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Lagrangian Surfaces with Circle Symmetry in the Complex Two-Space

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Introduction

The hyper-Kähler structure of the complex 2-space $\mathbb{C}^2$ produces an interesting correspondence between minimal Lagrangian surfaces and holomorphic curves. Indeed, the correspondence is given by exchanging the orthogonal complex structure J to another one on the Euclidean 4-space $\mathbb{R}^4$ [ChMo]. More generally, every Lagrangian conformal immersion of a Riemann surface M into $\mathbb{C}^2$ can be transformed to a map from M (possibly from a covering of M) satisfying a Dirac-type equation with a specific potential term. It is an extension of the Cauchy–Riemann equation. This result is based on the following obvious facts. First, every immersion of M into $\mathbb{C}^2$ is canonically identified with a section of the product vector bundle $M \times \mathbb{C}^2$ over M. Second, the "plus" part of the spin bundle $\mathbb{S}$ (associated to the canonical spin$^{\mathbb{C}}$-structure induced from the complex structure on M) is isomorphic to the product complex line bundle $M \times \mathbb{C}$. In fact, the operator associated with the Dirac-type equation is the bona fide Dirac operator on the direct sum of the spin bundle $\mathbb{S}$ and its conjugate spin bundle $\bar{\mathbb{S}}$ (see Section 1). Incidentally, it is now known that every conformal immersion of a Riemann surface into $\mathbb{R}^4$ or $\mathbb{R}^3$ can be expressed by a quaternionic analogue of the Cauchy–Riemann equation, and the quaternionic approach has advanced study for the surface theory (see [BFLPP]). However, we will study Lagrangian surfaces with prescribed Lagrangian angle in $\mathbb{C}^2$, and description in terms of the complex numbers seems to be suitable for this study.

In previous papers [A1; A2] we gave explicitly the transformation and Dirac-type equation just described in terms of the Lagrangian angle. For a given Lagrangian conformal immersion $f: M \to \mathbb{C}^2$, the Lagrangian angle β is defined as an $(\mathbb{R}/2\pi\mathbb{Z})$-valued function on M. It presents the self-dual part of the generalized Gauss map as for the surface in $\mathbb{R}^4$ ($\cong \mathbb{C}^2$). The mean curvature vector is given by $J\nabla\beta$. In particular, the angle β is constant if f is a minimal immersion. We proved that, if the half $\beta/2$ of Lagrangian angle is also well-defined globally on M, then f is represented by a solution of the Dirac-type equation on M with potential term of the complex derivative of β. (The corresponding results in [A1; A2] are not correct without the assumption on β.)

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On the other hand, a conformal immersion of M into the Euclidean 3-space $\mathbb{R}^3$ induces an adapted framing—that is, a pair of plus spinors locally defined over M. The adapted framing satisfies the Dirac equation with real-valued potential described by the mean curvature (see e.g. [KoT; KuS]). This equation closely resembles the aforementioned Dirac-type equation associated with a Lagrangian immersion. Indeed, for every helicoidal surface in $\mathbb{R}^3$, we can regard the Dirac equation as the Dirac-type equation associated with a Lagrangian surface in $\mathbb{C}^2$. Hence we obtain a correspondence from the set of helicoidal surfaces to a specific class of Lagrangian surfaces with circle symmetry. In this paper we use this correspondence to construct explicitly such Lagrangian surfaces with circle symmetry in $\mathbb{C}^2$. The Gauss map for a helicoidal surface in $\mathbb{R}^3$ coincides with the anti–self-dual part of the generalized Gauss map for the corresponding Lagrangian surface in $\mathbb{C}^2$.

Now we notice that every helicoidal surface with a fixed pitch in $\mathbb{R}^3$ is represented by the prescribed mean curvature function H of one variable s, and it corresponds to a plane curve with arc length parameter s and curvature $H/2$. Therefore we can construct a Lagrangian surface with prescribed Lagrangian angle function $\beta = \beta(s)$ in $\mathbb{C}^2$ from such any plane curve. For example, from a round circle centered at the origin, we have a right circular cylinder parameterized as a helicoidal surface with any pitch in $\mathbb{R}^3$, and moreover we obtain a Hopf torus in a round 3-sphere in $\mathbb{C}^2$. From a round circle through the origin, we obtain a round 2-sphere in $\mathbb{R}^3$ and then the Whitney sphere in $\mathbb{C}^2$.

These examples of Lagrangian surfaces in $\mathbb{C}^2$ are of conformal Maslov form; that is, the anti–self-dual part of the generalized Gauss map for the surface is a harmonic map to the unit 2-sphere. All compact Lagrangian surfaces with conformal Maslov form in $\mathbb{C}^2$ have been classified by Castro and Urbano [CU1] (cf. [RU]). Our construction guarantees that all such Lagrangian surfaces are given from the adapted framings for constant mean curvature surfaces of revolution and their associate helicoidal surfaces. Moreover, each of them corresponds to a round circle in the plane $\mathbb{C}$.

It is well known that the generalized Gauss map of a surface in $\mathbb{R}^4$ is harmonic if and only if the mean curvature vector is parallel. In [CU1] the authors also mentioned the class of Lagrangian surfaces in $\mathbb{C}^2$ each of whose generalized Gauss maps has harmonic self-dual; that is, the Lagrangian angle is harmonic. In other words, each surface in this class is a stationary point of the area functional under smooth Hamiltonian variations—that is, Hamiltonian stationary. Hélein and Romon [HRo1; HRo2] have studied Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$ from the viewpoint of the theory of integrable systems. In particular, they classified all Hamiltonian stationary Lagrangian tori. Applying our construction, we give some noncompact Hamiltonian stationary Lagrangian surfaces with circle symmetry in $\mathbb{C}^2$. The simplest example is given from the adapted framing of a right circular cone in $\mathbb{R}^3$, and hence it corresponds to a logarithmic spiral curve in $\mathbb{C}$. However, all examples constructed here are not complete.

In Section 1, we review the representation formula for Lagrangian surfaces in $\mathbb{C}^2$, including a new observation from the viewpoint of spinor calculus. In Section 2, we explain briefly about adapted framings of surfaces in $\mathbb{R}^3$ and helicoidal surfaces in $\mathbb{R}^3$. Then we give the correspondence from the set of helicoidal surfaces in $\mathbb{R}^3$ into the set of Lagrangian surfaces with circle symmetry in $\mathbb{C}^2$. Furthermore, we derive the explicit construction of Lagrangian surfaces with circle symmetry from the correspondence. In Sections 3 and 4, we apply the construction to the classes of conformal Maslov form and of Hamiltonian stationarity, respectively. Finally, in Section 5, we remark on the flat Lagrangian surfaces constructed by this method.

1. Representation Formula for Lagrangian Surfaces in $\mathbb{C}^2$

We consider $\mathbb{C}^2$ as the Euclidean 4-space $(\mathbb{R}^4, \langle\cdot,\cdot\rangle)$ with the standard complex structure $J(x_1,x_2,x_3,x_4) = (-x_3,-x_4,x_1,x_2)$, that is, a complex vector $\mathbf{x} = (x_1+\mathbf{i}x_3, x_2+\mathbf{i}x_4) \in \mathbb{C}^2$ is identified with the real vector $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$. Let M be a Riemann surface (without boundary) and $f\colon M \to \mathbb{C}^2$ a Lagrangian conformal immersion. Let $\{\mathsf{e}_1, \mathsf{e}_2\}$ be an oriented orthonormal basis of each tangent space f_*T_pM. Regarding e_1 and e_2 as the complex column vectors in $\mathbb{C}^2$, we obtain that $|\det(\mathsf{e}_1,\mathsf{e}_2)| = 1$. Then we can define a function $\beta\colon M \to \mathbb{R}/2\pi\mathbb{Z}$ by

$$\mathsf{e}_1 \wedge \mathsf{e}_2 = e^{\mathbf{i}\beta(p)}\mathbf{e}_1^{\mathbb{C}} \wedge \mathbf{e}_2^{\mathbb{C}} \quad \text{for} \ \ p \in M,$$

where $\mathbf{e}_1^{\mathbb{C}} = (1,0)$ and $\mathbf{e}_2^{\mathbb{C}} = (0,1) \in \mathbb{C}^2$. We call β the *Lagrangian angle* function for f.

On the other hand, regarding e_1 and e_2 as the real vectors in $\mathbb{R}^4$, we define the normalization $[\mathsf{e}_1 \wedge \mathsf{e}_2]$ of their real wedge product and identify it with the real plane $\mathcal{G}(p)$ that is parallel to the tangent plane f_*T_pM in $\mathbb{R}^4$. Thus we obtain the *generalized Gauss map* $\mathcal{G}\colon M \to G_2(\mathbb{R}^4)$ of the immersed surface in $\mathbb{R}^4$, where $G_2(\mathbb{R}^4)$ stands for the Grassmann manifold of oriented 2-planes in $\mathbb{R}^4$. According to the direct sum decomposition of the real wedge product space $\bigwedge^2(\mathbb{R}^4)$ between the self-dual subspace $\bigwedge^2_+ (\cong \mathbb{R}^3)$ and the anti–self-dual subspace $\bigwedge^2_- (\cong \mathbb{R}^3)$, $\mathcal{G}$ can be decomposed into the self-dual part $\mathcal{G}_+ = [(\mathsf{e}_1\wedge\mathsf{e}_2)^+]$ and the anti–self-dual part $\mathcal{G}_- = [(\mathsf{e}_1 \wedge \mathsf{e}_2)^-]$. Then the generalized Gauss map $\mathcal{G}$ can be identified with the pair of maps to the unit 2-sphere S^2:

$$\mathcal{G} = (\mathcal{G}_-, \mathcal{G}_+)\colon M \to S^2 \times S^2.$$

The assumption that the immersion f is Lagrangian implies that the image of $\mathcal{G}_+$ is contained in the equator of S^2, and then it is represented as follows (through the stereographic projection) in terms of the Lagrangian angle β:

$$\mathcal{G}_+ = e^{\mathbf{i}\beta}\colon M \to S^1 \subset \mathbb{C} \cup \{\infty\} \cong S^2.$$

Now we assume that the half $\beta/2$ of the Lagrangian angle β is well-defined globally on M as a $(\mathbb{R}/2\pi\mathbb{Z})$-valued function. Namely, the square root $\sqrt{\mathcal{G}_+} = e^{\mathbf{i}\beta/2}\colon M \to S^1$ is well-defined. Let us define a map $F = (F_1, F_2)\colon M \to \mathbb{C}^2$ by

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\mathbf{i}\beta/2} & \mathbf{i}e^{\mathbf{i}\beta/2} \\ \mathbf{i}e^{-\mathbf{i}\beta/2} & e^{\mathbf{i}\beta/2} \end{pmatrix} \begin{pmatrix} f_1^{\mathbb{C}} \\ f_2^{\mathbb{C}} \end{pmatrix}. \tag{1.1}$$

Then F satisfies the following Dirac-type equation with potential term:

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ \overline{F_2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \beta_z & 0 \\ 0 & \beta_{\bar z} \end{pmatrix}\begin{pmatrix} F_1 \\ \overline{F_2} \end{pmatrix}, \tag{1.2}$$

where z is an isothermal coordinate defined locally on M. However, equation (1.2) is well-defined globally on M.

Now define a pair $S = (S_1\,dz, S_2\,dz)$ of $(1,0)$-forms on M by

$$\begin{pmatrix} S_1 \\ \overline{S_2} \end{pmatrix} = \left[\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \beta_z & 0 \\ 0 & \beta_{\bar z} \end{pmatrix}\right]\begin{pmatrix} \overline{F_1} \\ F_2 \end{pmatrix}. \tag{1.3}$$

Then the induced metric on M is given by

$$f^*\,d\mathrm{s}^2 = (|S_1|^2 + |S_2|^2)|dz|^2,$$

and the anti–self-dual part of the generalized Gauss map is described as

$$\mathcal{G}_- = [S_1; S_2] \colon M \to \mathbb{C}P^1 \cong S^2.$$

Here the 2-sphere S^2 is identified with the complex projective line $\mathbb{C}P^1$. Moreover, S also satisfies the following Dirac-type equation, which is similar to (1.2):

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix}\begin{pmatrix} S_1 \\ \overline{S_2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\beta_{\bar z} & 0 \\ 0 & -\beta_z \end{pmatrix}\begin{pmatrix} S_1 \\ \overline{S_2} \end{pmatrix}. \tag{1.4}$$

From the viewpoint of spinor calculus, we will explain that $(F_1, \overline{F_2})$ and $(\overline{S_1}\,d\bar z, S_2\,dz)$ can be regarded as plus and minus spinors of a spin bundle on M, respectively. The conformal structure on M and the immersion f induce a unique Kähler structure on M. Let $\mathbb{S}$ be the spin bundle $\mathbb{S}^+ \oplus \mathbb{S}^- = \underline{\mathbb{C}}_M \oplus K_M^{-1}$ associated to the canonical spin$^{\mathbb{C}}$-structure on M, where $\underline{\mathbb{C}}_M$ and K_M^{-1} respectively denote the trivial complex line bundle and the anti-canonical line bundle. Then the (twisted) Dirac operator $\partial\!\!\!/_A \colon \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ associated to the Chern connection A is given by

$$\partial\!\!\!/_A = 2\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix}.$$

Let $\bar{\mathbb{S}}$ denote the conjugate spin bundle $\overline{\underline{\mathbb{C}}_M \oplus K_M^{-1}} = \underline{\mathbb{C}}_M \oplus K_M$ and let $\overline{\partial\!\!\!/_A} \colon \Gamma(\bar{\mathbb{S}}) \to \Gamma(\bar{\mathbb{S}})$ be its conjugate Dirac operator,

$$\overline{\partial\!\!\!/_A} = 2\begin{pmatrix} 0 & \partial_{\bar z} \\ -\partial_z & 0 \end{pmatrix}.$$

Then the sections $(F_1, \overline{F_2}) \in \Gamma(\underline{\mathbb{C}}_M \oplus \underline{\mathbb{C}}_M)$ and $(\overline{S_1}\,d\bar z, S_2\,dz) \in \Gamma(K_M^{-1} \oplus K_M)$ are respectively plus and minus spinors of the direct sum $\mathbb{S} \oplus \bar{\mathbb{S}}$. Moreover, the pair of Dirac-type equations (1.2) and (1.4) are equivalent to the following linear equation, which is associated with the Dirac operator $\partial\!\!\!/_A \oplus \overline{\partial\!\!\!/_A}$,

$$\partial\!\!\!/_A \oplus \overline{\partial\!\!\!/_A} \colon \Gamma((\underline{\mathbb{C}}_M \oplus \underline{\mathbb{C}}_M) \oplus (K_M^{-1} \oplus K_M)) \to \Gamma((\underline{\mathbb{C}}_M \oplus \underline{\mathbb{C}}_M) \oplus (K_M^{-1} \oplus K_M));$$

$$2\begin{pmatrix} 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & \partial_{\bar z} \\ -\partial_{\bar z} & 0 & 0 & 0 \\ 0 & -\partial_z & 0 & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ \overline{F_2} \\ \overline{S_1} \\ S_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \beta_{\bar z} \\ 0 & 0 & -\beta_z & 0 \\ 0 & \beta_{\bar z} & 0 & 0 \\ -\beta_z & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ \overline{F_2} \\ \overline{S_1} \\ S_2 \end{pmatrix}.$$

If M is compact, then the index of $(\not{\partial}_A \oplus \overline{\not{\partial}_A})^+ \colon \Gamma(\underline{\mathbb{C}}_M \oplus \underline{\mathbb{C}}_M) \to \Gamma(K_M^{-1} \oplus K_M)$ is calculated as

$$\begin{aligned}\operatorname{index}(\not{\partial}_A \oplus \overline{\not{\partial}_A})^+ &= \operatorname{index} \not{\partial}_A^+ + \operatorname{index}(\overline{\not{\partial}_A})^+ \\ &= \tfrac{1}{2}c_1(K_M^{-1})[M] - \tfrac{1}{2}c_1(K_M)[M] = c_1(K_M^{-1})[M] = \chi(M)\end{aligned}$$

(see [LaMi; M] for details). Here $c_1(K_M^{-1})$ and $\chi(M)$ denote the first Chern class of K_M^{-1} and the Euler characteristic of M, respectively. It should be pointed out that the index of $(\not{\partial}_A \oplus \overline{\not{\partial}_A})^+$ is not zero if M is not a torus. This implies the following interesting result. When $\chi(M)$ is positive (namely, $M = S^2$), the Dirac-type equation (1.2) always has a nonzero solution for any $\beta \in C^\infty(S^2)$. From Theorem A we shall obtain a Lagrangian conformal immersion of S^2 into $\mathbb{C}^2$ with Lagrangian angle β, provided S defined by (1.3) is not zero everywhere on S^2. On the other hand, if $\chi(M)$ is negative (i.e., if the genus of M is at least 2) then the Dirac-type equation (1.4) always has a nonzero solution for any $e^{\mathbf{i}\beta/2} \in C^\infty(M, S^1)$ (more generally, for any $e^{\mathbf{i}\beta} \in C^\infty(M, S^1)$). Then we can apply such a solution S corresponding to $e^{\mathbf{i}\beta} \in C^\infty(M, S^1)$ to the latter Weierstrass-type representation formula. If S is not zero everywhere on M then we obtain a complete Lagrangian immersion from the universal cover $\tilde{M}$ of M into $\mathbb{C}^2$ whose Lagrangian angle is the lifting $\tilde{\beta}$ of β.

The Dirac-type equation (1.4) is the integrability condition for the equation

$$df = \big(1/\sqrt{2}\big)e^{\mathbf{i}\beta/2}\{(-S_2, S_1)\,dz - \mathbf{i}(\overline{S_1}, \overline{S_2})\,d\bar{z}\}. \tag{1.5}$$

Using this fact, we can reproduce the Weierstrass-type representation formula proved in [HRo1] for Lagrangian surfaces in $\mathbb{C}^2$. This formula is a natural extension of the classical Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$.

On the other hand, we took advantage of (1.1) and (1.2) to represent Lagrangian surfaces in $\mathbb{C}^2$ without the procedure of integration.

THEOREM A [A1; A2] (Representation formula for inducing Lagrangian surfaces). *Given a* $(\mathbb{R}/4\pi\mathbb{Z})$*-valued smooth function* β *on* M, *let* $F = (F_1, F_2) \colon M \to \mathbb{C}^2$ *be a solution of the Dirac-type equation* (1.2) *and put*

$$f = \big(1/\sqrt{2}\big)e^{\mathbf{i}\beta/2}(F_1 - \mathbf{i}\overline{F_2}, F_2 + \mathbf{i}\overline{F_1}). \tag{1.6}$$

Assume that the pair of $(1,0)$*-forms* $(S_1\,dz, S_2\,dz)$ *defined by* (1.3) *is nonzero everywhere on* M. *Then the map* $f \colon M \to \mathbb{C}^2$ *is a Lagrangian conformal immersion with Lagrangian angle* β.

REMARK. In general, when β is given as a $(\mathbb{R}/2\pi\mathbb{Z})$-valued function on M, the maps $e^{\mathbf{i}\beta/2}$ and F are defined only on a suitable covering of M. However, if the right-hand side of (1.6) is well-defined on M then we can obtain a Lagrangian immersion f of M with Lagrangian angle β.

For a Lagrangian conformal immersion $f \colon M \to \mathbb{C}^2$ with Lagrangian angle β, the mean curvature vector $\vec{H}$ is given by $\vec{H} = \frac{1}{2}J\nabla\beta$. Then the Dirac-type equation (1.2) for a minimal Lagrangian surface is essentially the pair of Cauchy–Riemann equations. Hence, the representation formula (Theorem A) is a generalization of the following well-known result.

COROLLARY B [ChMo]. *Every minimal Lagrangian orientable surface in* $\mathbb{C}^2$ *can be represented as a holomorphic curve in* $\mathbb{C}^2$ *by exchanging the orthogonal complex structure on* $\mathbb{R}^4$.

Minimal Lagrangian surfaces in $\mathbb{C}^2$ that correspond to the holomorphic curve $(z, 1/z)$ in $\mathbb{C}^2$ are called the *Lagrangian catenoids* by Castro and Urbano in [CU2], where they proved that the only Lagrangian catenoids are minimal Lagrangian surfaces with circle symmetry in $\mathbb{C}^2$.

From now on, we consider only nonminimal Lagrangian surfaces in $\mathbb{C}^2$ whose Lagrangian angles are not constant.

2. Lagrangian Surfaces in $\mathbb{C}^2$ with Circle Symmetry

Both Dirac-type equations (1.2) and (1.4) are of the following type with potential term $p = p(z, \bar{z})$, provided we fix a coordinate z:

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \overline{\Phi_2} \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \overline{\Phi_2} \end{pmatrix}. \tag{2.1}$$

We recall that this type of equation with real-valued potential $p = p(z, \bar{z})$ appears in theory of surfaces in $\mathbb{R}^3$ as the integrability condition of the Weierstrass–Kenmotsu–Konopelchenko representation formula (see e.g. [KoT]). Hence we can obtain local solutions of (2.1) with real-valued potential $p = p(z, \bar{z})$ as adapted framings of conformal immersed surfaces in $\mathbb{R}^3$. In this section, we first review briefly this fact. Second, we mention helicoidal surfaces in $\mathbb{R}^3$ whose adapted framings satisfy equation (2.1) with real-valued potential p of one real variable $x = \mathrm{Re}(z)$. Finally, applying the solutions to the previous representation formula, we can explicitly give Lagrangian surfaces in $\mathbb{C}^2$, each with Lagrangian angle depending only on a real coordinate x and with periodicity for the other real coordinate, $y = \mathrm{Im}(z)$. In fact, we obtain a one-parameter family of Lagrangian surfaces with prescribed Lagrangian angle $\beta = \beta(x)$ and with circle symmetry in $\mathbb{C}^2$, where the one-parameter arises from the pitch of helicoidal surfaces.

2.1. Adapted Framings of Surfaces in $\mathbb{R}^3$

Here we identify $\mathbb{C}^2$ with the linear hull $\mathbb{R} \cdot \mathrm{SU}(2)$ of the special unitary group $\mathrm{SU}(2)$ by the map

$$\mathbf{x} = (x_1^{\mathbb{C}}, x_2^{\mathbb{C}}) = (x_1 + \mathbf{i}x_3, x_2 + \mathbf{i}x_4) \mapsto \underline{\mathbf{x}} = \begin{pmatrix} x_1^{\mathbb{C}} & -\overline{x_2^{\mathbb{C}}} \\ x_2^{\mathbb{C}} & \overline{x_1^{\mathbb{C}}} \end{pmatrix}.$$

Consider the Euclidean 3-space $\mathbb{R}^3$ as the real subspace in $\mathbb{C}^2$ defined by $x_1 \equiv 0$; then the standard basis of $\mathbb{R}^3$ consists of $\mathbf{e}_2 = (0, 1)$, $\mathbf{e}_3 = (\mathbf{i}, 0)$, and $\mathbf{e}_4 = (0, \mathbf{i})$. Now $\mathrm{SU}(2)$ acts on $\mathbb{R}^3$ as follows:

$$\mathtt{h} \cdot \mathbf{x} = \mathtt{h}\underline{\mathbf{x}}\mathtt{h}^*, \quad \mathtt{h} \in \mathrm{SU}(2), \ \mathbf{x} = (\mathbf{i}x_3, x_2 + \mathbf{i}x_4) \in \mathbb{R}^3 \subset \mathbb{C}^2.$$

It therefore acts on the orthonormal frame bundle over $\mathbb{R}^3$ transitively and isometrically. In fact, SU(2) is the double cover of the rotational group SO(3) of $\mathbb{R}^3$.

Let M be a Riemann surface and $z = x + \mathbf{i}y$ an isothermal coordinate on each neighborhood U of M. Let $\varphi\colon M \to \mathbb{R}^3$ be a conformal immersion with induced metric $\varphi^* d\mathrm{s}^2 = e^{2\lambda}|dz|^2$. Denote the unit normal vector field of the immersed surface by N; that is, $N\colon M \to S^2$ is the Gauss map. We denote by H_φ the mean curvature of φ.

Now we can choose uniquely the following map on U (up to ± 1-multiplication):

$$\underline{\Phi} = \begin{pmatrix} \Phi_1 & -\overline{\Phi_2} \\ \Phi_2 & \overline{\Phi_1} \end{pmatrix} \colon U \to 2e^{\lambda/2} \cdot \mathrm{SU}(2) \subset \mathbb{R} \cdot \mathrm{SU}(2) \ \ (\cong \mathbb{C}^2)$$

satisfying

$$\varphi_x = \tfrac{1}{4}\underline{\Phi} \cdot \mathbf{e}_4, \quad \varphi_y = \tfrac{1}{4}\underline{\Phi} \cdot \mathbf{e}_2, \quad N = \tfrac{1}{4}e^{\lambda}\underline{\Phi} \cdot \mathbf{e}_3.$$

Note that $\Phi_1\sqrt{dz}$ and $\Phi_2\sqrt{dz}$ are spinors of the plus spin bundle on U (see [AAk; KuS] for details). For simplicity, we call this map $\Phi = (\Phi_1, \Phi_2)\colon U \to \mathbb{C}^2$ the *adapted framing* of φ. The adapted framing Φ satisfies the Dirac equation (2.1) with real-valued potential $p = -\frac{1}{2}e^{\lambda}H_\varphi$. Moreover, this Dirac equation is just the integrability condition for the immersed surface φ in $\mathbb{R}^3$ (see e.g. [KoT; KuS]).

2.2. Helicoidal Surfaces in $\mathbb{R}^3$

Here we describe surfaces in $\mathbb{R}^3$ that are invariant under helicoidal motions with fixed axis and pitch (cf. [BiK; dCD; Ke2]). Any helicoidal surface with axis x_2 and pitch h_0 $(\in \mathbb{R})$ is locally parameterized as follows:

$$\varphi(r,\theta) = (\varphi_2, \varphi_3, \varphi_4) = (h(r) + h_0\theta, r\cos\theta, r\sin\theta), \quad r \neq 0.$$

When $h_0 = 0$, helicoidal surfaces are surfaces of revolution. We denote by $\mathcal{H}(h_0)$ the set of all isometry classes of such helicoidal surfaces with pitch h_0.

For any helicoidal surface $\varphi \in \mathcal{H}(h_0)$, we can take an orthogonal coordinate (s,t) such that the components of the first fundamental form and the mean curvature H_φ are functions of only one variable s: $(r,\theta) = (r(s), \theta(s,t))$. Conversely, the generating curve $(h(r), r)$ of any helicoidal surface $\varphi \in \mathcal{H}(h_0)$ can be determined in terms of the prescribed mean curvature function $H_\varphi(s)$. In fact, we can obtain the following result.

Proposition 1 (cf. [Ke1, Rem. 1]). *Suppose we are given a C^1-function $\beta(s)$ on an interval $I \subset \mathbb{R}$. Let $\mathcal{H}(h_0; \beta(s))$ be the subset in $\mathcal{H}(h_0)$ consisting of helicoidal surfaces $\varphi = \varphi(s,t)$ with mean curvature $H_\varphi = -\frac{1}{2}\beta'(s)$. Let $\mathcal{C}(\beta(s))$ be the set of all curves in $\mathbb{C} \setminus \{0\}$ parameterized by arc length $s \in I$ and with curvature $\beta'(s)$ (up to rotations around $0 \in \mathbb{C}$). Then there exists a bijection between the sets $\mathcal{H}(h_0; \beta(s))$ and $\mathcal{C}(\beta(s))$.*

The bijection, which maps a given curve

$$\gamma(s) = \left(\int \cos\beta(s)\,ds\right) + \mathbf{i}\left(\int \sin\beta(s)\,ds\right) \in \mathcal{C}(\beta(s))$$

to a helicoidal surface

$$\varphi(s,t) = \big(h(s) + h_0\theta(s,t), r(s)\cos\theta(s,t), r(s)\sin\theta(s,t)\big)$$

in $\mathcal{H}(h_0; \beta(s))$, is defined as follows. First, using the polar coordinate on $\mathbb{C}$, we represent $\gamma(s) \in \mathcal{C}(\beta(s))$ as

$$\gamma(s) = r(s)e^{\mathbf{i}\sigma(s)}.$$

Next, put

$$L(s) = \sqrt{r(s)^2 + h_0^2},$$
$$\iota(s) = \int \frac{1}{r(s)L(s)} \sin(\sigma(s) - \beta(s))\, ds.$$

Then the functions $h(s)$ and $\theta(s,t)$ of corresponding helicoidal surface are determined by

$$h(s) = \int \frac{L(s)}{r(s)} \sin(\sigma(s) - \beta(s))\, ds,$$
$$\theta(s,t) = t - h_0\iota(s).$$

The induced metric on the helicoidal surface $\varphi = \varphi(s,t)$ constructed as just shown is given by

$$\varphi^* d\mathrm{s}^2 = ds^2 + L(s)^2\, dt^2.$$

Here we can globally define an isothermal coordinate $z = x + \mathbf{i}y$ by $x = \int L(s)^{-1}\, ds$ and $y = t$, and then

$$\varphi^* d\mathrm{s}^2 = L^2|dz|^2.$$

Hence the adapted framing $\Phi = (\Phi_1(s,t), \Phi_2(s,t))$ of φ satisfies the Dirac equation (2.1) with potential $p = \frac{1}{4}L\beta'(s) = \frac{1}{2}\beta_z$, that is, equation (1.2) for $F = \Phi$:

$$\begin{aligned}
F_1 = \Phi_1(s,t) &= \sqrt{2(L(s)+h_0)}\sin\left(\frac{\beta(s) - \sigma(s) + \theta(s,t)}{2} + \frac{\pi}{4}\right)\\
&\quad - \mathbf{i}\sqrt{2(L(s)-h_0)}\sin\left(\frac{\beta(s) - \sigma(s) - \theta(s,t)}{2} + \frac{\pi}{4}\right);\\
F_2 = \Phi_2(s,t) &= -\sqrt{2(L(s)+h_0)}\cos\left(\frac{\beta(s) - \sigma(s) + \theta(s,t)}{2} + \frac{\pi}{4}\right)\\
&\quad - \mathbf{i}\sqrt{2(L(s)-h_0)}\cos\left(\frac{\beta(s) - \sigma(s) - \theta(s,t)}{2} + \frac{\pi}{4}\right).
\end{aligned}$$

2.3. One-Parameter Family of Lagrangian Surfaces in $\mathbb{C}^2$ with a Prescribed Lagrangian Angle Function on an Interval

For the preceding solution

$$F = (F_1, F_2) = (\Phi_1(s,t), \Phi_2(s,t))\colon I \times \mathbb{R}/4\pi\mathbb{Z} \to \mathbb{C}^2$$

of equation (1.2), we can define a map $S = (S_1(s,t), S_2(s,t))$ by (1.3). This map then satisfies

$$S = -(\mathbf{i}/2)F, \tag{2.2}$$

and hence $|S|^2 = |F|^2/4 = L \neq 0$. Then, applying this F to the representation formula, we obtain a Lagrangian conformal immersion $f(s,t)\colon I \times \mathbb{R}/4\pi\mathbb{Z} \to \mathbb{C}^2$ with prescribed Lagrangian angle $\beta(s)$ as follows. Put:

$$\eta(s) = \tan^{-1}\sqrt{\frac{L(s)+h_0}{L(s)-h_0}};$$

$$\theta_1(s) = \tfrac{1}{2}h_0\iota(s) + \eta(s), \qquad \theta_2(s) = \tfrac{1}{2}h_0\iota(s) - \eta(s).$$

Moreover, define a curve $\mathbf{p}(s)$ in $\mathbb{C}^2$ by

$$\mathbf{p}(s) = \sqrt{2L(s)}e^{\mathbf{i}(\sigma(s)/2-\pi/4)}(\cos\theta_1(s) - \mathbf{i}\sin\theta_2(s), -\sin\theta_1(s) - \mathbf{i}\cos\theta_2(s)).$$

Then

$$f(s,t) = \mathbf{p}(s)\cdot R(t),$$

where

$$R(t) = \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix}\colon \mathbb{R}/4\pi\mathbb{Z} \to \mathrm{SO}(2).$$

This Lagrangian surface in $\mathbb{C}^2$ is invariant under the action of the special orthogonal group SO(2) on $\mathbb{C}^2$, and hence it has a circle symmetry. The induced metric is given by

$$f^*\,d\mathrm{s}^2 = L(s)^{-1}\,ds^2 + L(s)\,dt^2 = L|dz|^2,$$

and hence the Gauss curvatures $K = K(s,t)$ of $f(s,t)$ and $K_\varphi = K_\varphi(s,t)$ of $\varphi(s,t)$ are related as

$$K = -2L^{-1}K_\varphi = 2L^{-2}L_{ss}. \tag{2.3}$$

It follows from (2.2) that the anti–self-dual part $\mathcal{G}_- = \mathcal{G}_-(s,t)$ of the generalized Gauss map of $f(s,t)$ coincides with the Gauss map $N = N(s,t)$ of $\varphi(s,t)$ up to isometries of S^2.

To sum up, the foregoing construction for f in terms of φ (or (h_0,β)) gives a correspondence from the set of helicoidal surfaces in $\mathbb{R}^3$ into the set of Lagrangian surfaces with circle symmetry in $\mathbb{C}^2$. To put it more precisely, we obtain the following main construction. Here we denote by f_{h_0} the Lagrangian immersion f constructed as before but for a fixed “pitch” h_0.

THEOREM 2. *Suppose we are given a C^1-function $\beta(s)$ on an interval $I \subset \mathbb{R}$ and a plane curve $\gamma(s) \in \mathcal{C}(\beta(s))$. Then there exists a one-parameter family $\{f_{h_0}\}_{h_0\in\mathbb{R}}$ of Lagrangian conformal immersions $f_{h_0}\colon I \times \mathbb{R}/4\pi\mathbb{Z} \to \mathbb{C}^2$ with Lagrangian angle $\beta(s)$. Moreover, if the curve $\gamma(s)$ is complete then each Lagrangian immersion f_{h_0} is also complete.*

REMARK 1. When $h_0 = 0$, the Lagrangian immersion f_0 so constructed can be rewritten as

$$f_0(s,t) = \tilde{\gamma}(s)(\cos(t/2), \sin(t/2)), \tag{2.4}$$

where $\tilde{\gamma}(s)$ is the curve in $\mathbb{C}\setminus\{0\}$ defined by $\tilde{\gamma}(s) = r(s)e^{\mathbf{i}\sigma(s)/2}$. It can also be obtained by applying the method of Ros and Urbano [RU], who took advantage of the Hopf fibration $S^{2n-1} \to \mathbb{C}P^{n-1}$ to provide examples of Lagrangian submanifolds in the complex n-space $\mathbb{C}^n$.

Lagrangian catenoids are obtained by applying Theorem 2 to straight lines in $\mathbb{C} \setminus \{0\}$. The family corresponds to the one-parameter family of minimal helicoidal surfaces in $\mathbb{R}^3$ associated to the catenoid.

3. Lagrangian Tori in $\mathbb{C}^2$ with Conformal Maslov Form

Let $f\colon M \to \mathbb{C}^2$ be a Lagrangian conformal immersion with Lagrangian angle β. We denote the induced metric by $f^* ds^2 = e^{2u}|dz|^2$ for each isothermal coordinate z defined locally on M. Suppose that the Maslov form $d\beta$ is conformal—namely, that $(e^{-2u}\beta_{\bar z}) f_z$ defines a complex vector field X on M globally. This implies that the anti–self-dual part $\mathcal{G}_-$ of the generalized Gauss map of f is a harmonic map to the unit 2-sphere. See [CU1], where Castro and Urbano classified compact orientable Lagrangian surfaces with conformal Maslov form in $\mathbb{C}^2$. In this case, M is either the sphere or a torus. The only Lagrangian sphere in $\mathbb{C}^2$ with conformal Maslov form is the well-known *Whitney sphere*. Every Lagrangian torus in $\mathbb{C}^2$ with conformal Maslov form is obtained from a solution of the sinh-Gordon equation $u'' + \sinh 4u = 0$. Each solution u gives the induced metric as well as the case of constant mean curvature surfaces of revolution in $\mathbb{R}^3$ (i.e., Delaunay surfaces).

We shall now proceed to give these Lagrangian surfaces more explicitly by applying the method of Section 2.

Let M be a torus $\mathbb{C}/\Gamma$ and $f\colon M \to \mathbb{C}^2$ a Lagrangian conformal immersion with conformal Maslov form. Let $z = x + \mathbf{i}y$ be the standard complex coordinate in $\mathbb{C}$. We can assume (as in [CU1]) that $X = -H_0 f_z$ for a positive real constant H_0. Then the Lagrangian angle β depends only on the variable x (and the isothermal coefficient e^{2u} also depends only on x). Now we can define maps $F = (F_1, F_2)\colon \mathbb{C} \to \mathbb{C}^2$ and $S = (S_1, S_2)\colon \mathbb{C} \to \mathbb{C}^2$ by (1.1) and (1.3), respectively. Then S (resp. F) is a solution of equation (2.1) with real-valued potential $p = p(x) = -\frac{1}{4}\beta_x = -\frac{1}{2}e^{2u(x)}H_0 = -\frac{1}{2}(|S_1|^2 + |S_2|^2)H_0$ $\big($resp. $p = p(x) = \frac{1}{4}\beta_x\big)$. This fact implies that there exists a conformal immersion $\varphi\colon \mathbb{C} \to \mathbb{R}^3$ whose adapted framing is $2\mathbf{i}S$ and whose mean curvature is constant H_0. It follows from the result in [KoT] that this φ is congruent to a constant mean curvature helicoidal surface. From the relation (2.2), F is regarded as the adapted framing of a helicoidal surface with constant mean curvature H_0. We remark that every such helicoidal surface with a fixed pitch h_0 corresponds to a unit-speed curve in $\mathbb{C} \setminus \{0\}$ of curvature $-2H_0$, namely, a round circle of radius $1/2H_0$.

For a round circle of radius $1/2H_0$ with center at $\mathbf{i}B(2H_0)^{-1} \in \mathbb{C}$, we may choose the following clockwise parameterization $\gamma = \gamma_B(s)$ by the arc length s:

$$\gamma_B(s) = \frac{1}{2H_0}[-\cos(2H_0 s) + \mathbf{i}\{\sin(2H_0 s) + B\}].$$

Put

$$\beta = \beta(s) = -2H_0 s + \pi/2;$$

then $\gamma_B(s) \in \mathcal{C}(\beta(s))$. Here we may assume that $B \geq 0$. For a pitch h_0 and a circle $\gamma_B(s) \in \mathcal{C}(\beta(s))$, we construct the helicoidal surface $\varphi(s,t) = \varphi_{h_0,B}(s,t)$ as in Section 2.2 and also the Lagrangian surface $f(s,t) = f_{h_0,B}(s,t)$ as in Section 2.3. Then $\varphi_{h_0,B}(s,t)$ has constant mean curvature and $f_{h_0,B}(s,t)$ is of conformal Maslov form. Here we note that

$$r(s) = r_B(s) = \frac{1}{2H_0}\sqrt{1 + 2B\sin(2H_0 s) + B^2},$$

$$\sin(\sigma(s) - \beta(s)) = \frac{1 + B\sin(2H_0 s)}{2H_0 r_B(s)} \quad \text{for } \sigma(s) = \sigma_B(s).$$

If $h_0 = 0$ then the constant mean curvature surfaces of revolution,

$$\varphi_{0,B}(s,t) = \left(h_{0,B}(s) = \int \frac{1 + B\sin(2H_0 s)}{2H_0 r_B(s)}\, ds,\ r_B(s)\cos t,\ r_B(s)\sin t \right),$$

are classified with respect to B as follows (cf. [E; Ke2]).

(1) If $B = 0$, then $\varphi_{0,0}(s,t)$ is a right circular cylinder.

(2) If $0 < B < 1$, then $\varphi_{0,B}(s,t)$ is an embedded Delaunay surface known as an *unduloid*; the embedded generating curve $(h_{0,B}(s), r_B(s))$ is obtained as the locus of one focus of an ellipse rolling along the axis.

(3) If $B = 1$, then $\varphi_{0,1}(s,t)$ $(-\pi/4H_0 \leq s \leq 3\pi/4H_0)$ is a sphere. (Note that this immersion is extendable to the poles corresponding to $r_B(s) = 0$.)

(4) If $B > 1$, then $\varphi_{0,B}(s,t)$ is a nonembedded Delaunay surface known as a *nodoid*; the nonembedded generating curve $(h_{0,B}(s), r_B(s))$ is obtained as the locus of the focus of an hyperbola rolling along the axis.

Then also the Lagrangian surfaces $f_{0,B}(s,t)$ with conformal Maslov form in $\mathbb{C}^2$ are accordingly classified into four types. In the representation (2.4) for $f_{0,B}(s,t)$, the curve $\tilde{\gamma}(s) = \tilde{\gamma}_B(s)$ in $\mathbb{C}$ is congruent to one of the Cassinian ovals depicted in Figure 1, where in each case the locus of points for which the product of the distances from two fixed points $\left(\pm\sqrt{B/2H_0}, 0\right)$ is a constant $1/2H_0$.

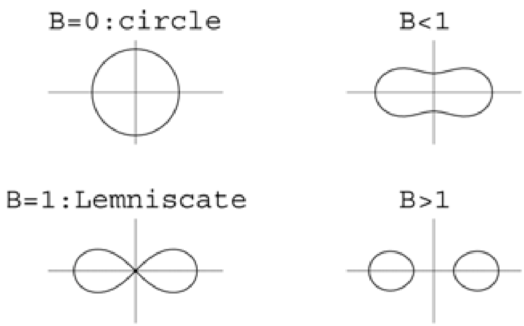

Figure 1 Cassinian ovals

If $B = 1$ then $f_{0,1}(s,t)$ is a Lagrangian immersion of a sphere known as the *Whitney sphere*. When $B \neq 1$, it is clear from Figure 1 that $f_{0,B}(s,t)$ defines a Lagrangian immersion of a torus.

In fact, if $B \neq 1$, then $f_{h_0,B}(s,t)$ (where h_0 need not be zero) has the following double periodicity:

$$f_{h_0,B}(s,t+4\pi) = f_{h_0,B}(s,t), \qquad f_{h_0,B}(s+a,t) = f_{h_0,B}(s,t-b),$$

where $a = 2\pi/H_0$ for $0 \leq B < 1$ and $a = \pi/H_0$ for $B > 1$ and where b is the constant defined by $b = h_0(\iota(s+a) - \iota(s))$ for $\iota(s) = \iota_{h_0,B}(s)$. Then we can obtain the following theorem.

THEOREM 3. *For a constant h_0 and a nonnegative constant $B \neq 0$, define the constants a and b as before. Let $\Gamma_{h_0,B}$ be the lattice generated by $\{(a,b),(0,4\pi)\}$. Then $f_{h_0,B}$ is a Lagrangian immersion with conformal Maslov form from the torus $\mathbb{C}/\Gamma_{h_0,B}$ into $\mathbb{C}^2$. Conversely, every Lagrangian immersion with conformal Maslov form from a torus into $\mathbb{C}^2$ is congruent to one of such $f_{h_0,B}$.*

REMARK 2. In [CU1] it is proved that all Lagrangian immersions with conformal Maslov form from tori into $\mathbb{C}^2$ are embedding.

REMARK 3. For $B = 0$, the image of each Lagrangian immersion $f_{h_0,0}$ of a torus is contained in a 3-dimensional round sphere S^3. This surface is obtained by lifting a closed curve on S^2 under the Hopf fibration $S^3 \to S^2$; that is, $f_{h_0,0}$ is a *Hopf torus* in S^3. In [W] it is proved that the only Hopf tori are compact Lagrangian surfaces lying in $S^3 \subset \mathbb{C}^2$. We note that every Hopf torus $f_{h_0,0}$ corresponds to a right circular cylinder in $\mathbb{R}^3$ parameterized by $\varphi_{h_0,0}$. In particular, $f_{0,0}$ is congruent to the *Clifford torus* in S^3 or its rectangular counterpart.

4. Hamiltonian Stationary Lagrangian Surfaces in $\mathbb{C}^2$

In this section we take up Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$. For every such surface, the self-dual part $\mathcal{G}_+ = e^{\mathbf{i}\beta}\colon M \to S^1 \subset S^2$ of the generalized Gauss map is a harmonic map to the unit 2-sphere. Namely, the Lagrangian angle $\beta\colon M \to \mathbb{R}/2\pi\mathbb{Z}$ satisfies $\beta_{z\bar{z}} = 0$, where z is an isothermal coordinate on M (cf. [HRo2]).

Let $f_{h_0}(s,t)\colon I \times \mathbb{R}/4\pi\mathbb{Z} \to \mathbb{C}^2$ be a Lagrangian immersion constructed via its Lagrangian angle $\beta(s)$ and a plane curve $\gamma(s) \in \mathcal{C}(\beta(s))$, as in Section 2. We assume that $f_{h_0}(s,t)$ is not minimal. It is easy to obtain the following result.

LEMMA 4.1. *The immersion $f_{h_0}(s,t)$ is Hamiltonian stationary if and only if the curvature $\kappa(s) = \beta'(s)$ of the plane curve $\gamma(s) = r(s)e^{\mathbf{i}\sigma(s)}$ is $a(r^2+h_0^2)^{-1/2} = a/L(s)$ for any nonzero constant a.*

We assume that $a > 0$ (and $h_0 \geq 0$). When $(h_0, a) = (0,1)$, the equation $\kappa = 1/r$ implies that γ is a round circle centered at the origin; that is, f_0 is the Clifford torus or its rectangular counterpart. This f_0 is also of conformal Maslov form and hence of parallel mean curvature vector. Now let $(h_0, a) \neq (0,1)$. By solving the ordinary differential equation of second order,

$$\kappa = \kappa\left(\sigma, r, \frac{dr}{d\sigma}, \frac{d^2 r}{d\sigma^2}\right) = \frac{a}{L} \quad \text{with } L = \sqrt{r^2 + h_0^2},$$

we obtain:

$$\frac{d\sigma}{dL} = \frac{aL}{(L + h_0)\sqrt{(L - h_0)\{(1 - a^2)L + (1 + a^2)h_0\}}},$$
$$\frac{ds}{dL} = \frac{L}{\sqrt{(L - h_0)\{(1 - a^2)L + (1 + a^2)h_0\}}}.$$

Case I: $h_0 = 0$. The curve γ is a logarithmic spiral:

$$\sigma = \left(a/\sqrt{1 - a^2}\right)\log cr \quad (L = r > 0),$$

where $0 < a < 1$ and c is any positive constant. Moreover, we can put $\beta = \sigma + c'$ for a constant $c' \in \mathbb{R} \setminus \pi\mathbb{Z}$.

Case II: $h_0 > 0$.

(a) For $0 < a < 1$, the curve γ is a spiral started from the origin with

$$\beta = \frac{2a}{\sqrt{1 - a^2}} \log\left(\sqrt{L - h_0} + \sqrt{L + a_1 h_0}\right),$$

$$\beta - \sigma = \sin^{-1}\frac{(2a^2 - 1)L - (2a^2 + 1)h_0}{L + h_0} + \text{const.},$$

where $L \geq h_0$ and $a_1 = (1 + a^2)/(1 - a^2) > 0$.

(b) For $a = 1$, γ is also a spiral started from the origin with

$$\beta = \frac{2}{\sqrt{2h_0}}\sqrt{L - h_0},$$

$$\beta - \sigma = \sin^{-1}\sqrt{\frac{L - h_0}{L + h_0}} + \text{const.},$$

where again $L \geq h_0$.

(c) For $a > 1$, γ is a curve of finite length with

$$\beta = \frac{a}{\sqrt{a^2 - 1}} \sin^{-1}\frac{(a^2 - 1)L - a^2 h_0}{h_0},$$

$$\beta - \sigma = \sin^{-1}\frac{(2a^2 - 1)L - (2a^2 + 1)h_0}{L + h_0} + \text{const.},$$

where $a_2 h_0 \geq L \geq h_0$ and $a_2 = (a^2 + 1)/(a^2 - 1) > 1$.

Put $I' = (h_0, \infty) \subset \mathbb{R}$ for $0 < a \leq 1$ or $I' = (h_0, a_2 h)$ for $a > 1$. Applying the construction in Section 2 to each curve $\gamma = \gamma_{h_0,a}$ just described on I', we have the following.

PROPOSITION 4. *For each curve $\gamma = \gamma_{h_0,a}$, the Lagrangian immersion*

$$f_{h_0} = f_{h_0,a}(L,t) \colon I' \times \mathbb{R}/4\pi\mathbb{Z} \to \mathbb{C}^2$$

is Hamiltonian stationary. The induced metric is given by

$$f^*_{h_0,a}\, d\mathrm{s}^2 = \frac{L}{(L-h_0)\{(1-a^2)L + (1+a^2)h_0\}}\, dL^2 + L\, dt^2.$$

REMARK 4. When $h_0 = 0$, the Hamiltonian stationary Lagrangian surface $f_{0,a}$ $(0 < a < 1)$ is given by

$$f_{0,a}(r,t) = \tilde{\gamma}_a(r)(\cos(t/2), \sin(t/2)), \quad r > 0,\ t \in \mathbb{R}/4\pi,$$

where $r = L$ and $\tilde{\gamma}_a = re^{\mathbf{i}\sigma/2}$ is a logarithmic spiral. The corresponding surface of revolution $\varphi = \varphi_{0,a}$ is generated by a half-line $(h(r), r) = (cr, r)$ (here c is a nonzero constant); that is, $\varphi_{0,a}$ is a right circular cone in $\mathbb{R}^3$. Because such cones are flat, it follows from (2.3) that the Hamiltonian stationary Lagrangian surface $f_{0,a}$ is flat.

REMARK 5. Every Hamiltonian stationary Lagrangian surface $f_{h_0,a}$ constructed in this manner is neither compact nor complete. On the other hand, the following immersion gives a simple example of a complete Hamiltonian stationary Lagrangian cylinder:

$$f(s,t) = (\mathbf{i}s, e^{\mathbf{i}t}) \colon \mathbb{R} \times S^1 \to \mathbb{R}^3 \subset \mathbb{C}^2.$$

Its Lagrangian angle is given by $\beta = \beta(t) = t + \pi/2$. This example contrasts with the previous $f_{h_0,a}(s,t) \colon I \times S^1 \to \mathbb{C}^2$, whose Lagrangian angle depends only on s.

HAMILTONIAN STATIONARY LAGRANGIAN TORI. Hélein and Romon [HRo1; HRo2] have given explicit parameterizations of all Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2$. Applying their idea to the representation formula in Section 1, we can reproduce the parameterizations. Every harmonic map β of a given torus $\mathbb{C}/\Gamma$ to $\mathbb{R}/2\pi\mathbb{Z}$ is described as the form $\beta(z) = 2\pi\langle\beta_0, z - z_0\rangle$ for an element β_0 of the dual lattice Γ^* to Γ. Here z denotes the standard complex coordinate in $\mathbb{C}^2$. Hence, any solution $F = (F_1, F_2)$ of the Dirac-type equation (1.2) satisfies $-\triangle F_j = \pi^2|\beta_0|^2 F_j$ on $\mathbb{C}$ $(j = 1, 2)$, where $\triangle = 4\partial_z\partial_{\bar z}$. If F_1 and F_2 are 2Γ-periodic (i.e., $\mathbb{C}$-valued eigenfunctions on $\mathbb{C}/2\Gamma$ of $-\triangle$), then the map f given by (1.6) is well-defined on $\mathbb{C}/2\Gamma$. Among such maps f, we look for immersions with Γ-periodicity. Then every Hamiltonian stationary Lagrangian conformal immersion of the torus $\mathbb{C}/\Gamma$ is parameterized in terms of $\beta_0 \in \Gamma^* \setminus \{0\}$ as follows:

$$f(z) = \sum_{\delta \in \Gamma^*_{\beta_0}} e^{2\pi\mathbf{i}\langle\delta + \beta_0/2,\, z - z_0\rangle} \left(C_\delta \left(1 - \frac{2\delta}{\beta_0}\right), \mathbf{i}\overline{C_{-\delta}} \left(1 - \frac{\beta_0}{2\delta}\right) \right),$$

where $\Gamma^*_{\beta_0} = \{\delta \in \beta_0/2 + \Gamma^* \mid |\delta| = |\beta_0|/2,\ \delta^2 \neq \beta_0^2/4\}$ and where not all constants C_δ $(\in \mathbb{C})$ are zero.

5. A Note on Flat Lagrangian Surfaces in $\mathbb{C}^2$

In Sections 3 and 4, we referred to typical examples of flat Lagrangian surfaces. Indeed, in Remark 4 we constructed flat Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$ from (the adapted framings of) the right circular cones in $\mathbb{R}^3$. In Remark 3 we noted that the Hopf tori in $S^3 \subset \mathbb{C}^2$ correspond to the right circular cylinder, and hence they are flat Lagrangian surfaces with conformal Maslov form. In this last section, we comment on the other flat Lagrangian surfaces with circle symmetry.

From (2.3), the flatness for a Lagrangian immersion f_{h_0} $(h_0 > 0)$ constructed in Section 2 implies that the corresponding plane curve $\gamma = re^{\mathbf{i}\sigma}$ is parameterized by arc length s as

$$r = \sqrt{L^2 - h_0^2} = \sqrt{(as+b)^2 - h_0^2}$$

for some constants $0 < a < 1$ and b. By suitably changing the parameter of the flat Lagrangian surface f_{h_0}, we reparameterize it as follows:

$$f_{h_0}(u,v) = \mathbf{p}(u) \cdot R(v) \colon \mathbb{R} \times \mathbb{R}/4\pi\mathbb{Z} \to \mathbb{C}^2;$$

$$\mathbf{p}(u) := \sqrt{2L(u)}\,(e^{\mathbf{i}(\eta(u)-\psi(u))}, e^{-\mathbf{i}(\eta(u)+\psi(u))}),$$

$$L(u) := \sqrt{\frac{u^2+h_0^2}{1-a^2}}, \qquad \eta(u) := \tan^{-1}\sqrt{\frac{L(u)+h_0}{L(u)-h_0}},$$

$$\begin{aligned}\psi(u) := {}& \frac{1}{2a}\sqrt{1-a^2}\log\bigl[2\sqrt{1-a^2}\bigl(\sqrt{1-a^2}L(u)+u\bigr)\bigr] \\ & -\frac{1}{4}\operatorname{sgn}(u)\cos^{-1}\left[\frac{(2a^2-1)u^2+a^2h_0^2}{u^2+a^2h_0^2}\right].\end{aligned}$$

It is known that every flat surface in $\mathbb{R}^3$ is parameterized as a developable surface and has singularities—except for planes and right circular cylinders. Namely, the flat helicoidal surface φ_{h_0} corresponding to each foregoing curve γ has singularities. Likewise, the flat Lagrangian immersion $f_{h_0}(u,v)$ is degenerate at $u = 0$.

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