. Indeed, in Theorem 26 we prove that if X is any finite measure space supporting an ergodic transformation, then $s(X) \in \{0, \mu(X)\}$.

THEOREM 26. *If $(X, \mathcal{A}, \mu)$ is a finite measure space that supports an ergodic transformation, then $s(X) \in \{0, \mu(X)\}$.*

Proof. Let $F : X \to X$ be ergodic, and suppose that $s(X) > 0$. We must show that $s(X) = \mu(X)$. By Theorem 20, there exist a set $U \subset X \times I$ such that U_x is of full measure in I for every $x \in X$ and a set $B \subset I$ of full outer measure such that, for every $y \in B$, $\mu^*(U^y) < \mu(X)$.

Define $\phi(x, y) = (F(x), y)$ and put

$$S = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \phi^{-n}(U).$$

Then, for every $x \in X$, S_x has full measure. However, for each $y \in B$, $S^y = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} F^{-n}(U^y)$ is of less than full outer measure for every $y \in B$ and is also F-invariant. Since F is ergodic, it follows that $\mu(S^y) = 0$ for every $y \in B$. Hence S is a weak 0-1 set and so, by Corollary 21, $s(X) = \mu(X)$. □

THEOREM 27. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space that supports an ergodic transformation. Then the following statements are equivalent.*

(i) *There is no weak 0-1 set in* $X \times I$.
(ii) $s(X) = 0$.
(iii) $\underline{\int}_X \underline{\int} f(x,y)\,dy\,d\mu(x) \le \underline{\int}\,\overline{\int}_X f(x,y)\,d\mu(x)\,dy$ *for every* $f\colon (X \times I) \to I$.

Proof. Suppose that (i) holds. By Corollary 21 we have $s(X) < \mu(X)$, and by Theorem 26 we obtain $s(X) = 0$. The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follow directly from the definitions. □

We remark that a statement equivalent to (iii) but in arbitrary finite measure spaces can be found in [4, Thm. 4].

6. No Weak 0-1 Set Implies Almost Complete Transfer of Density a.e.

The goal of this section is to prove Theorem 13. In Section 4 we proved the implication (i) $\Rightarrow$ (ii). Therefore, it would be natural to infer (iii) from (ii) the same way as Theorem 6 was inferred from Theorem 5.

Suppose (ii) holds, and let $E \subset H$ be such that $d_*(E,x,\pi/2) = 1$ for every $x \in \mathbb{R}$. Let C be the set of pairs $(x,\theta) \in \mathbb{R} \times (0,\pi)$ such that $\bar{d}^*(E,x,\theta) = 1$. We have to prove that C_x is of full measure for a.e. x. Now, if (ii) is true, then the section C^θ is of full measure for every fixed θ. However, since C is not measurable, this does not imply that C_x is of full measure for a.e. x. Indeed, there exists a set $C \subset \mathbb{R}^2$ such that C^y is the complement of a singleton for every y but $\lambda_*(C_x) = 0$ for every x. Since the existence of such a set can be proved in ZFC, the preceding argument cannot prove the implication (ii) $\Rightarrow$ (iii) of Theorem 12.

We overcome this difficulty by following a different argument proving Theorem 6. We first give a proof of Theorem 6 that is independent of the previous argument, and then we adapt it to the nonmeasurable case.

Second Proof of Theorem 6. Let $E \subset H$ be a Borel set such that $d(E,x,\pi/2) = 1$ for every $x \in \mathbb{R}$. It is enough to show that, for a.e. $x \in I$, $\bar{d}(E,x,\theta) = 1$ for almost every θ. Suppose this is not true. Then there exist an $\varepsilon > 0$ and a measurable set $A \subset I$ with $\lambda(A) > \varepsilon$ such that, for every $x \in A$, $\bar{d}(E,x,\theta) < 1-\varepsilon$ for every $\theta \in \Theta_x$ with $\lambda(\Theta_x) > \varepsilon$.

It follows that there exist measurable sets $B \subset A$ and $\Theta'_x \subset \Theta_x$ and also a positive number h_0 such that $\lambda(B) > \varepsilon$, $\lambda(\Theta'_x) > \varepsilon$ for every $x \in B$, and

$$\frac{\lambda(E \cap L(x,\theta,h))}{h} < 1-\varepsilon$$

for every $x \in B$, $\theta \in \Theta'_x$, and $0 < h < h_0$. We define H_n as the set of points $x \in I$ such that

$$\frac{\lambda(E_x \cap [0,h])}{h} > 1 - \frac{\varepsilon^3}{4} \tag{18}$$

for every $0 < h < 1/n$. Then H_n is measurable for every n, $H_1 \subset H_2 \subset \cdots$, and it follows (from our assumption on the set E) that $\bigcup_{n=1}^\infty H_n = I$. Then $\lambda(H_n) \to 1$, and we can select an n such that $\lambda(H_n) > 1-\varepsilon$. Since $\lambda(H_n \cap B) > 0$, there is a point $x \in H_n \cap B$ such that x is a density point of H_n. For computational convenience we assume that $x = 0$, and we denote Θ'_0 simply by Θ. Since $0 = x \in B$,

$$\frac{\lambda(E \cap L(0, \theta, h))}{h} < 1 - \varepsilon \tag{19}$$

for every $\theta \in \Theta$ and $0 < h < h_0$. Also, since 0 is a density point of H_n, we can fix a positive number h such that $h < \min(h_0/2, 1/n)$ and

$$\lambda(H_n \cap [0, h]) > \left(1 - \frac{\varepsilon^3}{4}\right) \cdot h. \tag{20}$$

Let $f(x, y) = 1$ if $(x, y) \in E \cap ([0, h] \times [0, h])$, and let $f(x, y) = 0$ otherwise. Applying the change of variable $y = x \tan\theta$, we obtain

$$\begin{aligned}\int_0^h \int_0^h f(x, y)\, dy\, dx &= \int_0^h \int_0^{\pi/2} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, d\theta\, dx \\ &= \int_0^{\pi/2} \int_0^h x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, dx\, d\theta. \end{aligned} \tag{21}$$

On the one hand, if $x \in H_n \cap (0, h]$ is fixed then

$$\int_0^h f(x, y)\, dy = \lambda(E_x \cap [0, h]) > \left(1 - \frac{\varepsilon^3}{4}\right) \cdot h$$

by (18); hence

$$\int_0^h \int_0^h f(x, y)\, dy\, dx \geq \left(1 - \frac{\varepsilon^3}{4}\right)^2 \cdot h^2 > \left(1 - \frac{\varepsilon^3}{2}\right) \cdot h^2. \tag{22}$$

On the other hand, if $0 < \theta < \pi/2$ is fixed then the points $(x, x\tan\theta)$ $(x \geq 0)$ run through the half-line $L(0, \theta)$. Since $f(x, y) = 0$ outside the square $[0, h] \times [0, h]$, the last integral of (21) is equal to

$$\int_0^{\pi/2} \int_0^{r(\theta)} t \cdot f(t\cos\theta, t\sin\theta)\, dt\, d\theta,$$

where $r(\theta)$ denotes the length of the segment $L(0, \theta) \cap ([0, h] \times [0, h])$. Note that

$$\int_0^{\pi/2} \frac{r^2(\theta)}{2}\, d\theta = \lambda_2([0, h] \times [0, h]) = h^2$$

by the area formula when polar coordinates are used.

If $\theta \in \Theta$, then it follows from (19) and from $r(\theta) < 2h \leq h_0$ that the measure of the set $\{t \in [0, r(\theta)] : f(t\cos\theta, t\sin\theta) = 1\}$ is at most $(1 - \varepsilon) \cdot r(\theta)$. Therefore,

$$\int_0^{r(\theta)} t \cdot f(t\cos\theta, t\sin\theta)\, dt \leq \int_{\varepsilon \cdot r(\theta)}^{r(\theta)} t\, dt = \frac{r^2(\theta)}{2} \cdot (1 - \varepsilon^2).$$

If $\theta \in (0, \pi/2) \setminus \Theta$, then

$$\int_0^{r(\theta)} t \cdot f(t\cos\theta, t\sin\theta)\, dt \leq \int_0^{r(\theta)} t\, dt = \frac{r^2(\theta)}{2}$$

and so

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{r(\theta)} t\cdot f(t\cos\theta, t\sin\theta)\,dt\,d\theta \\
&\quad\le \int_\Theta \frac{r^2(\theta)}{2}\cdot(1-\varepsilon^2)\,d\theta + \int_{(0,\pi/2)\setminus\Theta}\frac{r^2(\theta)}{2}\,d\theta \\
&\quad= \int_0^{\pi/2}\frac{r^2(\theta)}{2}\,d\theta - \int_\Theta \frac{r^2(\theta)}{2}\varepsilon^2\,d\theta \\
&\quad\le h^2 - \frac{h^2}{2}\cdot\varepsilon^2\cdot\lambda(\Theta) \\
&\quad\le \left(1-\frac{\varepsilon^3}{2}\right)\cdot h^2
\end{aligned}\tag{23}$$

because $\lambda(\Theta) > \varepsilon$. This estimate contradicts (22) and (21), which completes the proof. □

Now we turn to the proof of Theorem 13. Suppose that there is no weak 0-1 set. By Theorem 27, this implies $s(I) = 0$.

It is enough to show that if $E \subset H$ is such that $d_*(E, x, \pi/2) = 1$ for every $x \in \mathbb{R}$ then, for a.e. $x \in I$, $\bar{d}^*(E, x, \theta) = 1$ for almost every θ. Suppose this is not true. Then there exist an $\varepsilon > 0$ and a set $A \subset I$ such that $\lambda^*(A) > \varepsilon$, and $\bar{d}^*(E, x, \theta) < 1-\varepsilon$ for every $\theta \in \Theta_x$ with $\lambda^*(\Theta_x) > \varepsilon$.

It follows that there exist sets $B \subset A$ and $\Theta_x' \subset \Theta_x$ as well as a positive number h_0 such that $\lambda^*(B) > \varepsilon$, $\lambda^*(\Theta_x') > \varepsilon$ for every $x \in B$, and

$$\frac{\lambda^*(E \cap L(x,\theta,h))}{h} < 1-\varepsilon$$

for every $x \in B$, $\theta \in \Theta_x'$, and $0 < h < h_0$. We define H_n as the set of points $x \in I$ such that

$$\frac{\lambda_*(E_x \cap [0,h])}{h} > 1 - \frac{\varepsilon^3}{16} \tag{24}$$

for every $0 < h < 1/n$. Then $H_1 \subset H_2 \subset \cdots$, and it follows from our assumption on the set E that $\bigcup_{n=1}^\infty H_n = I$. Hence $\bigcup_{n=1}^\infty (H_n \cap B) = B$ and so $\lim_{n\to\infty}\lambda^*(H_n \cap B) = \lambda^*(B) > \varepsilon$. Moreover, by Theorem 23 we have $\lim_{n\to\infty} s(H_n) = s(I) = 0$. We may therefore fix an n such that $\lambda^*(H_n \cap B) > \varepsilon$ and $s(H_n) < \varepsilon^{10}/2^{13}$.

Let Z denote the set of points $x \in H_n$ such that

$$\limsup_{h\to 0+}\frac{s(H_n \cap [x, x+h])}{h} > \frac{\varepsilon^9}{2^{13}}.$$

By Lemma 24, we have $\lambda^*(Z) \le \varepsilon$. Since $\lambda^*(H_n \cap B) > \varepsilon$, the set $(H_n \cap B) \setminus Z$ has positive outer measure. Let $x \in (H_n \cap B) \setminus Z$ be an outer density point of $(H_n \cap B) \setminus Z$. For computational convenience we assume that $x = 0$, and we denote Θ_0' simply by Θ.

Since $0 = x \in B$, we have

$$\frac{\lambda^*(E \cap L(0,\theta,h))}{h} < 1 - \varepsilon \tag{25}$$

for every $\theta \in \Theta$ and $0 < h < h_0$. Also, since 0 is an outer density point of H_n and since $0 \in H_n \setminus Z$, we can fix a positive number h such that $h < \min(h_0/2, 1/n)$,

$$\lambda^*(H_n \cap [0,h]) > \left(1 - \frac{\varepsilon^3}{16}\right)h, \tag{26}$$

and

$$\frac{s(H_n \cap [0,h])}{h} < \frac{\varepsilon^9}{2^{12}}. \tag{27}$$

For the remainder of this proof we fix ε, h, and Θ with the properties (25), (26), and (27). Note that (24) holds for every $x \in H_n$.

Put $\eta = \varepsilon^3/16$, and let f denote the characteristic function of the set $E \cap ([\eta h, h] \times [0,h])$. We claim that

$$\begin{aligned} &\underline{\int}_A \underline{\int}_0^{\pi/2} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, d\theta\, dx \\ &\qquad \leq \underline{\int}_0^{\pi/2} \overline{\int}_A x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, dx\, d\theta + \frac{h\pi}{\eta^2} \cdot s(A) \end{aligned} \tag{28}$$

for every $A \subset [0,h]$. Indeed, if $f(x, x\tan\theta) \neq 0$ then $x \geq \eta h$ and $x \tan\theta \leq h$, whence $\tan\theta \leq 1/\eta$. Thus $\cos^{-2}\theta = 1 + \tan^2\theta \leq 2/\eta^2$ and $x\cos^{-2}\theta \leq 2h/\eta^2$. Therefore, (28) follows from Lemma 19.

We now apply (28) with $A = H_n \cap [0,h]$. By (27) we have $s(H_n \cap [0,h]) < (\varepsilon^9/2^{12}) \cdot h$, so

$$\begin{aligned} &\underline{\int}_{H_n \cap [0,h]} \underline{\int}_0^{\pi/2} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, d\theta\, dx \\ &\qquad < \underline{\int}_0^{\pi/2} \overline{\int}_{H_n \cap [0,h]} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, dx\, d\theta + \frac{\varepsilon^3}{4} \cdot h^2. \end{aligned} \tag{29}$$

Next we estimate the left-hand side of (29) from below. Let $x \in H_n \cap [\eta h, h]$ be fixed. Since $f(x, x\tan\theta) = 0$ unless $x\tan\theta \leq h$, we do not change the value of the inner integral when changing the limits of integration to $\underline{\int}_0^{(\arctan h)/x}$. Applying the change of variable $y = x\tan\theta$ (and Lemma 17), by (24) we obtain

$$\begin{aligned} \underline{\int}_0^{\pi/2} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, d\theta &= \underline{\int}_0^h f(x,y)\, dy = \lambda_*(E_x \cap [0,h]) \\ &> \left(1 - \frac{\varepsilon^3}{16}\right)h. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\underline{\int}_{H_n\cap[0,h]} \underline{\int}_0^{\pi/2} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, d\theta\, dx \\
&\quad \geq \underline{\int}_{H_n\cap[\eta h,h]} \underline{\int}_0^{\pi/2} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, d\theta\, dx \\
&\quad \geq \left(1-\frac{\varepsilon^3}{8}\right)^2 \cdot h^2 > \left(1-\frac{\varepsilon^3}{4}\right) \cdot h^2
\end{aligned}$$

by (26). Observe that the first integral is over the measure space $H_n \cap [0,h]$ of total measure $\mu(H_n \cap [0,h]) = \lambda^*(H_n \cap [0,h])$ and that the second integral is over a μ-measurable subset of that measure space. Also, we have

$$\lambda^*(H_n \cap [\eta h, h]) \geq \lambda^*(H_n \cap [0,h]) - \eta h \geq \left(1-\frac{\varepsilon^3}{8}\right)h.$$

Finally, we estimate the right-hand side of (29) from above. We begin by giving an upper estimate for the integral on the right-hand side of (29). We have

$$\begin{aligned}
&\underline{\int}_0^{\pi/2} \overline{\int}_{H_n\cap[0,h]} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, dx\, d\theta \\
&\quad \leq \underline{\int}_0^{\pi/2} \overline{\int}_{[0,h]} x \cdot \cos^{-2}\theta \cdot f(x, x\tan\theta)\, dx\, d\theta \\
&\quad = \underline{\int}_0^{\pi/2} \overline{\int}_0^{r(\theta)} t \cdot f(t\cos\theta, t\sin\theta)\, dt\, d\theta,
\end{aligned}$$

where the last equality uses the linear substitution $x = t\cos\theta$ in the inner integral. Note that $r(\theta)$ denotes the length of the segment $L(0,\theta) \cap ([0,h] \times [0,h])$. Let

$$V(\theta) = \overline{\int}_0^{r(\theta)} t \cdot f(t\cos\theta, t\sin\theta)\, dt$$

for every $\theta \in [0, \pi/2)$. Then, arguing much as in the proof of Theorem 6, we find

$$V(\theta) \leq \int_{\varepsilon\cdot r(\theta)}^{r(\theta)} t\, dt = \frac{r^2(\theta)}{2} \cdot (1-\varepsilon^2)$$

for every $\theta \in \Theta$. If $U\colon [0,\pi/2) \to \mathbb{R}$ is measurable and if $U \leq V$ on $[0, \pi/2)$, then $U(\theta) \leq r^2(\theta) \cdot (1-\varepsilon^2)/2$ on a measurable set containing Θ. Therefore, the computation in (23) gives

$$\underline{\int}_0^{\pi/2} \overline{\int}_0^{r(\theta)} t \cdot f(t\cos\theta, t\sin\theta)\, dt\, d\theta \leq \left(1-\frac{\varepsilon^3}{2}\right) \cdot h^2.$$

Hence the right-hand side of (29) is not greater than

$$\left(1-\frac{\varepsilon^3}{2}\right)h^2 + \frac{\varepsilon^3}{4} \cdot h^2 = \left(1-\frac{\varepsilon^3}{4}\right)h^2.$$

This estimate contradicts (29) and (30), which completes the proof. □

7. Construction of the Examples

7.1. Proof of Theorem 3

We begin by constructing the following preliminary example.

THEOREM 28. *There exists a closed set $E \subset H$ such that*:

(i) $D(E,x) = 0$ *for every* $x \in \mathbb{R}$; *and*
(ii) $\bar{d}(E,x,\theta) = 1$ *for every* $x \in \mathbb{R}$ *and* $\theta \in (0,\pi)$.

Proof. This construction is a refinement of [3, Exm. 3]. Let $[\alpha_k,\beta_k] \subset (0,\pi)$ be a closed interval for every $k = 1,2,\dots$, let $(m_k)_{k=1}^{\infty}$ be a sequence of positive integers, and let $(a_n)_{n=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers converging to zero.

We define a trapezoid $T_{n,i}$ for every positive integer n and for every $i \in \mathbb{Z}$ as follows. Put $n_1 = 0$ and $n_k = m_1 + \cdots + m_{k-1}$ for every $k \geq 2$. Then there is a unique k such that $n_k \leq n < n_{k+1}$. Let $T_{n,i}$ be the trapezoid bounded by the lines $L((i-1)a_{n_{k+1}},\beta_k)$, $L(ia_{n_{k+1}},\alpha_k)$, $y = a_{n-1}$, and $y = a_n$. For $n = n_k + r$ we define $A_n = \bigcup\{T_{n,i} : i \in \mathbb{Z}, i \equiv r \pmod{m_k}\}$. Then A_n is a closed subset of the strip $S_n = \{(x,y) : a_n \leq y \leq a_{n-1}\}$ for every n. Let $E = \bigcup_{n=1}^{\infty} A_n \cup \{(x,y) : y = 0\}$, in which case E is a closed subset of H.

First we will show that—under suitable conditions imposed on the sequences $([\alpha_k,\beta_k])$, (m_k), and (a_n)—the set E satisfies the requirements of the theorem. Suppose that

(A) every $\theta \in (0,\pi)$ belongs to infinitely many intervals $[\alpha_k,\beta_k]$ and
(B) $a_n/a_{n-1} \to 0$ as $n \to \infty$.

Then (ii) is satisfied. Indeed, let $x \in \mathbb{R}$ and $\theta \in (0,\pi)$ be arbitrary, and let $\theta \in [\alpha_k,\beta_k]$. Choose an integer i such that $x \in [(i-1)a_{n_{k+1}}, ia_{n_{k+1}}]$ and suppose that $i = q \cdot m_k + r$, where $0 \leq r < m_k$. If $n = n_k + r$, then $n_k \leq n < n_{k+1}$ and it follows (from the definitions of $T_{n,i}$, A_n, and E) that

$$L(x,\theta) \cap E \supset L(x,\theta) \cap T_{n,i} = L(x,\theta) \cap S_n. \tag{31}$$

Therefore, putting $h_{n-1} = a_{n-1}/\sin\theta$, we obtain

$$\frac{\lambda(L(x,\theta,h_{n-1}) \cap E)}{h_{n-1}} \geq \frac{a_{n-1} - a_n}{a_{n-1}}.$$

Since there are infinitely many such n, from (B) it follows that $\bar{d}(E,x,\theta) = 1$.

Now suppose that

(C) $m_k \to \infty$ as $k \to \infty$,
(D) $m_k \cdot a_{n_{k+1}} < a_{n_{k+1}-1}$ for every $k \geq k_0$, and
(E) $(a_{n_k-1}/a_{n_{k+1}}) \cdot \varepsilon_k \to 0$ as $k \to \infty$;

here $\varepsilon_k = \cot\alpha_k - \cot\beta_k$. We prove that these conditions imply (i). If $n_k \leq n < n_{k+1}$ and $a_n < y < a_{n-1}$ then, for every i, the length of the segment $(T_{n,i})^y$ is at most

$$a_{n_{k+1}} + a_{n-1} \cdot \varepsilon_k \le a_{n_{k+1}} + a_{n_k-1} \cdot \varepsilon_k.$$

It follows from the definition of A_n that if $a_n < y < a_{n-1}$ then the set $(A_n)^y$ is periodic with period $p_k = m_k \cdot a_{n_{k+1}}$. By (D), we have $p_k < a_{n_{k+1}-1}$ for every $k \ge k_0$. We also have $E^y = (A_n)^y$ because $A_m \subset S_m$ for every $m \ne n$. Therefore, the measure of E^y in every interval of length p_k is at most $a_{n_{k+1}} + a_{n_k-1} \cdot \varepsilon_k$; hence the relative density of E^y in every interval of length $\ge p_k$ is at most

$$2 \cdot \frac{a_{n_{k+1}} + a_{n_k-1} \cdot \varepsilon_k}{m_k \cdot a_{n_{k+1}}} \le \frac{2}{m_k} + 2 \cdot \frac{a_{n_k-1}}{a_{n_{k+1}}} \cdot \varepsilon_k = b_k.$$

Thus, by (C) and (E), $\lim_{k\to\infty} b_k = 0$.

Let $K \ge k_0$, $x \in \mathbb{R}$, and $0 < h < a_{n_K}$ be arbitrary. We can estimate the relative density of E in $[x, x+h] \times [0, h]$ as follows. Suppose $0 < y < h$ is such that $y \ne a_i$ for every i. If $a_n < y < a_{n-1}$, where $n_k \le n < n_{k+1}$, then $k \ge K \ge k_0$ and we have $h > a_n \ge a_{n_{k+1}-1} > p_k$. Hence the relative density of E^y in $[x, x+h]$ is at most b_k. The relative density of E^y in $[x, x+h]$ is therefore at most $c_K = \max_{k \ge K} b_k$ for every $0 < y < h$ satisfying $y \ne a_i$ $(i = 1, 2, \dots)$, and so the relative density of E in $[x, x+h] \times [0, h]$ is at most c_K as well. Since $c_K \to 0$ as $K \to \infty$, this implies (i).

In order to complete the proof, we must construct the sequences $([\alpha_k, \beta_k])$, (m_k), and (a_n) such that conditions (A)–(E) are satisfied.

It is easy to see, from the divergence of the series $\sum_{k=1}^{\infty} 1/k$, that there are intervals I_k $(k = 1, 2, \dots)$ with $|I_k| = 1/k$ for every k and such that every real number belongs to infinitely many of the intervals I_k. Let $\alpha_k, \beta_k \in (0, \pi)$ be such that $[\cot \beta_k, \cot \alpha_k] = I_k$. The $\cot x$ function establishes a strictly decreasing bijection between $(0, \pi)$ and $\mathbb{R}$, so clearly (A) is true. Note that $\varepsilon_k = \cot \alpha_k - \cot \beta_k = 1/k$ for every k.

We define $m_k = 1$ for $k \le 20$ and $m_k = [\log \log k]$ for $k > 20$. Then (m_k) is a sequence of positive integers satisfying (C). Put $n_1 = 0$ and $n_k = m_1 + \dots + m_{k-1}$ for every $k \ge 2$. We shall define the numbers a_n inductively. First put $a_0 = 1$. If $k \ge 1$ and a_{n_k} has been defined, then we put $a_n = a_{n_k} / \max(2, \log 2k)^{n - n_k}$ for every $n_k < n \le n_{k+1}$. It should be clear that the sequence (a_n) defined in this way is strictly decreasing and satisfies (B).

We have $a_{n_{k+1}-1}/a_{n_{k+1}} = \max(2, \log 2k) \ge m_k$ if $k \ge k_0$, so (D) must hold. Finally, if k is large enough then

$$\begin{aligned} \frac{a_{n_k-1}}{a_{n_{k+1}}} &= \frac{a_{n_k-1}}{a_{n_k}} \cdot \frac{a_{n_k}}{a_{n_{k+1}}} = \log 2(k-1) \cdot (\log 2k)^{m_k} \\ &< e^{(1+m_k)\cdot \log\log 2k} \le e^{(1+\log\log k)\cdot \log\log 2k} \\ &< e^{(\log k)/2} = k^{1/2}. \end{aligned}$$

Therefore,

$$\frac{a_{n_k-1}}{a_{n_{k+1}}} \cdot \varepsilon_k < \frac{1}{k^{1/2}} \to 0,$$

which proves (E). $\square$

Proof of Theorem 3. We use notation from the proof of Theorem 28. Put $c_k = \max_{j \geq k} b_j$ for every $k \geq 1$. Then clearly (c_k) is a decreasing sequence that converges to zero.

We shall construct an open set G_n for every n as follows. Suppose $n_k \leq n < n_{k+1}$. As shown in the proof of Theorem 28,

$$\lambda((A_n)^y \cap [0, p_k]) \leq b_k \cdot p_k \leq c_k \cdot p_k$$

for every $a_n < y < a_{n-1}$; hence

$$\lambda_2(A_n \cap ([0, p_k] \times \mathbb{R})) \leq c_k \cdot p_k \cdot (a_{n-1} - a_n). \tag{32}$$

If we set $B_n = \{x \in (0, p_k) : \lambda((A_n)_x) > \sqrt{c_k} \cdot (a_{n-1} - a_n)\}$, then (32) implies that $\lambda(B_n) < p_k \cdot \sqrt{c_k}$. Let C_n be an open subset of $\mathbb{R}$ such that C_n is periodic modulo p_k, $B_n \subset C_n$, and $\lambda(C_n \cap [0, p_k]) < p_k \cdot \sqrt{c_k}$. We define

$$d_n = \min\bigl(\max\bigl(a_n, \sqrt[4]{c_k} \cdot a_{n-1}\bigr), a_{n-1}\bigr)$$

and

$$G_n = (\mathbb{R} \times (a_n, d_n)) \cup (C_n \times (a_n, a_{n-1})).$$

Let $F = E \setminus \bigcup_{n=1}^{\infty} G_n$ and $G = (\operatorname{int} H) \setminus F$. We prove that the set G satisfies the requirements of the theorem. Clearly, G is an open set. For every $x \in \mathbb{R}$ and $n = 1, 2 \ldots$, we have $F_x \cap (a_n, a_{n-1}) = \emptyset$ if $x \in C_n$ and $\lambda((A_n)_x) \leq \sqrt{c_k} \cdot (a_{n-1} - a_n)$ if $x \notin C_n$. This implies that $\lambda(F_x \cap [a_n, a_{n-1}]) \leq \sqrt{c_k} \cdot (a_{n-1} - a_n)$ for every n and so $\lambda(F_x \cap [0, a_n]) \leq \sqrt{c_k} \cdot a_n$ for every n, where k is determined by $n_k \leq n < n_{k+1}$.

On the one hand, if $a_n \leq h \leq d_n$ then we thereby obtain

$$\frac{\lambda(F_x \cap [0, h])}{h} \leq \frac{\lambda(F_x \cap [0, a_n])}{a_n} \leq \sqrt{c_k};$$

on the other hand, if $d_n < h < a_{n-1}$ then

$$\begin{aligned} \frac{\lambda(F_x \cap [0, h])}{h} &\leq \frac{\lambda(F_x \cap [0, a_n]) + \lambda(F_x \cap [a_n, a_{n-1}])}{h} \\ &\leq \frac{\lambda(F_x \cap [0, a_n])}{a_n} + \frac{\sqrt{c_k} \cdot a_{n-1}}{d_n} \\ &\leq \sqrt{c_k} + \sqrt[4]{c_k}. \end{aligned}$$

Since $\sqrt{c_k} + \sqrt[4]{c_k} \to 0$ as $k \to \infty$, it follows that $d(F, x, \pi/2) = 0$ and that $d(G, x, \pi/2) = 1$ for every $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ and $\theta \neq \pi/2$ be fixed. We put $h_n = a_n / \sin\theta$ for every n. In the proof of Theorem 28 we saw that (31) holds for infinitely many n. For every such n we have

$$\frac{\lambda(L(x, \theta, h_{n-1}) \cap F)}{h_{n-1}} \geq \frac{a_{n-1} - a_n}{a_{n-1}} - \frac{\lambda(L(x, \theta, h_{n-1}) \cap G_n)}{h_{n-1}};$$

therefore, to show $\bar{d}(F, x, \theta) = 1$ it is enough to prove that

$$\lim_{n\to\infty} \frac{\lambda(L(x,\theta,h_{n-1}) \cap G_n)}{h_{n-1}} = 0. \tag{33}$$

We have $L(x,\theta,h_{n-1}) \cap G_n = U_n \cup V_n$, where $U_n = L(x,\theta,h_{n-1}) \cap (\mathbb{R} \times (a_n, d_n))$ and $V_n = L(x,\theta,h_{n-1}) \cap (C_n \times (a_n, a_{n-1}))$.

It is easy to see that $\lambda(U_n)/h_{n-1} \le d_n/a_{n-1} \le \sqrt[4]{c_k} \to 0$ as $n \to \infty$. Indeed, for large n we have $d_n = \max\big(a_n, \sqrt[4]{c_k} \cdot a_{n-1}\big)$, in which case the statement follows from $a_n/a_{n-1} \to 0$ and $\sqrt[4]{c_k} \to 0$.

Now observe that $\lambda(V_n)/h_{n-1}$ equals the relative density of C_n in the interval J, where J is the projection of $L(x,\theta) \cap (\mathbb{R} \times (a_n, a_{n-1}))$ onto $\mathbb{R}$. The length of J is equal to $|\cot\theta| \cdot (a_{n-1} - a_n) \ge |\cot\theta| \cdot a_{n-1}/2$. Hence, for n large enough,

$$a_{n-1} \ge a_{n_{k+1}-2} = \log 2k \cdot a_{n_{k+1}-1} \ge \log 2k \cdot p_k$$

by (D) and so $|J| \ge p_k$ for $n \ge n_0$. Since the relative density of C_n in every interval of length $\ge p_k$ is at most $2 \cdot \sqrt{c_k}$, it follows that $\lambda(V_n)/h_{n-1} \to 0$ if $n \to \infty$. This proves (33). Consequently, $\bar{d}(F, x, \theta) = 1$ and $\underline{d}(G, x, \theta) = 0$, completing the proof of the theorem. □

7.2. Proof of Theorem 4

For $x \in \mathbb{R}$ and $-\pi < \theta < \pi$, let $L(x, \theta, *)$ denote the line segment of direction θ extending between the x-axis and the line $y = 1$. In [5, Exm. 1], Kinney constructs the following example.

KINNEY'S SET. There is a closed null set $K \subset [0,1]^2$ such that, for every $x \in [0,1]$, there exists an $\alpha_x \in [\pi/2, \cos^{-1}(-1/3)]$ with $L(x, \alpha_x, *) \subset K$.

Let T_1 and T_2 be the affine transformations $T_1(x, y) = (x/2 + y/2, y)$ and $T_2(x, y) = (x/2 - y/2 + 1/2, y)$, and define $B_0 = T_1(K) \cup T_2(K)$. Then $B_0 \subset [0,1]^2$ is again a closed null set and, by the nature of K, for every $x \in [0,1]$ there is a $\theta_x \in (-\pi, \pi)\backslash\{\pi/2\}$ with $L(x, \theta_x, *) \subset B_0$. Finally, for each $n \in \mathbb{Z}$ let $B_n = B_0 + (n, 0)$ and set $B = \bigcup_{n\in\mathbb{Z}} B_n$.

Since $B \subset H$ is closed and null, it is easy to see that $A = \{x \in \mathbb{R} : B_x$ is not null$\}$ is a linear F_σ null set. Let A^* be a G_δ null set containing A, and define

$$E = (\mathbb{R}^2\backslash B) \cup (A \times \mathbb{R}), \qquad E^* = (\mathbb{R}^2\backslash B) \cup (A^* \times \mathbb{R}).$$

In summary, we have the following statements.

1. $E \in F_\sigma$ and $E^* \in G_\delta$.
2. For every $x \in \mathbb{R}$, $L(x, \theta_x, *) \cap E$ and $L(x, \theta_x, *) \cap E^*$ are of linear measure 0.
3. For every $x \in \mathbb{R}$, E_x and E^*_x are of full linear measure.

This completes the proof of Theorem 4.

7.3. Proofs of Theorems 7 and 8

Proof of Theorem 7. We use transfinite induction to construct a set $E \subset \mathbb{R}^2$ that contains a point from every nonempty vertical perfect set but contains at most two points from every nonvertical line. The conclusion of Theorem 7 follows immediately from these properties.

Let $\mathfrak{P}$ consist of the nonempty vertical perfect sets in $\mathbb{R}^2$—in other words, perfect sets that are situated on some vertical line. Then $\mathrm{card}(\mathfrak{P}) = 2^\omega$ and we enumerate $\mathfrak{P} = \{S_\alpha : \alpha < 2^\omega\}$. To begin the induction, select a point $s_0 \in S_0$ and put $s_0 \in E$.

Now suppose that $\alpha < 2^\omega$ and that s_β has been selected for every $\beta < \alpha$. Since pairs of distinct points from $E_\alpha = \{s_\beta : \beta < \alpha\}$ determine $\mathrm{card}(E_\alpha^2)$ lines and since $\mathrm{card}(E_\alpha) \leq \mathrm{card}(\alpha) < 2^\omega = \mathrm{card}(S_\alpha)$, it follows that S_α contains a point that is not on any of those lines. Select such a point, s_α, and assign it to E. This completes the induction.

Next suppose that $\{s_\alpha, s_\beta, s_\gamma\} \subset E$, where $\alpha < \beta < \gamma$ and no two of these points lie on the same vertical line. Then s_γ is not on the line determined by s_α and s_β, so these three points are not collinear. However, E does contain a point from every vertical perfect set; hence E has full outer measure on every vertical line. □

Proof of Theorem 8. Here we construct a set F that contains a point from every nonvertical linear perfect set but has at most one point from every vertical line. The complement of F will have linear inner measure 0 on every nonvertical line yet will be missing at most one point from every vertical line. This $\mathbb{R}^2 \setminus F$, then, is the set E of Theorem 8.

To construct F, we enumerate the nonvertical and nonempty linear perfect sets as $\{P_\alpha : \alpha < 2^\omega\}$, and we start the induction by selecting a point $p_0 \in P_0$.

Suppose now that $\alpha < 2^\omega$ and that a point p_β has been selected for every $\beta < \alpha$. Because P_α is a nonvertical and nonempty perfect set, it is not covered by fewer than 2^ω vertical lines. But $\mathrm{card}(\alpha) < 2^\omega$ and so there is a point $p_\alpha \in P_\alpha$ such that $L(p_\alpha, \pi/2) \cap \{p_\beta : \beta < \alpha\} = \emptyset$. In this way, a point p_α is defined for every $\alpha < 2^\omega$. If we set $F = \{p_\alpha : \alpha < 2^\omega\}$, then it is clear that F satisfies the requirements. □

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