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Extremal Elliptic Surfaces and Infinitesimal Torelli

Remke Kloosterman

1. Introduction

An extremal elliptic surface over $\mathbb{C}$ is an elliptic surface such that the rank of the Néron–Severi group equals $h^{1,1}$ and there are finitely many sections.

Their main application, until now, is in the classification of singular fibers on certain elliptic surfaces: Once a configuration of singular fibers on an extremal elliptic surface is known, one can construct from this configuration many other configurations of singular fibers on elliptic surfaces, where the genus of the base curve and the geometric and arithmetic genus of the surface remain fixed. In [S] these operations are called elementary transformations and are, a priori, valid only for $K3$ surfaces. Actually, all elementary transformations are combinations of twisting and deformations of the J-map (terminology from [M1, VIII.2; M2]) and hence are valid for any elliptic surface.

The classification of singular fibers on a rational elliptic surface was given more than ten years ago (see [M2; OSh; P]). Recently there has been given a classification of all singular fibers of elliptic $K3$ surfaces with a section (see [S]). From the classification of configurations of singular fibers on rational surfaces (see [M2]) and $K3$ surfaces (see [S]), we know that any configuration can be obtained from an extremal configuration using elementary transformations.

In this paper we give a complete classification of extremal elliptic surfaces with constant j-invariant (Theorem 3.3). We use this classification to prove the following.

Theorem 1.1. *Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface without multiple fibers. Assume that $p_g(X) > 1$. Then X does not satisfy infinitesimal Torelli if and only if $j(\pi)$ is constant and π is extremal.*

Kiĭ [K, Thm. 2] proved infinitesimal Torelli for elliptic surfaces without multiple fibers and nonconstant j-invariant. Saitō [Sa] proved in a different way infinitesimal Torelli for elliptic surfaces without multiple fibers and j-invariant different from 0 and 1728.

For elliptic surfaces with nonconstant j-invariant, we will give the following structure theorem.

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THEOREM 1.2. *Suppose $\pi : X \to C$ is an elliptic surface without multiple fibers and nonconstant j-invariant. Then the following statements are equivalent.*

(1) *π is extremal.*
(2) *$j(\pi)$ is not ramified outside* $0, 1728, \infty$; *the only possible ramification indices above* 0 *are* $1, 2, 3$ *and above* 1728 *are* $1, 2$; *and π has no fibers of type II, III, IV, or I_0^*.*
(3) *There exists an elliptic surface with $\pi' \colon X' \to C$ with $j(\pi') = j(\pi)$; the fibration π' has no fibers of type II^*, III^*, or IV^* and at most one fiber of type I_0^*; and π' has precisely $2p_g(X) + 4 - 4g(C)$ singular fibers.*

We will give a similar theorem that includes the Mordell–Weil group. Let $m, n \in \mathbb{Z}_{\geq 1}$ be such that $m|n$ and $n > 1$. Let $X_m(n)$ be the modular curve parameterizing triples $((E, O), P, Q)$, such that (E, O) is an elliptic curve, $P \in E$ is a point of order m, and $Q \in E$ is a point of order n.

If $(m, n) \notin \{(1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\}$ then there exists a universal family for $X_m(n)$, which we denote by $E_m(n)$. Denote by $j_{m,n} \colon X_m(n) \to \mathbb{P}^1$ the map usually called j.

From the results of [Sh1, Secs. 4 & 5] it follows that $E_m(n)$ is an extremal elliptic surface. The following theorem explains how to construct many examples of extremal elliptic surfaces with a given torsion group.

THEOREM 1.3. *Fix $m, n \in \mathbb{Z}_{\geq 1}$ such that $m|n$ and $(m, n) \notin \{(1, 1), (1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\}$, and let $j \in \mathbb{C}(C)$ be nonconstant. Then there exists a unique elliptic surface $\pi : X \to C$ with $j(\pi) = j$, and $MW(\pi)$ has $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ as a subgroup if and only if* (a) *j is of* $(3, 2)$*-type, not ramified outside* $0, 1728, \infty$, *and* (b) *$j = j_{n,m} \circ g$ for some $g \colon C \to X_m(n)$.*

The extremal elliptic surface with $j(\pi) = j$ and $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ is a subgroup of $MW(\pi)$, which is the unique surface with only singular fibers of type I_ν.

If $\pi : X \to \mathbb{P}^1$ is an extremal semistable rational elliptic surface, then X is determined by the configuration of singular fibers (see [MP, Thm. 5.4]). It seems that this statement applies only to *rational* elliptic surfaces, for if X is a $K3$ surface then a similar statement does not hold.

THEOREM 1.4. *There exist pairs of extremal semistable elliptic $K3$ surfaces, $\pi_i \colon X_i \to \mathbb{P}^1$ $(i = 1, 2)$, such that $MW(\pi_1) \cong MW(\pi_2)$, the configuration of singular fibers of the π_i coincide, and X_1 and X_2 are nonisomorphic.*

This gives a negative answer to [ATZ, Question 0.2]. The essential ingredient for the proof comes from [SZ, Table 2].

The paper is organized as follows. Section 2 contains some definitions and several standard facts. In Section 3 we give a list of extremal elliptic surfaces with constant j-invariant, surfaces that behave differently than the nonconstant ones. There are exactly five infinite families of extremal elliptic surfaces with constant

j-invariant (three of dimension 1, one of dimension 2, and one of dimension 3). In Section 4 we explain this different behavior by proving Theorem 1.1.

In Section 5 we explain how twisting can reduce the problem of classification. In Section 6 we link the ramification of the j-map and the number of singular fibers of a certain elliptic surface. This, combined with the results of Section 5, gives a proof of Theorem 1.2. Section 7 contains a proof of the version with the description of the group of sections (Theorem 1.3), and in Section 8 we prove Theorem 1.4. In Section 9 we give a classification of extremal elliptic surfaces with $g(C) = p_g(X) = q(X) = 1$. Section 10 contains a proof of the fact that, for any positive k, there exist elliptic surfaces with only one singular fiber; the singular fiber must be of type I_{12k} or I_{12k-6}^*, and both occur.

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2. Preliminaries and Conventions

Assumption 2.1. By a *curve* we mean a nonsingular projective complex curve. By a *surface* we mean a nonsingular projective complex surface.

Definition 2.2. An *elliptic surface* is a triple (π, X, C) with X a surface, C a curve, and π a morphism $X \to C$ such that almost all fibers are irreducible genus-1 curves and X is relatively minimal. We denote by $j(\pi)\colon C \to \mathbb{P}^1$ the rational function such that $j(\pi)(P)$ equals the j-invariant of $\pi^{-1}(P)$ whenever $\pi^{-1}(P)$ is nonsingular.

A *Jacobian elliptic surface* is an elliptic surface together with a section $\sigma_0\colon C \to X$ to π. The set of sections of π is an abelian group, with σ_0 as the identity element. Denote this group by $MW(\pi)$.

An *extremal elliptic surface* is an elliptic surface such that $\rho(X) = h^{1,1}(X)$ and $MW(\pi)$ is empty or finite. Let $\mathbb{L}$ be the fundamental line bundle $[R^1\pi_*\mathcal{O}_X]^{-1}$.

Assumption 2.3. All elliptic surface are without multiple fibers.

Remark 2.4. To an elliptic surface $\pi: X \to C$ we can associate its Jacobian fibration $\mathrm{Jac}(\pi)\colon \mathrm{Jac}(X) \to C$. The Hodge numbers $h^{p,q}$, the Picard number $\rho(X)$, the type of singular fibers of π, and $\deg(\mathbb{L})$ are all invariant under taking the Jacobian fibration.

Remark 2.5. If P is a point on C such that $\pi^{-1}(P)$ is singular, then $j(\pi)(P)$ behaves as follows.

Kodaira type of fiber over P	$j(\pi)(P)$
I_0^*	$\neq\infty$
I_ν, I_ν^* $(\nu > 0)$	∞
II, IV, IV^*, II^*	0
III, III^*	1728

DEFINITION 2.6. Let X be a surface and let C and C_1 be curves. Let $\varphi\colon X \to C$ and $f\colon C_1 \to C$ be two morphisms. Then we denote by $X \times_C C_1$ the smooth, relatively minimal model of the ordinary fiber product of X and C_1.

We use the line bundle $\mathbb{L}$ only to simplify notation. Note that $\deg(\mathbb{L}) = p_g(X) + 1 - g(C) = p_a(X) + 1$. (See [M1, Lemma IV.1.1].)

Recall the following theorem.

THEOREM 2.7 [Sh2, Thm. 1.3 & Cor. 5.3]. *Let $\pi : X \to C$ be a Jacobian elliptic surface such that* $\deg(\mathbb{L}) > 0$. *Then the Néron–Severi group of X is generated by the classes of $\sigma_0(C)$, a fiber, the components of the singular fibers not intersecting $\sigma_0(C)$, and the generators of the Mordell–Weil group. Moreover, let S be the set of points such that $\pi^{-1}(P)$ is singular and let $m(P)$ be the number of irreducible components of $\pi^{-1}(P)$. Then*

$$\rho(X) := \operatorname{rank}(NS(X)) = 2 + \sum_{P\in S}(m(P) - 1) + \operatorname{rank}(MW(\pi)).$$

DEFINITION 2.8. Suppose $\pi : X \to C$ is an elliptic fibration. Denote by $\Lambda(\mathrm{Jac}(\pi))$ the subgroup of the Néron–Severi group of $\mathrm{Jac}(\pi)$ generated by the classes of $\sigma_0(C)$, a fiber, and the components of the singular fibers not intersecting $\sigma_0(C)$. Let $\rho_{tr}(\pi) = \operatorname{rank}\Lambda(\mathrm{Jac}(\pi))$.

Note that if $\pi : X \to \mathbb{P}^1$ has $\deg(\mathbb{L}) = 0$ then $\rho_{tr} = 2$, although Theorem 2.7 does not apply.

DEFINITION 2.9. Let $\pi : X \to C$ be an elliptic surface. Define:

(a) $a(\pi)$ as the number of fibers of type II^*, III^*, IV^*;
(b) $b(\pi)$ as the number of fibers of type II, III, IV;
(c) $c(\pi)$ as the number of fibers of type I_0^*;
(d) $d(\pi)$ as the number of fibers of type I_ν^* with $\nu > 0$; and
(e) $e(\pi)$ as the number of fibers of type I_ν, $\nu > 0$.

DEFINITION 2.10. Let $\pi : X \to C$ be an elliptic surface and let $P \in C(\mathbb{C})$. Define $v_P(\Delta_P)$ as the valuation at P of the minimal discriminant of the Kodaira–Néron model, which equals the topological Euler characteristic of $\pi^{-1}(P)$.

PROPOSITION 2.11. *Let $\pi : X \to C$ be an elliptic surface. Then*

$$\sum_{P\in C(\mathbb{C})} v_P(\Delta_P) = 12\deg(\mathbb{L}).$$

Proof. This follows from Noether's formula (see [M1, III.4.4]). □

PROPOSITION 2.12. *For any elliptic surface $\pi : X \to C$ that is not a product, we have*

$$h^{1,1}(X)-\rho_{tr}(\pi) = 2(a(\pi)+b(\pi)+c(\pi)+d(\pi))+e(\pi)-2\deg(\mathbb{L})-2+2g(C).$$

Proof. Recall from [Mi, Lemma IV.1.1] that

$$h^{1,1} = 10\deg(\mathbb{L}) + 2g(C).$$

From Kodaira's classification of singular fibers and Proposition 2.11 it follows that

$$\rho_{tr}(\pi) = 2 + 12\deg(\mathbb{L}) - 2(a(\pi) + b(\pi) + c(\pi) + d(\pi)) - e(\pi).$$

Combining these finishes the proof. □

COROLLARY 2.13. *Let $\pi : X \to C$ be an elliptic surface (not a product) with constant j-invariant. Then π is extremal if and only if π has* $\deg(\mathbb{L}) + 1 - g(C)$ *singular fibers.*

Proof. If j is constant, then $e(\pi) = d(\pi) = 0$. □

3. Constant j-Invariant

In this section we give a list of all extremal elliptic surfaces with constant j-invariant.

LEMMA 3.1. *Suppose $\pi : X \to C$ is an extremal elliptic surface such that $j(\pi)$ is constant. Then $g(C) \le 1$.*

Proof. If $j(\pi)$ is constant then $v_P(\Delta_P) \le 10$ for every point P. Hence, by Proposition 2.11 and Corollary 2.13 it follows that

$$12\deg(\mathbb{L}) = \sum_{P|\pi^{-1}(P)\text{ singular}} v_P(\Delta_p) \le 10(\deg(\mathbb{L}) + 1 - g(C)),$$

and from this it follows that $g(C) \le 1$. □

LEMMA 3.2. *Suppose $\pi : X \to C$ is an extremal elliptic surface such that $j(\pi)$ is constant. Then one of the following occurs.*

(1) $g(C) = 1$; $\deg(\mathbb{L}) = 0$, *and X is a hyperelliptic surface.*
(2) $g(C) = 0$; $j(\pi) \ne 0, 1728$; $\deg(\mathbb{L}) = 1$.
(3) $g(C) = 0$; $j(\pi) = 0$; $1 \le \deg(\mathbb{L}) \le 5$.
(4) $g(C) = 0$; $j(\pi) = 1728$; $1 \le \deg(\mathbb{L}) \le 3$.

Proof. Suppose $g(C) = 1$. Then π has $\deg(\mathbb{L})$ singular fibers, so

$$12\deg(\mathbb{L}) = \sum v_P(\Delta_P) \le 10\deg(\mathbb{L}).$$

Therefore, $\deg(\mathbb{L}) = 0$ and so X is a hyperelliptic surface.

Suppose $g(C) = 0$. If $\deg(\mathbb{L}) = 0$ then π is birational to a projection from a product, so by definition π is not extremal.

Suppose $\deg(\mathbb{L}) > 0$. Assume $j(\pi) \neq 0, 1728$. Then all singular fibers are of type I_0^*. Since the Euler characteristic of such a fiber is 6, Proposition 2.11 implies that there are exactly $2\deg(\mathbb{L})$ singular fibers. Applying Corollary 2.13 yields

$$2\deg(\mathbb{L}) = \deg(\mathbb{L}) + 1,$$

and from this we know that $\deg(\mathbb{L}) = 1$.

Assume $j(\pi) = 1728$; then all singular fibers are of type III, I_0^*, III^*. From this it follows that $v_P(\Delta_P) \leq 9$. By Proposition 2.11 and Proposition 2.12 we have

$$12\deg(\mathbb{L}) \leq 9(a(\pi) + b(\pi) + c(\pi)) = 9\deg(\mathbb{L}) + 9,$$

so $1 \leq \deg(\mathbb{L}) \leq 3$.

If we assume $j(\pi) = 0$ then similarly we obtain $1 \leq \deg(\mathbb{L}) \leq 5$. □

THEOREM 3.3. *Suppose $\pi : X \to C$ is an elliptic surface with $j(\pi)$ constant. Then π is extremal if and only if either* (a) *C is a curve of genus* 1 *and* $\mathrm{Jac}(X)$ *is a hyperelliptic surface or* (b) $C \cong \mathbb{P}^1$ *and* $\mathrm{Jac}(\pi)$ *has a model that is isomorphic to one of* (i), (ii), *or* (iii) *as follows.*

(i) $(j(\pi) = 0)$ $y^2 = x^3 + f(t)$, *where $f(t)$ comes from the following table* (*the left-hand side indicates the positions of the singular fibers, and $\alpha, \beta, \gamma \in \mathbb{C} - \{0, 1\}$ are pairwise distinct*).

II	IV	I_0^*	IV^*	II^*	p_g	$f(t)$
0				∞	0	t
	0		∞		0	t^2
		$0, \infty$			0	t^3
	1			$0, \infty$	1	$t^5(t-1)^2$
		1	0	∞	1	$t^4(t-1)^3$
			$0, 1, \infty$		1	$t^4(t-1)^4$
		α		$0, 1, \infty$	2	$t^5(t-1)^5(t-\alpha)^3$
			$\alpha, 1$	$0, \infty$	2	$t^5(t-1)^4(t-\alpha)^4$
			β	$\alpha, 0, 1, \infty$	3	$t^5(t-1)^5(t-\alpha)^5(t-\beta)^4$
				$0, 1, \infty, \alpha, \beta, \gamma$	4	$t^5(t-1)^5(t-\alpha)^5(t-\beta)^5(t-\gamma)^5$

(ii) $(j(\pi) = 1728)$ $y^2 = x^3 + g(t)x$, *where $g(t)$ comes from the following table* $(\alpha \neq 0, 1)$.

III	I_0^*	III^*	p_g	$g(t)$
0		∞	0	t
	$0, \infty$		0	t^2
	1	$0, \infty$	1	$t^3(t-1)^2$
		$0, 1, \infty, \alpha$	2	$t^3(t-1)^3(t-\alpha)^3$

(iii) $(j(\pi) \neq 1728)$ $y^2 = x^3 + at^2x + t^3$, *with singular fibers of type I_0^* at $t = 0$ and $t = \infty$.*

Proof. The three cases listed follow directly from Corollary 2.13 and Lemma 3.2. Since all are very similar, we discuss only the case $j(\pi) = 0$ and $p_g = 2$. In this case, $\mathrm{Jac}(\pi)$ has a model isomorphic to

$$y^2 = x^3 + f(t),$$

with f a polynomial such that $13 \leq \deg(f) \leq 18$, and $v_P(f) \leq 5$ for all finite P. At all zeros of f there is a singular fiber. If $\deg(f) < 18$ then the fiber over $t = \infty$ is also singular.

If π is extremal then from Corollary 2.13 it follows that π has exactly four singular fibers. Assume that the fibers with the highest Euler characteristic are over $t = \infty, 0, 1$. Since $5+5+5+3 = 5+5+4+4$ are the only two ways of writing 18 as a sum of four positive integers smaller than 6, we obtain (after applying an isomorphism, if necessary) that f equals either

$$t^5(t-1)^5(t-\alpha)^3 \text{ or } t^5(t-1)^4(t-\alpha)^4. \qquad \square$$

REMARK 3.4. Note that all extremal elliptic surfaces with constant j-invariant and $p_g(X) > 1$ have moduli.

4. Infinitesimal Torelli

In the previous section we gave examples of families of elliptic surfaces with maximal Picard number. In this section we prove that these surfaces are counterexamples to infinitesimal Torelli. Moreover, we give a complete solution for infinitesimal Torelli for elliptic surfaces (with a section) over $\mathbb{P}^1$.

Suppose X is a smooth complex algebraic variety. Then the first-order deformations of X are parameterized by $H^1(X, \Theta_X)$, with Θ_X the tangent bundle of X. The isomorphism $H^{p,q}(X) = H^q(X, \Omega^p)$ and the contraction map $\Theta_X \otimes_{\mathcal{O}_X} \Omega_X^p \to \Omega_X^{p-1}$ give a cup product map,

$$H^1(X, \Theta_X) \otimes H^{p,q}(X) \to H^{p-1,q+1}(X),$$

and from this one obtains the infinitesimal period map

$$\delta_k\colon H^1(X, \Theta_X) \to \bigoplus_{p+q=k} \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)).$$

The (holomorphic) map δ_k is closely related to the period map. Assume that $\varphi\colon \mathcal{X} \to \mathcal{B}$ is a proper, smooth, surjective holomorphic map between complex manifolds with connected fibers and that, for all $t \in \mathcal{B}$, the vector space $H^k(X_t, \mathbb{C})$ carries a Hodge structure of weight k with $X_t := \varphi^{-1}(t)$. Fix a point $0 \in \mathcal{B}$, and let U be a small simply connected open neighborhood of 0 such that the Kodaira–Spencer at the point 0 is injective. Define

$$\mathcal{P}^{p,k}\colon U \to \mathrm{Grass}\left(\sum_{i \geq p} h^{i,k-i}, H^k(X_0)\right)$$

by sending t to

$$\left(\bigoplus_{i\geq p} H^{i,k-i}(X_t)\right) \subset H^k(X_0,\mathbb{C})$$

via the identification $H^k(X_0,\mathbb{C}) \cong H^k(X_t,\mathbb{C})$. (Note that U is simply connected.) Then the differential of $\bigoplus_d \mathcal{P}^{p,k}$ is injective if and only if δ_k is injective. (See [Sa, Sec. 2] or [V, Chap. 10].)

We say that X satisfies infinitesimal Torelli if and only if $\delta_{\dim(X)}$ is injective. Note that, if X is an elliptic surface with a section and base $\mathbb{P}^1$, then δ_k ($k \neq \dim X$) is the zero-map.

If X is a rational surface, then there is no variation of Hodge structures. If X is a $K3$ surface, then infinitesimal Torelli follows from [PjS]. This section will focus on the case $p_g(X) > 1$.

There is an easy sufficient condition (see [K; LWP]) for checking infinitesimal Torelli for manifolds with divisible canonical bundle. The following result is a direct consequence of [LWP, Thm. 1′].

THEOREM 4.1. *Let X be a compact Kähler n-manifold with $p_g(X) > 1$. Let $\mathcal{L}$ be a line bundle such that*

(1) $\mathcal{L}^{\otimes k} = \Omega_X^n$ *for some* $k > 0$;
(2) *the linear system corresponding to $\mathcal{L}$ has no fixed components of codimension* 1; *and*
(3) $H^0(X, \Omega_X^{n-1} \otimes \mathcal{L}) = 0$.

Then δ_n is injective.

We want to apply the above theorem when X is an elliptic surface and $\mathcal{L} = \mathcal{O}_X(F)$.

LEMMA 4.2. *Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface. Assume that X is not birational to a product $C \times \mathbb{P}^1$. Then, for $n > 0$,*

$$\dim H^0(X,\Omega^1(nF)) = \begin{cases} n-1 & \text{if } j(\pi) \text{ is not constant,}\\ n-1+\max(0,n+d+1) & \text{if } j(\pi) \text{ is constant,}\end{cases}$$

where $d = \deg(\mathbb{L}) - \#\{P \in C(\mathbb{C}) \mid \pi^{-1}(P) \text{ singular}\}$.

Proof. By [Sa, Prop. 4.4(I)] we know that $\pi_*\Omega_X^1 \cong \Omega_{\mathbb{P}^1}^1$ if $j(\pi)$ is not constant, which gives the first case.

If $j(\pi)$ is constant then we have the following exact sequence (by [Sa, Prop. 4.4(II)]):

$$0 \to \Omega^1_{\mathbb{P}^1} \to \pi_*\Omega^1_X \to \mathcal{O}_{\mathbb{P}^1}(d) \to 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}^1}(n)$ gives

$$\begin{aligned}\dim H^0(X,\Omega^1_X(nF)) &= \dim H^0(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}(n)) + \dim H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(n+d))\\ &= n-1+\max(0,n+d+1),\end{aligned}$$

using that $\dim H^1(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}(n)) = 0$. □

COROLLARY 4.3. *Suppose $\pi : X \to \mathbb{P}^1$ is an elliptic surface (cf. Assumption 2.3) such that $p_g(X) > 1$, and suppose that $j(\pi)$ is nonconstant or π is not extremal. Then X satisfies infinitesimal Torelli.*

Proof. Let F be a smooth fiber of π. Note that $\mathcal{O}(F)^{\otimes(p_g(X)-1)} = \Omega^2_X$ (see [Sh2, Thm. 2.8]) and $|F|$ is the elliptic fibration, hence without base-points.

We claim that $\dim H^0(X, \Omega^1(F)) = 0$. If this is not the case, then by Lemma 4.2 $j(\pi)$ is constant and π has at most $\deg(\mathbb{L}) + 1$ singular fibers. From Lemma 2.13 it follows that $j(\pi)$ is extremal.

Apply now Theorem 4.1 with $\mathcal{L} = \mathcal{O}(F)$. $\square$

PROPOSITION 4.4. *Let $\varphi\colon \mathcal{X} \to \mathcal{B}$ be a family of surfaces with $p_g(X_0) > 0$ and such that, for all $t \in \mathcal{B}$, the set $\{t' \mid X_t \cong X_{t'}\}$ is zero-dimensional. Let r be the Picard number of a generic member of the family φ. Suppose that either $p_g(X_0) > 1$ and*

$$\dim \mathcal{B} \geq \tfrac{1}{2} p_g(X)(h^{1,1} - r)$$

or $p_g(X_0) = 1$ and

$$\dim \mathcal{B} \geq (h^{1,1} - r).$$

Then, for all t, the surface X_t does not satisfy infinitesimal Torelli.

COROLLARY 4.5. *Let $\psi\colon \mathcal{X} \to \mathcal{B}$ be a nontrivial family of surfaces such that $\rho(X_t) = h^{1,1}(X_t)$ for all t. Then X_t does not satisfy infinitesimal Torelli.*

COROLLARY 4.6. *Let $\pi : X \to \mathbb{P}^1$ be an extremal elliptic surface with constant j-invariant and $p_g(X) > 1$. Then X does not satisfy infinitesimal Torelli.*

Proof. From Remark 3.4 it follows that X is a member of a positive-dimensional family of surfaces with $r = h^{1,1}(X_0)$. $\square$

Proof of Proposition 4.4. Fix a base point $0 \in \mathcal{B}$ and let X be isomorphic to the fiber over 0. For any $t \in \mathcal{B}$, denote by X_t the fiber over t. Let Λ be a lattice of rank r such that $\Lambda \hookrightarrow NS(X_0)$; we can fix the identification of Hodge structures $H^2(X_0, \mathbb{C}) \cong H^2(X_t, \mathbb{C})$ such that $\Lambda \hookrightarrow NS(X_t)$.

Let $T(X_t)$ be the orthogonal complement of the image of Λ in $H^2(X_t, \mathbb{Z})$. Then $T(X_t) \otimes \mathbb{C}$ carries a sub-Hodge structure, and we consider variation of this Hodge structure (cf. [vGT, Sec. 6]). Since the variation of the Hodge structure on $H^2(X, \mathbb{C})$ is determined by the variation of the sub-Hodge structures Λ and T, and since Λ remains of pure type $(1, 1)$, we have the following diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}_k \arrow[d] & \\
\mathcal{B} \arrow[r, "\psi"] & \mathcal{B}_k \arrow[d, "\rho"] & \\
& H^1(X_0, \Theta_{X_0}) \arrow[r, "\delta_2"] & \mathrm{Hom}(H^{2,0}(X_0), H^{1,1}(X_0)) \oplus \mathrm{Hom}(H^{1,1}(X_0), H^{0,2}(X_0)) \\
& \mathrm{Im}(\rho\psi) \arrow[u] \arrow[r] & \mathrm{Hom}(T^{2,0}(X_0), T^{1,1}(X_0)) \oplus \mathrm{Hom}(T^{1,1}(X_0), T^{0,2}(X_0)) \arrow[u]
\end{tikzcd},$$

where $\mathcal{X}_k$ is the Kuranishi family of X_0 and $\mathcal{B}_k$ is its base.

From this diagram one deduces $\dim \operatorname{Im}(\rho\psi) \leq 2(h^{1,1} - r)h^{2,0}$. Using Serre duality and a result of Griffiths, one can show (cf. [vGT, Sec. 6]) that $\operatorname{Im}(\rho\psi)$ has dimension at most $\frac{1}{2}(h^{1,1} - r)h^{2,0}$ if $h^{2,0} > 1$ and has dimension at most $h^{1,1} - r$ if $h^{2,0} = 1$. $\square$

REMARK 4.7. The referee pointed out that the factorization

$$\mathcal{B} \to \operatorname{Hom}(T^{2,0}(X_0), T^{1,1}(X_0)) \to \operatorname{Hom}(H^{2,0}(X_0), H^{1,1}(X_0))$$

can be proven as follows.

Let $t \in B_k$ be a direction for which the image of Λ remain divisors. Then the kernel of a cup product with t consists of the classes of type (1, 1) infinitesimally fixed in that direction.

THEOREM 4.8. *Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface* (*cf. Assumption 2.3*), *and assume that $p_g(X) > 1$. Then X does* not *satisfy infinitesimal Torelli if and only if $j(\pi)$ is constant and π is extremal.*

Proof. Combine Corollary 4.3 and Corollary 4.6. $\square$

REMARK 4.9. Note that the hyperelliptic surfaces form a family of elliptic surfaces with $p_g(X) = 0$, so they do not satisfy infinitesimal Torelli.

REMARK 4.10. In [C, Sec. 4], Chakiris gives different but invalid formulas for $\dim H^0(X, \Omega^1(nF))$. He uses them to deduce a formula for $\dim H^1(X, \Theta_X)$, which he then uses to prove that generic global Torelli holds. Even with the use of these incorrect formulas his proof of generic global Torelli seems to remain valid—after a small modification. His formulas would imply that infinitesimal Torelli holds for any elliptic surface with a section. The same erroneous formulas lead Beauville to state [B2, p. 13], in a survey paper on Torelli problems, that infinitesimal Torelli holds for *arbitrary* elliptic surface with a section. Theorem 4.8 shows instead that this is true only under the condition that $j(\pi)$ is not constant or π is not extremal.

The argument used to prove that several elliptic surfaces satisfy infinitesimal Torelli relies heavily on $C = \mathbb{P}^1$. Saitō proved (using other techniques) that, if $\pi : X \to C$ is an elliptic surface with nonconstant j-invariant, then X satisfies infinitesimal Torelli. We consider now the case that the j-invariant is constant, and we try to find surfaces for which infinitesimal Torelli does not hold.

LEMMA 4.11. *Let $\varphi \colon \mathcal{X} \to \mathcal{B}$ be a family of elliptic surfaces with $p_g(X_0) > 1$, constant j-invariant, and s singular fibers. Let $g(C)$ be the genus of the base curve of a generic member of this family. Suppose we have that $\{t' \mid X_t \cong X_{t'}\}$ is zero-dimensional for all t and that*

$$\dim \mathcal{B} > (s - \deg(\mathbb{L}) + g(C) - 1)h^{2,0}.$$

Then, for all t, X_t does not satisfy infinitesimal Torelli.

Table 1

$g(C)$	$p_g(X)$	$\deg(\mathbb{L})$	$s_{\max}$	$\left[\frac{6}{5}\deg(\mathbb{L})\right]$	$\left[\frac{4}{3}\deg(\mathbb{L})\right]$
1	2	2	3	3	3
1	3	3	4	4	4
1	4	4	5	5	–
1	5	5	6	6	–
2	2	1	2	2	2

Proof. Note that $\rho(X_t) \geq \rho_{tr}(\pi)$ for all $t \in \mathcal{B}$ and

$$h^{1,1}(X) - \rho_{tr}(\pi) = 2s - 2\deg(\mathbb{L}) - 2 + 2g(C).$$

Apply now Proposition 4.4. □

EXAMPLE 4.12. Let $\psi : \mathcal{X} \to \mathcal{B}$ be a maximal-dimensional family of elliptic surfaces with $2\deg(\mathbb{L})$ singular fibers of type I_0^*. Then $\dim \mathcal{B} = 3g - 3 + 2\deg(\mathbb{L}) + 1$.

We have that $h^{2,0} > 1$ and

$$\dim \mathcal{B} > (s - \deg(\mathbb{L}) + g(C) - 1)h^{2,0}$$

holds if and only if $h^{2,0} = 2$ and $g(C) \in \{1, 2, 3\}$ or $h^{2,0} = 3$ and $g(C) = 4$. In all these cases, any member of the family ψ is a counterexample to infinitesimal Torelli.

EXAMPLE 4.13. Let $\psi : \mathcal{X} \to \mathcal{B}$ be a maximal family of elliptic surfaces with j-invariant 0 or 1728 and with s singular fibers. Then $\dim \mathcal{B} = 3g - 3 + s$.

Using $h^{2,0} = \deg(\mathbb{L}) + g(C) - 1$, we obtain that

$$\dim \mathcal{B} > (s - \deg(\mathbb{L}) + g(C) - 1)h^{2,0}$$

holds if and only if

$$s < \frac{\deg(\mathbb{L})^2 - g(C)^2 + 5g(C) - 4)}{\deg(\mathbb{L}) + g(C) - 2}.$$

From Noether's condition, the smallest s that is possible is $\frac{6}{5}\deg \mathbb{L}$. From this we can deduce

$$\deg(\mathbb{L}) < -3g(C) + 6 + \sqrt{4g(C)^2 - 11g(C) + 16}.$$

All combinations of $g(X)$ and $p_g(X)$ satisfying the previous conditions are mentioned in Table 1. There $s_{\max}$ denotes the maximum number of singular fibers such that the maximal family with $s_{\max}$ singular fibers has $\dim(\mathcal{B})$ larger than the upper bound for the dimension of the period domain. The columns with $\left[\frac{6}{5}\deg(\mathbb{L})\right]$ and $\left[\frac{4}{3}\deg(\mathbb{L})\right]$ denote the minimal number of singular fibers for an elliptic surface with j-invariant 0 or 1728, whenever this is smaller than $s_{\max}$.

5. Twisting

In this section we study the behavior of $h^{1,1}(X) - \rho_{tr}(X)$ under twisting when X is a Jacobian elliptic surface.

To a Jacobian elliptic surface $\pi : X \to C$ we can associate an elliptic curve in $\mathbb{P}^2_{\mathbb{C}(C)}$ corresponding to the generic fiber of π. Denote this elliptic curve by E_1. Conversely, to an elliptic curve $E'/\mathbb{C}(C)$ we can associate an elliptic surface $\pi' \colon X' \to C$.

Let $\pi_i : X_i \to C$ $(i = 1, 2)$ be two Jacobian elliptic surfaces such that $j(\pi_i) \neq 0, 1728$. Then there exists an isomorphism $\varphi \colon X_1 \to X_2$ such that $\pi_2\varphi = \pi_1$ if and only if the associated elliptic curves E_1 and E_2 are isomorphic over $\mathbb{C}(C)$. This statement is equivalent to $j(E_1) = j(E_2)$, and the quotient of the minimal discriminants of $E_1/\mathbb{C}(C)$ and $E_2/\mathbb{C}(C)$ is a twelfth power (in $\mathbb{C}(C)^*$).

If $j(E_1) = j(E_2)$, then (under our assumptions) the quotient of the minimal discriminants is a sixth power, say u^6. From this it follows that E_1 and E_2 are isomorphic over $\mathbb{C}(C)\big(\sqrt{u}\big)$. We call E_2 the *twist* of E_1 by u and denote this by $E_1^{(u)}$. Actually, we are not interested in the function u but rather in the places at which the valuation of u is odd.

DEFINITION 5.1. Let $\pi : X \to C$ be a Jacobian elliptic surface. Fix $2n$ points $P_i \in C(\mathbb{C})$. Let $E/\mathbb{C}(C)$ be the Weierstrass model of the generic fiber of π.

A Jacobian elliptic surface $\pi' \colon X' \to C$ is called a *quadratic twist* of π by $(P_1, \dots, P_n)$ if the Weierstrass model of the generic fiber of π' is isomorphic to $E^{(f)}$, where $E^{(f)}$ denotes the quadratic twist of E by f in the aforementioned sense and $f \in \mathbb{C}(C)$ is a function such that $v_{P_i}(f) \equiv 1 \bmod 2$ and $v_Q(f) \equiv 0 \bmod 2$ for all $Q \notin \{P_i\}$.

A **-minimal twist* of π is a twist $\tilde{\pi} : \tilde{X} \to C$ such that none of the fibers are of type II^*, III^*, IV^*, or I_ν^* and at most one fiber is of type I_0^*.

The existence of the twist follows easily from the fact that $\mathrm{Pic}^0(C)$ is divisible. If $g(C) > 0$ and we fix $2n$ points $P_1, \dots, P_{2n}$, then there exist $2^{2g(C)}$ twists by $(P_1, \dots, P_{2n})$.

Note that a *-minimal twist is a twist for which $p_g(X)$ (and $p_a(X)$) is minimal. Later on we will introduce another notion of minimality: the twist for which $h^{1,1}(X) - \rho_{tr}(\pi)$ is minimal. A *-minimal twist of an elliptic curve need not be unique, but the configuration of the singular fibers of any two *-minimal twists of the same surface are equal.

If P is one of the $2n$ distinguished points, then the fiber of P changes in the following way (see [M1, V.4]).

$$I_\nu \leftrightarrow I_\nu^* \ (\nu \geq 0), \quad II \leftrightarrow IV^*, \quad III \leftrightarrow III^*, \quad IV \leftrightarrow II^*.$$

LEMMA 5.2. *Let $\pi : X \to C$ be a Jacobian elliptic surface, and let $P_i \in C(\mathbb{C})$ be $2n$ points. Let $\pi' \colon X' \to C$ be a twist by (P_i). Then*

$$h^{1,1}(X') - \rho_{tr}(\pi') = h^{1,1}(X) - \rho_{tr}(\pi) + \sum_{i=1}^{2n} c_{P_i},$$

with

$$c_{P_i} = \begin{cases} 1 & \textit{if } \pi^{-1}(P_i) \textit{ is of type } I_0,\ IV^*,\ III^*,\ \textit{or } II^*; \\ 0 & \textit{if } \pi^{-1}(P_i) \textit{ is of type } I_\nu \textit{ or } I_\nu^* \textit{ with } \nu > 0; \\ -1 & \textit{if } \pi^{-1}(P_i) \textit{ is of type } II,\ III,\ IV,\ \textit{or } I_0^*. \end{cases}$$

Proof. Suppose $\pi^{-1}(P_i)$ is of type I_0. Then $\pi'^{-1}(P_i)$ is of type I_0^*. The Euler characteristic of this fiber is 6, so this point causes $h^{1,1}$ to increase by 5. An I_0^* fiber has four components not intersecting the zero-section. Hence ρ_{tr} increases by 4.

The other fiber types can be handled similarly. □

LEMMA 5.3. *Suppose we have a Jacobian elliptic surface $\pi : X \to C$ with nonconstant j-invariant. There exist finitely many twists π' of π such that the nonnegative integer $h^{1,1} - \rho_{tr}$ is minimal under twisting. These twists are characterized by $b(\pi') = c(\pi') = 0$.*

Proof. It is easy to see that there are at most finitely many twists with $c(\pi') = 0$. Hence it suffices to show that $b(\pi') = c(\pi') = 0$ if $h^{1,1}(X') - \rho_{tr}(\pi')$ is minimal under twisting. From Lemma 5.2 it follows that we need only show that, for any elliptic surface, there exists a twist with $b(\pi') = c(\pi') = 0$.

Consider a *-minimal twist $\tilde{\pi} : \tilde{X} \to C$. Note that $e(\tilde{\pi}) > 0$ (otherwise the j-invariant would be constant).

Suppose $b(\tilde{\pi}) + c(\tilde{\pi})$ is even. Twist by all points with a fiber of type II, III, IV, or I_0^*. The new elliptic surface has $b = c = d = 0$.

Suppose $b(\tilde{\pi}) + c(\tilde{\pi})$ is odd. Twist by all points with a fibers of type II, III, IV, or I_0^* and one point with a fiber of type I_ν. The new elliptic surface has $b = c = 0$ and $d = 1$. □

REMARK 5.4. The classification (in [SZ]) of extremal $K3$ surfaces is a classification of the root lattices corresponding to the singular fibers. In general one can *not* decide which singular fibers correspond to these lattices, since each of the pairs (I_1, II), (I_2, III), and (I_3, IV) gives rise to the same lattice (A_0, A_1, A_2). From Lemma 5.3 it follows that this is not a problem when π is extremal.

PROPOSITION 5.5. *Let $\pi_1 : X_1 \to C$ be an elliptic surface with $j(\pi_1)$ nonconstant. Let $\pi : X \to C$ be a *-minimal twist of* $\mathrm{Jac}(\pi_1)$ *with associated line bundle $\mathbb{L}$. Let $\tilde{\pi} : \tilde{X} \to C$ be a twist for which $h^{1,1} - \rho_{tr}$ is minimal. Then*

$$h^{1,1}(\tilde{X}) - \rho_{tr}(\tilde{X}) = 2g(C) - 2\deg(\mathbb{L}) - 2 + \#\{\textit{singular fibers for } \pi\}.$$

Proof. From Proposition 2.12 and Lemmas 5.3 and 5.2 we have that $\deg(\tilde{\mathbb{L}}) = \deg(\mathbb{L}) + (d(\tilde{\pi}) + a(\tilde{\pi}) - c(\pi))/2$, $d(\tilde{\pi}) + e(\tilde{\pi}) = e(\pi)$, and $a(\tilde{\pi}) = b(\pi)$. This yields

$$\begin{aligned} h^{1,1}(\tilde{X}) - \rho_{tr}(\tilde{X}) &= 2g(C) - 2\deg(\tilde{\mathbb{L}}) - 2 + 2(a(\tilde{\pi}) + d(\tilde{\pi})) + e(\tilde{\pi}) \\ &= 2g(C) - 2\deg(\mathbb{L}) - 2 + b(\pi) + c(\pi) + e(\pi). \end{aligned}$$

Finally note that $a(\pi) = d(\pi) = 0$. □

COROLLARY 5.6. *Let $\tilde{\pi} : \tilde{X} \to C$ be an elliptic surface with $j(\pi)$ nonconstant. Then $\tilde{\pi}$ is extremal if and only if $\tilde{\pi}$ has no fibers of type II, III, IV, or I_0^*, and the $*$-minimal twist of its Jacobian $\pi : X \to C$ has $2\deg(\mathbb{L}) + 2 - 2g(C)$ singular fibers.*

6. Configurations of Singular Fibers

In order to apply the results of the previous section, we need to know which elliptic surfaces have a minimal twist with $2\deg(\mathbb{L}) + 2 - 2g(C)$ singular fibers. We shall need the following definition.

DEFINITION 6.1. A function $f : C \to \mathbb{P}^1$ is called *of* $(3, 2)$-*type* if the ramification indices of the points in the fiber of 0 are at most 3 and in the fiber of 1728 are at most 2.

For the connection between functions of $(3, 2)$-type and certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$ see [B1].

PROPOSITION 6.2. *Let $\pi : X \to C$ be an elliptic surface with $j(\pi)$ nonconstant and such that π is a $*$-minimal twist. Then $j(\pi)$ is of $(3, 2)$-type and is not ramified outside $0, 1728, \infty$ if and only if there are $2\deg(\mathbb{L}) + 2 - 2g(C)$ singular fibers.*

Proof. Denote by

(a) n_2 the number of fibers of π of type II,
(b) n_3 the number of fibers of π of type III,
(c) n_4 the number of fibers of π of type IV,
(d) n_6 the number of fibers of π of type I_0^*, and
(e) m_ν the number of fibers of π of type I_ν $(\nu > 0)$.

Let $r = \sum \nu m_\nu$.

The ramification of $j(\pi)$ is as follows (using [M1, Lemma IV.4.1]). Above 0 we have n_2 points with ramification index 1 modulo 3, n_4 points with index 2 modulo 3, and at most $(r - n_2 - 2n_4)/3$ points with index 0 modulo 3. In total, we have at most $n_2 + n_4 + (r - n_2 - 2n_4)/3$ points in $j(\pi)^{-1}(0)$. Above 1728 we have n_3 points with index 1 modulo 2 and at most $(r - n_3)/2$ points with index 0 modulo 2. So $j(\pi)^{-1}(1728)$ has at most $n_3 + (r - n_3)/2$ points. Above ∞ we have $\sum m_\nu$ points.

These statements together imply that

$$\#j(\pi)^{-1}(0) + \#j(\pi)^{-1}(1728) + \#j(\pi)^{-1}(\infty) \leq \tfrac{2}{3}n_2 + \tfrac{1}{2}n_3 + \tfrac{1}{3}n_4 + \tfrac{5}{6}r + \sum m_\nu,$$

with equality if and only if $j(\pi)$ is of $(3, 2)$-type.

Hurwitz's formula implies that

$$r + 2 - 2g(C) \leq \#j(\pi)^{-1}(0) + \#j(\pi)^{-1}(1728) + \#j(\pi)^{-1}(\infty),$$

with equality if and only if there is no ramification outside 0, 1728, and ∞. Hence

$$r \le 12g(C) - 12 + 4n_2 + 3n_3 + 2n_4 + 6\sum m_\nu$$

and, by Proposition 2.11,

$$\sum in_i + r = 12\deg(\mathbb{L}).$$

Substituting gives

$$2\deg(\mathbb{L}) + 2 - 2g(C) \le n_2 + n_3 + n_4 + n_6 + \sum m_\nu,$$

with equality if and only if (a) $j(\pi)$ is not ramified outside 0, 1728, and ∞ and (b) $j(\pi)$ is of (3, 2)-type. □

This enables us to prove our next theorem.

THEOREM 6.3. *Suppose $\pi : X \to C$ is an elliptic surface with nonconstant j-invariant. Then the following three statements are equivalent.*

(1) *π is extremal.*
(2) *$j(\pi)$ is of $(3, 2)$-type, not ramified outside $0, 1728, \infty$, and π has no fibers of type II, III, IV, or I_0^*.*
(3) *The minimal twist π' of* $\mathrm{Jac}(\pi)$ *has* $2\deg(\mathbb{L}) + 2 - 2g(C)$ *singular fibers.*

Proof. Apply Proposition 6.2 to Corollary 5.6. □

REMARK 6.4. Frederic Mangolte pointed out to me that the equivalence of (1) and (2) was already proved in [N].

REMARK 6.5. Consider functions $f : C \to \mathbb{P}^1$ up to automorphisms of C. If we fix the ramification indices above 0, 1728, ∞ and demand that f be unramified at any other point, then there are only finitely many f with that property. A small deformation of f in $\mathrm{Mor}_d(C, \mathbb{P}^1)$, the moduli space of morphisms $C \to \mathbb{P}^1$ of degree d, has more critical values.

Hence the j-invariants of any extremal elliptic surface lie "discretely" in $\mathrm{Mor}_d(C, \mathbb{P}^1)$. By Lemma 5.3, to any j-invariant there correspond only finitely many extremal elliptic surfaces (by Lemma 5.3). Therefore, extremal elliptic surfaces over $\mathbb{P}^1$ with geometric genus n and nonconstant j-invariant lie discretely in the moduli space of elliptic surfaces over $\mathbb{P}^1$ with geometric genus n.

Suppose that $\pi : X \to \mathbb{P}^1$ has $2p_g(X) + 4$ singular fibers of type I_ν and no other singular fibers. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a cyclic morphism ramified at two points P such that $\pi^{-1}(P)$ is singular. Then Fastenberg [F, Thm. 2.1] has proved that the base-changed surface has Mordell–Weil rank 0. In fact, she proved that the base-changed surface is extremal. The first surface is also extremal (by Proposition 5.6). A slightly more general variant is the following.

EXAMPLE 6.6. Suppose $\pi: X \to C$ is an extremal elliptic surface, and let $f: C' \to C$ be a finite morphism. Then the base-changed elliptic surface π': $X' \to C'$ is extremal if f is not ramified outside the set of points P such that $\pi^{-1}(P)$ is multiplicative or potential multiplicative.

In that case, the composition $j': C' \to C \xrightarrow{j} \mathbb{P}^1$ is not ramified outside 0, 1728, and ∞, and the ramification indices above 0 and 1728 are at most 3 and 2. Moreover, there are no fibers of type II, III, IV, or I_0^*.

An easy calculation shows that all elliptic surfaces for which Fastenberg's results [F, Thm. 1] hold are either extremal elliptic surfaces or have a twist that is extremal. Moreover, these surfaces have no fibers of type I_0^* and so all elliptic surfaces for which her results hold lie discretely in the moduli spaces mentioned above.

7. Mordell–Weil Groups of Extremal Elliptic Surfaces

It remains to classify which Mordell–Weil groups can occur. However, we can give only a partial answer to this problem.

Let $m, n \in \mathbb{Z}_{\geq 1}$ be such that $m|n$ and $n > 1$. Recall from Section 1 that $X_m(n)$ is the modular curve parameterizing triples $((E, O), P, Q)$ such that (E, O) is an elliptic curve, $P \in E$ is a point of order m, and $Q \in E$ is a point of order n.

If $(m, n) \notin \{(1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\}$ then there exists a universal family for $X_m(n)$, which we denote by $E_m(n)$. Denote by $j_{m,n}: X_m(n) \to \mathbb{P}^1$ the map that is usually called j.

From the results of [Sh1, Secs. 4 & 5] it follows that $E_m(n)$ is an extremal elliptic surface. The following theorem explains how to construct many examples of extremal elliptic surfaces with a given torsion group.

THEOREM 7.1. *Fix $m, n \in \mathbb{Z}_{\geq 1}$ such that $m|n$ and $(m, n) \notin \{(1, 1), (1, 2), (2, 2), (1, 3), (1, 4), (2, 4)\}$, and let $j \in \mathbb{C}(C)$ be nonconstant. Then there exists a unique elliptic surface $\pi: X \to C$ with $j(\pi) = j$, and $MW(\pi)$ has $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ as a subgroup if and only if* (a) *j is of* $(3, 2)$*-type, not ramified outside* $0, 1728, \infty$, *and* (b) *$j = j_{n,m} \circ g$ for some $g: C \to X_m(n)$.*

The extremal elliptic surface with $j(\pi) = j$ and $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ is a subgroup of $MW(\pi)$, which is the unique surface with only singular fibers of type I_ν.

Proof. Let $\pi: X \to C$ be an elliptic surface such that $MW(\pi)$ has a subgroup isomorphic to $G := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Then $j: C \to \mathbb{P}^1$ can be decomposed in $g: C \to X_m(n)$ and $j_{m,n}: X_m(n) \to \mathbb{P}^1$, and X is isomorphic to $E_m(n) \times_{X_m(n)} C$. Conversely, for any base change π' of $\varphi_{m,n}: E_m(n) \to X_m(n)$, the group $MW(\pi')$ has G as a subgroup.

Moreover, since $\varphi_{m,n}$ has only singular fibers of type I_ν, the same holds for π'. An application of Theorem 6.3 concludes the proof. □

REMARK 7.2. Let $n = 2$ and $m \leq 2$. Then any elliptic surface with $j(\pi) = j_{m,n} \circ g$ has $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ as a subgroup of $MW(\pi)$.

8. Uniqueness

Artal Bartolo, Tokunaga, and Zhang [ATZ] expect that an extremal elliptic surface is determined by the configuration of the singular fibers and the Mordell–Weil group. More precisely, they raise the following question.

QUESTION 8.1. Suppose $\pi_1 \colon X_1 \to \mathbb{P}^1$ and $\pi_2 \colon X_2 \to \mathbb{P}^1$ are extremal semistable elliptic surfaces such that $MW(\pi_1) \cong MW(\pi_2)$ and the configurations of singular fibers of π_1 and π_2 are the same.

Are X_1 and X_2 isomorphic? If so, is there then an isomorphism that respects the fibration and the zero-section?

By [MP, Thm. 5.4], this is true if X_1 and X_2 are rational elliptic surfaces. If X_1 and X_2 are $K3$ surfaces, then the question is answered by the following theorem.

THEOREM 8.2. *There exist precisely nineteen pairs* $(\pi_1 \colon X_1 \to \mathbb{P}^1, \pi_2 \colon X_2 \to \mathbb{P}^1)$ *of extremal elliptic $K3$ surfaces such that π_1 and π_2 have the same configuration of singular fibers, $MW(\pi_1)$ and $MW(\pi_2)$ are trivial, and X_1 and X_2 are not isomorphic. Of these pairs, thirteen are semistable. There is also a unique pair for which this statement holds with $MW(\pi_1) = MW(\pi_2) = \mathbb{Z}/2\mathbb{Z}$ that is not semistable.*

Proof. From [SZ, Table 2] there exist nineteen pairs of surfaces (X_1, X_2) such that the transcendental lattices of X_1 and X_2 lie in distinct $\mathrm{SL}_2(\mathbb{Z})$-orbits, they admit elliptic fibrations $\pi_i \colon X_i \to \mathbb{P}^1$ such that $MW(\pi_1) = MW(\pi_2) = 0$, and the contributions of the singular fibers as sublattices of the Néron–Severi lattice coincide.

From Remark 5.4 we know that, for extremal elliptic surfaces, the sublattices determine the singular fibers. Since the transcendental lattices are in different $\mathrm{SL}_2(\mathbb{Z})$-orbits, the surfaces are not isomorphic.

The rest of the statement follows from the same table in [SZ]. □

REMARK 8.3. From these surfaces one should be able to construct other pairs of extremal elliptic surfaces, with isomorphic Mordell–Weil groups and the same configuration of singular fibers, such that the geometric genus is higher than 1.

Start with two nonisomorphic extremal elliptic $K3$ surfaces with the same configuration of singular fibers and the same Mordell–Weil group. Then the j-invariants of both surfaces are unequal modulo automorphism of $\mathbb{P}^1$.

We can base-change both surfaces in such a way that the base-changed surfaces remain extremal (cf. Example 6.6); they have the same configuration of singular fibers and their j-invariants are unequal modulo an automorphism of $\mathbb{P}^1$.

The configuration of singular fibers gives restrictions on the possibilities for the torsion part of the Mordell–Weil group. One can hope that this is sufficient to prove that the Mordell–Weil groups are isomorphic.

Note that the base-changed surfaces are not isomorphic, since a surface that is not a $K3$ surface has at most one elliptic fibration.

REMARK 8.4 ("Case 49"). In [ATZ] it is proven that there are two elliptic surfaces with $MW(\pi_i) = \mathbb{Z}/5\mathbb{Z}$ and singular fibers $2I_1, I_2, 2I_5, I_{10}$ and there exists no isomorphism between them that respects the fibration. In [SZ] it is proven that both surfaces are isomorphic. For any other pair of (semistable) extremal elliptic $K3$ surfaces with $\#MW(\pi) > 4$ and the same singular fibers configuration, there exists an isomorphism that respects the fibration [ATZ, Thm. 0.4].

9. Extremal Elliptic Surfaces with $p_g = 1$ and $q = 1$

An elliptic surface with a section and $q = 1$ needs to have a genus-1 base curve. This implies that, for an extremal elliptic surface with $p_g = 1$ and $q = 1$, we have $\deg(\mathbb{L}) = 1$. The minimal twist of an extremal elliptic surface with $p_g = 1$ and $q = 1$ has two singular fibers.

The following table lists all possible pairs of fiber types such that the sum of the Euler characteristics is 12. Several of these surfaces are already described in the literature (see the references cited in the table's bottom row).

F_1	I_{11}	I_{10}	I_9	I_8	I_7	I_6	I_{10}	I_9	I_8	I_6
F_2	I_1	I_2	I_3	I_4	I_5	I_6	II	III	IV	I_0^*
		[AIR4]	[AIR3]	[AIR1]		[AIR2]				

REMARK 9.1. Note that the six configurations with two I_ν fibers are already extremal. The configurations with one additive and one multiplicative fiber are not extremal. If we twist by the two points with a singular fiber we obtain an extremal elliptic surface, but then the degree of $\mathbb{L}$ is 2—except for the $I_6\, I_0^*$, whose extremal twist has one singular fiber of type I_6^*.

PROPOSITION 9.2. *All these configurations exist except $I_7\, I_5$.*

Proof. This is a consequence of the following lemmas. Note that the existence of elliptic surfaces over $\mathbb{P}^1$ with the singular fibers mentioned hereunder follows from [P].

LEMMA 9.3. *The configurations with $I_k\, I_{12-k}$ exist for $k = 2, 4, 6$.*

Proof. Let $\pi : X \to \mathbb{P}^1$ be an elliptic surface with two III fibers, a fiber of type $I_{k/2}$, a fiber of type $I_{6-k/2}$, and no other singular fibers. Let $\varphi\colon C \to \mathbb{P}^1$ be a degree-2 cover ramified at the four points where the fiber of π is singular. Then $\pi'\colon X \times_{\mathbb{P}^1} C \to C$ has two fibers of type I_0^*, a fiber of type I_k, and a fiber of type I_{12-k}. Twisting by the two points with I_0^* fibers gives the desired configuration. $\square$

LEMMA 9.4. *The configurations $I_8\, IV$ and $I_6\, I_0^*$ exist.*

Proof. For the first claim, let $\pi : X \to \mathbb{P}^1$ be an elliptic surface with two III fibers, a fiber of type II, a fiber of type I_4, and no other singular fibers. Let $\varphi\colon C \to \mathbb{P}^1$ be a degree-2 cover ramified at the four points where the fiber of π is singular. Then $\pi'\colon X \times_{\mathbb{P}^1} C \to C$ has two fibers of type I_0^*, a fiber of type I_8, and a fiber of type IV. Twisting by the two points with a I_0^* fiber gives the desired configuration.

For the second, let $\pi: X \to \mathbb{P}^1$ be an elliptic surface with three III fibers, a fiber of type I_3, and no other singular fibers. Let $\varphi: C \to \mathbb{P}^1$ be a degree-2 cover ramified at the four points where the fiber of π is singular. Then $\pi': X \times_{\mathbb{P}^1} C \to C$ has three fibers of type I_0^* and a fiber of type I_6. Twisting by two points with a I_0^* fiber gives the desired configuration. □

For the other four configurations we have a different strategy. We simply show that the j-map with the right ramification indices exists. This is equivalent to giving the monodromy representation.

LEMMA 9.5. *The configurations $I_{11}\,I_1$, $I_9\,I_3$, $II\,I_{10}$, and $III\,I_9$ exist.*

Proof. For the two configurations of type $I_\mu\,I_\nu$ we need to find curves C and functions $j: C \to \mathbb{P}^1$ of degree 12 such that above ∞ there are two points with ramification indices μ and ν, all points above 0 have ramification index 3, and all points above 1728 have ramification index 2 (see [M1, Lemma IV.4.1]).

By the Riemann existence theorem it suffices to give two permutations σ_0, σ_1 in S_{12} such that σ_0 is the product of six disjoint 2-cycles, σ_1 is the product of four disjoint 3-cycles, $\sigma_0\sigma_1$ is the product of a μ-cycle and a ν-cycle, and the subgroup generated by σ_0 and σ_1 is transitive (see [M3, Cor. 4.10]).

For $I_1\,I_{11}$ we use

$$(1\,2\,3)(4\,5\,6)(7\,8\,9)(10\,11\,12) * (1\,3)(2\,4)(5\,7)(8\,10)(9\,11)(6\,12) = (1)(3\,2\,5\,8\,11\,7\,6\,10\,9\,12\,4)$$

For $I_3\,I_9$ we use

$$(1\,2\,3)(4\,5\,6)(7\,8\,9)(10\,11\,12) * (1\,6)(4\,9)(3\,7)(2\,10)(5\,12)(8\,11) = (1\,4\,7)(2\,11\,9\,5\,10\,3\,8\,12\,6)$$

Similarly, the existence of $II\,I_{10}$ follows from

$$(1\,2)(3\,4)(5\,6)(7\,8)(9\,10) * (1\,4\,7)(2\,5\,8)(3\,6\,9) = (1\,3\,5\,7\,2\,6\,10\,9\,4\,8),$$

and the existence of $III\,I_9$ follows from

$$(2\,3)(4\,5)(6\,7)(8\,9) * (1\,4\,7)(2\,5\,8)(3\,6\,9) = (1\,5\,9\,2\,4\,6\,8\,3\,7).$$ □

LEMMA 9.6. *The configuration $I_7\,I_5$ does not exist.*

Proof. A computer search informed us that the permutations needed for the existence of $I_7\,I_5$ do not exist. □

COROLLARY 9.7. *Let k_i be positive integers such that $\sum k_i = 12$ with $i \geq 2$, and if $i = 2$ then $(k_1, k_2) \neq (7, 5)$ or $(5, 7)$. Then there exist a curve C of genus 1 and an elliptic surface $\pi: X \to C$ such that the configuration of singular fibers of π is $\sum I_{k_i}$.*

Proof. Use the monodromy representation as in [M2, Rem. after Cor. 3.5]. □

10. Elliptic Surfaces with One Singular Fiber

In the previous section we proved that there exists an elliptic fibration with an I_6 and an I_0^* fiber. Twisting by the points with a singular fiber yields an elliptic surface with one singular fiber of type I_6^*. We will prove that, for fixed $\deg(\mathbb{L})$, there are two possible configurations that can be realized as elliptic surfaces.

PROPOSITION 10.1. *Suppose $\pi : X \to C$ is an elliptic surface with one singular fiber. The fiber is of type I_{12k-6}^* or I_{12k} for some $k \in \mathbb{Z}_{>0}$, and $g(C) \geq k$ in the first case and $g(C) \geq k+1$ in the second. Conversely, all these configurations occur.*

Proof. Since the Euler characteristic of the singular fiber is $12\deg(\mathbb{L})$, the only possible configurations are I_ν and I_ν^*.

Fix a rational elliptic surface $\pi : X \to \mathbb{P}^1$ with three fibers of type III and one fiber of type I_3. (The existence follows from [P].) Fix k a positive integer. Take a curve C such that $\varphi\colon C \to \mathbb{P}^1$ has degree $4k-2$ and is ramified at the four points that have the singular fibers—and above such a point there is exactly one point.

The base change $\pi'\colon X \times_{\mathbb{P}^1} C \to C$ has three fibers of type I_0^* and one fiber of type I_{12k-6}. Twisting by all four points with a singular fiber yields an elliptic surface with one singular fiber, and this fiber is of type I_{12k-6}^*. If we replace $4k-2$ by $4k$, we obtain an elliptic surface with one fiber of type I_{12k}.

For any elliptic surface with only one singular fiber, which is of type I_{12k-6}^*, the following statement holds: the j-map $C \to \mathbb{P}^1$ has degree $12k-6$, one point above ∞, at most $4k-2$ points above 0, and at most $6k-3$ points above 1728. This implies that the base curve has genus at least k. A similar argument shows that, if the only singular fiber is of type I_{12k}, then the base curve has genus at least $k+1$. □

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Department of Mathematics and Computer Science
University of Groningen
P.O. Box 800
9700 AV Groningen
The Netherlands

r.n.kloosterman@math.rug.nl