The Total Mean Curvature of Submanifolds in a Euclidean Space

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1. Introduction

Let M^n be an n-dimensional submanifold in a Euclidean space E^{n+p} of dimension $n+p$. Denote by R the normalized scalar curvature and by H the mean curvature of M^n.

Ōtsuki [O] introduced a kind of curvatures, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ for M^2 in E^{2+p}, and showed that they can be used to study the geometry of surfaces in higher-dimensional Euclidean space. Shiohama [S] proved that a complete oriented surface in E^{2+p} with $\lambda_\alpha = 0$ $(1 \leq \alpha \leq p)$ is a cylinder. Chen [C1] classified compact oriented surfaces in E^{2+p} with $\lambda_p \geq 0$.

In higher-dimensional cases, Chen [C3] introduced the notion of αth scalar curvatures, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ for M^n in E^{n+p}, and found a relationship between the αth scalar curvatures and the scalar curvature. When $n = 2$, it reduces to that introduced by Ōtsuki [O]. Chen [C3] also proved that a closed submanifold M^n $(n \geq 3)$ in E^{n+p} with $\int_{M^n} (\lambda_1)^{n/2}\, dV = c_n$ and $\lambda_\alpha = 0$ $(2 \leq \alpha \leq p)$ is an n-sphere, where c_n is the volume of the unit n-sphere and dV denotes the volume element of M^n.

In this paper, we give a further description of the behavior of the αth scalar curvatures and obtain some applications of them. In Section 2, we first prove that $\lambda_\alpha \leq 0$ $(2 \leq \alpha \leq p)$ for any submanifold M^n in E^{n+p}. Then we prove an inequality involving the integral of λ_1 for closed M^n in E^{n+p} with $R \geq 0$.

Suppose that M^n is closed in E^{n+p}. The total mean curvature of M^n is defined to be the integral $\int_{M^n} H^n\, dV$. An interesting and outstanding problem is to find the best possible lower bound of this integral in terms of the geometric or topologic invariants of M^n. A special case of this problem is the famous Willmore's conjecture. There have been many results obtained on this problem. In Section 3 we give an estimate of the total mean curvature for closed submanifolds in E^{n+p} with $R \geq 0$. The main result of this paper is the following theorem.

Theorem 3.1. *Let M^n be a closed submanifold in E^{n+p} with $R \geq 0$. Then*

$$\int_{M^n} H^n\, dV \geq 2\kappa_n c_{n-1} + \left\{1 - 2\kappa_n \left(\frac{c_{n-1}}{c_n}\right)\right\} \int_{M^n} R^{n/2}\, dV,$$

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where $\kappa_n = (\sqrt{\pi}/4)\Gamma\left(\frac{n}{4}\right)\big/\Gamma\left(\frac{n+2}{4}\right)$ *. Moreover, the equality holds if and only if* M^n *is imbedded as a hypersphere in an* $(n+1)$*-dimensional linear subspace of* E^{n+p}.

2. The αth Scalar Curvatures of M^n

Let $f: M^n \to E^{n+p}$ be an immersion of an n-dimensional closed manifold M^n into a Euclidean space E^{n+p}. Throughout this paper, we identify M^n with its immersed image in E^{n+p} and agree on the following index ranges: $1 \le i, j, k \le n$; $1 \le \alpha, \beta, \gamma \le p$; $n+1 \le r, s, t \le n+p$; $1 \le A, B, C \le n+p$.

Let $(e_1, \dots, e_{n+p})$ be a local orthonormal frame field in $T(E^{n+p})$ such that $(e_1, \dots, e_n)$ are tangent to M^n. Let $(\omega_1, \dots, \omega_{n+p})$ be the dual coframe. Then there is a unique connection 1-form (ω_{AB}), the Levi–Civita connection form, such that

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}.$$

Restricting these forms to M^n, we have $\omega_r = 0$ for all r. Hence $0 = d\omega_r = \sum_i \omega_{ri} \wedge \omega_i$ for all r. By Cartan's lemma we have $\omega_{ri} = \sum_j h^r_{ij}\omega_j$, where $h^r_{ij} = h^r_{ji}$ for all i, j, and r.

The first and second fundamental forms are

$$\mathrm{I} = \sum_i (\omega_i)^2 \quad \text{and} \quad \mathrm{II} = \sum_{i,j,r} h^r_{ij}\omega_i\omega_j e_r.$$

The shape operator A_e of M^n with respect to a normal vector e is the linear self-adjoint operator on $T(M^n)$ corresponding to the quadratic form $\mathrm{II}_e = \langle \mathrm{II}, e\rangle$. The matrix for A_{e_r} with respect to the base $\{e_1, \dots, e_n\}$ is $L_r = (h^r_{ij})_{n\times n}$. The mean curvature vector field ξ, the mean curvature H, and the square length of the second fundamental form S can be expressed as $\xi = \sum_r H_r e_r$, $H = |\xi|$, and $S = \sum_{i,j,r}(h^r_{ij})^2$, where $H_r = (1/n)\sum_i h^r_{ii}$ for every r. The Riemannian curvature tensor $\{R_{ijkl}\}$ and the normalized scalar curvature R can be expressed as

$$R_{ijkl} = \sum_r (h^r_{ik}h^r_{jl} - h^r_{il}h^r_{jk}), \qquad R = \sum_{i,j} \frac{R_{ijij}}{n(n-1)} = \frac{n^2H^2 - S}{n(n-1)}. \tag{2.1}$$

Let B_ν be the bundle of unit normal vectors of M^n in E^{n+p}. Then the $(p-1)$-form $d\sigma_{p-1} = \omega_{n+p,n+1} \wedge \cdots \wedge \omega_{n+p,n+p-1}$ can be regarded as a $(p-1)$-form on B_ν. On the other hand, the volume element of M^n can be written as $dV = \omega_1 \wedge \cdots \wedge \omega_n$. Hence the $(n+p-1)$-form $dV \wedge d\sigma_{p-1}$ can be regarded as the volume element of B_ν.

At an arbitrary point $(x, e) \in B_\nu$, denote $A_e = (A_{ij})$. We define the kth *mean curvature* $K_k(x, e)$ at (x, e) by

$$\det(\delta_{ij} + tA_{ij}) = 1 + \sum_k \binom{n}{k} K_k(x, e)t^k, \tag{2.2}$$

where δ_{ij} is the Kronecker delta, t is a parameter, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!};$$

$K_n(x, e)$ is the so-called *Lipschitz–Killing curvature* at (x, e). We call the integral

$$K_k^*(x) := \int_{S_x^{p-1}} |K_k(x, e)|^{n/k}\, d\sigma_{p-1}$$

over the sphere S_x^{p-1} of all unit normal vectors at x the kth *total absolute curvature of M^n at x*. The kth *total absolute curvature of M^n with respect to f* is defined by

$$\mathrm{TA}_k(f) := \frac{1}{c_{n+p-1}} \int_{M^n} K_k^*(x)\, dV.$$

For all $\mathrm{TA}_k(f)$, we have the following lemmas.

LEMMA 2.1 [CL2]. *Let $f\colon M^n \to E^{n+p}$ be an immersion of a closed manifold into E^{n+p}. Then $\mathrm{TA}_n(f) \geq \mu(M^n) \geq \beta(M^n)$, where $\mu(M^n)$ is the Morse number, $\beta_i(M^n)$ is the ith Betti number, and $\beta(M^n) = \sum_{i=0}^n \beta_i(M^n)$.*

LEMMA 2.2 [C2; CL1]. *Let $f\colon M^n \to E^{n+p}$ be an immersion of a closed manifold into E^{n+p}. Then $\mathrm{TA}_k(f) \geq 2$ for $k = 1, \dots, n$. The equality holds when and only when M^n is imbedded as either a hypersphere if $k < n$, or as a convex hypersurface if $k = n$, in an $(n+1)$-dimensional linear subspace of E^{n+p}.*

To describe how $K_2(x, e)$ depends on e, we take a local orthonormal frame field $\{e_A\}_{A=1}^{n+p}$ in a neighborhood of x as before. Then e and A_e can be expressed as $e = \sum_\alpha y_\alpha e_{n+\alpha}$ and $A_e = \sum_\alpha y_\alpha L_{n+\alpha}$, where $\sum_\alpha y_\alpha^2 = 1$. In this case, $K_2(x, e)$ is given by

$$K_2(x, e) = \binom{n}{2}^{-1} \sum_{i<j} (A_{ii} A_{jj} - A_{ij}^2). \tag{2.3}$$

Denote $\Pi = (\Pi_{\alpha\beta})_{p\times p}$ where

$$\Pi_{\alpha\beta} = \frac{1}{n(n-1)} \sum_{i,j} (h_{ii}^{n+\alpha} h_{jj}^{n+\beta} - h_{ij}^{n+\alpha} h_{ij}^{n+\beta}) \tag{2.4}$$

for all α and β. Then Π is symmetric and $K_2(x, e)$ can be expressed as

$$K_2(x, e) = \sum_{\alpha,\beta} \Pi_{\alpha\beta} y_\alpha y_\beta. \tag{2.5}$$

Choose a suitable local normal frame field $\{\bar{e}_{n+\alpha}\}_{\alpha=1}^p$ such that $K_2(x, e)$ can be rewritten as $K_2(x, e) = \sum_\alpha \lambda_\alpha \bar{y}_\alpha^2$, where $e = \sum_\alpha \bar{y}_\alpha \bar{e}_{n+\alpha}$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p$ are the eigenvalues of Π. We call such a frame field $(x; e_1, \dots, e_n; \bar{e}_{n+1}, \dots, \bar{e}_{n+p})$ a *Frenet–Ōtsuki frame*, and we call λ_α $(\alpha = 1, \dots, p)$ the αth *scalar curvature of M^n* in E^{n+p}. By means of the method of definition, we see that the λ_α are defined continuously on the whole manifold M^n and are differentiable on the open subset in which $\lambda_1 > \lambda_2 > \dots > \lambda_p$ (see [C3]).

Put $S_{n+\alpha,n+\beta} = \sum_{i,j} h_{ij}^{n+\alpha} h_{ij}^{n+\beta}$ and $S_{n+\alpha} = S_{n+\alpha,n+\alpha} = \sum_{i,j} (h_{ij}^{n+\alpha})^2$ for all α and β. Then $\Pi_{\alpha\beta}$ can be expressed as

$$\Pi_{\alpha\beta} = \frac{1}{n(n-1)} (n^2 H_{n+\alpha} H_{n+\beta} - S_{n+\alpha,n+\beta}) \tag{2.6}$$

for all α, β. Since the matrix $(H_{n+\alpha}H_{n+\beta})_{p\times p}$ is of rank 1 and $(S_{n+\alpha,n+\beta})_{p\times p}$ is positive semidefinite, it follows from (2.6) that there are at least $(p-1)$ numbers of the eigenvalues of Π that are less than or equal to zero. From the definition of λ_α, we obtain the following proposition.

PROPOSITION 2.1. *Let M^n $(n \geq 2)$ be a submanifold in E^{n+p} with $p \geq 2$. Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ are the αth scalar curvatures of M^n. Then $0 \geq \lambda_2 \geq \cdots \geq \lambda_p$ everywhere on M^n.*

Let M^n be a closed submanifold in E^{n+p} with $R \geq 0$. Chen [C3] established an equality for the submanifold M^{2m} in E^{2m+2} with $R = 0$ (see Corollary 2.1). In this section, we will prove an inequality that generalizes his result.

Before starting our discussion, we need the following lemma.

LEMMA 2.3. *Let c_n denote the volume of the unit sphere S^n. If $n \geq 2$, then*

$$\frac{c_{n-1}}{c_n} - \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right)} < 0. \tag{2.7}$$

Proof. Let $F(u) = 2\ln\Gamma(u) - \ln\Gamma(u-\frac{1}{4}) - \ln\Gamma(u+\frac{1}{4})$ and $\psi(u) = d\ln\Gamma(u)/du$. Since $\psi''(u) = -2\sum_{n=0}^{\infty}(1/(u+n)^3) < 0$ for $u > 0$, $\psi(u)$ is a concave function for $u > 0$. Hence $F'(u) = 2\{\psi(u) - \frac{1}{2}[\psi(u-\frac{1}{4}) + \psi(u+\frac{1}{4})]\} > 0$. It follows that $F(u)$ is a monotone increasing function for $u \geq 0$. It is easy to see that

$$\begin{aligned} F(1) &= \ln\frac{4}{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})} \\ &= \ln\left(\frac{4}{\pi}\sin\frac{\pi}{4}\right) = \ln\left(\frac{4}{\pi}\right) + \ln\left(\frac{\sqrt{2}}{2}\right) > \ln\left(\frac{\sqrt{2}}{2}\right). \end{aligned}$$

Therefore $F(u) \geq F(1) > \ln(\sqrt{2}/2)$ for $u \geq 1$, from which we have

$$\Gamma^2(u) > (\sqrt{2}/2)\Gamma(u-\tfrac{1}{4})\Gamma(u+\tfrac{1}{4}). \tag{2.8}$$

Since $c_{n-1} = 2(\sqrt{\pi})^n/\Gamma(n/2)$, using the "duplication formula of gamma function" we have

$$\frac{c_{n-1}}{c_n} - \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right)} = \frac{\sqrt{2}\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(\frac{n+3}{4}\right) - 2\Gamma^2\left(\frac{n+2}{4}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{4}\right)\Gamma\left(\frac{n+2}{4}\right)}. \tag{2.9}$$

Therefore (2.7) follows from (2.8) and (2.9) with $u = \frac{n+2}{4} \geq 1$. $\square$

Let us prove the main result of this section now. By using Proposition 2.1, we have

$$\begin{aligned} &|K_2(x,e)|^{n/2} \\ &\quad = |R\bar{y}_1^2 + \lambda_2(\bar{y}_2^2 - \bar{y}_1^2) + \cdots + \lambda_p(\bar{y}_p^2 - \bar{y}_1^2)|^{n/2} \\ &\quad \leq (R\bar{y}_1^2 - \lambda_2|\bar{y}_2^2 - \bar{y}_1^2| - \cdots - \lambda_p|\bar{y}_p^2 - \bar{y}_1^2|)^{n/2} \\ &\quad \leq \lambda_1^{n/2-1}(R|\bar{y}_1|^n - \lambda_2|\bar{y}_2^2 - \bar{y}_1^2|^{n/2} - \cdots - \lambda_p|\bar{y}_p^2 - \bar{y}_1^2|^{n/2}), \end{aligned} \tag{2.10}$$

where we have used that $f(u) = u^{n/2}$ is convex in $[0, +\infty)$.

We need the following spherical integrals

$$\int_{S_x^{p-1}} |\bar{y}_1|^n \, dS_{p-1} = \frac{2c_{n+p-1}}{c_n},$$
$$\int_{S_x^{p-1}} |\bar{y}_r^2 - \bar{y}_1^2|^{n/2} \, dS_{p-1} = \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right)} \left(\frac{c_{n+p-1}}{c_{n-1}} \right) \tag{2.11}$$

for $r = 2, \ldots, p$, which can be found in [W] or [C9, pp. 200, 228]. Integrating both sides of (2.10) with respect to e on S_x^{p-1}, we obtain

$$\frac{1}{c_{n+p-1}} K_2^*(x) \leq \lambda_1^{n/2-1} \left\{ \lambda_1 \cdot \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right) c_{n-1}} + \frac{2R}{c_{n-1}} \left[\frac{c_{n-1}}{c_n} - \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right)} \right] \right\}. \tag{2.12}$$

It is known that $R = \sum_\alpha \lambda_\alpha$ (see [C3]). From Proposition 2.1, we have $\lambda_1 \geq R \geq 0$. Therefore, it follows from Lemma 2.3 and (2.12) that

$$\frac{1}{c_{n+p-1}} K_2^*(x) \leq \lambda_1^{n/2} \cdot \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right) c_{n-1}} + \frac{2R^{n/2}}{c_{n-1}} \left\{ \frac{c_{n-1}}{c_n} - \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right)} \right\}. \tag{2.13}$$

Then we can prove our next proposition.

Proposition 2.2. *Let M^n $(n \geq 2)$ be a closed submanifold in E^{n+p} with $R \geq 0$. Then*

$$\int_{M^n} \lambda_1^{n/2} \, dV \geq \frac{\kappa_n c_{n-1}}{c_{n+p-1}} \int_{M^n} K_2^*(x) \, dV + \left\{ 1 - 2\kappa_n \left(\frac{c_{n-1}}{c_n} \right) \right\} \int_{M^n} R^{n/2} \, dV, \tag{2.14}$$

where $\kappa_n = (\sqrt{\pi}/4)\Gamma\left(\frac{n}{4}\right) \big/ \Gamma\left(\frac{n+2}{4}\right)$. Moreover, if the equality in (2.14) *holds then $\lambda_\alpha = 0$ $(2 \leq \alpha \leq p-1)$ on M^n.*

Proof. Integrating (2.13) on M^n, we obtain (2.14). Suppose that the equality in (2.14) holds. Then the relevant inequalities become equalities. Put $P = \{ x \in M^n \mid R(x) > 0 \}$. It follows that $\lambda_1 = R$ in P, from which we have $\lambda_2 = \cdots = \lambda_p = 0$ in P.

Set $Q = \{ x \in M^n \mid R(x) = 0 \}$. Then $M^n = P \cup Q$. It follows from the equality sign in (2.10) that at most one of $\lambda_2, \ldots, \lambda_p$ is not zero in Q. This, together with the fact $0 \geq \lambda_2 \geq \cdots \geq \lambda_p$, yields $\lambda_2 = \cdots = \lambda_{p-1} = 0$ in Q. From the foregoing deductions, we have $\lambda_2 = \cdots = \lambda_{p-1} = 0$ on $P \cup Q = M^n$. This completes the proof. □

In the special case when $n = 2m$, $p = 2$, and $R = 0$, the inequalities in (2.10) are in fact equalities, and so is that in (2.14). Thus Proposition 2.2 reduces to the following result of Chen.

Corollary 2.1 [C3]. *Suppose that M^{2m} is closed in E^{2m+2} with $R = 0$. Then*

$$\int_{M^n} (\lambda_1)^m \, dV = \frac{c_m}{2c_{m+1}} \int_{M^n} K_2^*(x) \, dV. \tag{2.15}$$

3. An Estimate of the Total Mean Curvature

Let $f: M^n \to E^{n+p}$ be an immersion of the closed manifold M^n into E^{n+p}. The total mean curvature of M^n with respect to f is defined by $\mathrm{TM}(f) = \int_{M^n} H^n\, dV$ and is a conformal invariant when $n = 2$. An interesting and outstanding problem is to characterize those immersions that minimize the functional $\mathrm{TM}(f)$. There are many results obtained on this problem. Chen [C4] proved that $\int_{M^n} H^n\, dV \ge c_n$, where the equality holds if and only if M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+p} when $n > 1$. He also obtained a lower bound of $\mathrm{TM}(f)$ in terms of the Betti number of M^n for f with $R \ge 0$ (see e.g. [C6]).

In this section, we obtain a sharp estimate of the total mean curvature for M^n with $R \ge 0$ in terms of R. For this purpose, we first prove the following lemma.

LEMMA 3.1. *Let M^n be a submanifold in E^{n+p}. If $\lambda_1(x) \ge 0$ at a point $x \in M^n$, then $H^2(x) \ge \lambda_1(x)$, where the equality sign holds at x when and only when x is a pseudo-umbilical point of M^n.*

Proof. Let $(x; e_1, \dots, e_n; \bar{e}_{n+1}, \dots, \bar{e}_{n+p})$ be a Frenet–Ōtsuki frame in a neighborhood of x. It follows that $n(n-1)\lambda_\alpha = 2\sum_{i<j}[h_{ii}^{n+\alpha}h_{jj}^{n+\alpha} - (h_{ij}^{n+\alpha})^2]$ for every α. Hence

$$\sum_{\alpha>1}\sum_{i,j}(h_{ij}^{n+\alpha})^2 = \sum_{\alpha>1}(nH_{n+\alpha})^2 - n(n-1)\sum_{\alpha>1}\lambda_\alpha.$$

On the other hand, we have

$$\sum_i (h_{ii}^{n+1})^2 = \frac{1}{n-1}\left\{\sum_{i<j}(h_{ii}^{n+1} - h_{jj}^{n+1})^2 + 2\sum_{i<j}h_{ii}^{n+1}h_{jj}^{n+1}\right\}.$$

Therefore,

$$\sum_{i,j}(h_{ij}^{n+1})^2 = n\lambda_1 + \frac{1}{n-1}\sum_{i<j}(h_{ii}^{n+1} - h_{jj}^{n+1})^2 + \frac{2n}{n-1}\sum_{i<j}(h_{ij}^{n+1})^2.$$

Using the fact $\sum_\alpha \lambda_\alpha = R$, we obtain

$$\begin{aligned} S &= \sum_{i,j}(h_{ij}^{n+1})^2 + \sum_{\alpha>1}\sum_{i,j}(h_{ij}^{n+\alpha})^2 \\ &= n^2\lambda_1 + \frac{2n}{n-1}\sum_{i<j}(h_{ij}^{n+1})^2 + \frac{1}{n-1}\sum_{i<j}(h_{ii}^{n+1} - h_{jj}^{n+1})^2 \\ &\quad + \sum_{\alpha>1}(nH_{n+\alpha})^2 - n(n-1)R. \end{aligned}$$

It is known that $n(n-1)R = n^2H^2 - S$. Therefore,

$$\begin{aligned} n^2H^2 = n^2\lambda_1 &+ \frac{2n}{n-1}\sum_{i<j}(h_{ij}^{n+1})^2 \\ &+ \frac{1}{n-1}\sum_{i<j}(h_{ii}^{n+1} - h_{jj}^{n+1})^2 + \sum_{\alpha>1}(nH_{n+\alpha})^2. \end{aligned} \tag{3.1}$$

From (3.1) we have $H^2 \geq \lambda_1$ and that $H^2 = \lambda_1$ if and only if $h^{n+1}_{ii} = h^{n+1}_{jj}$, $h^{n+1}_{ij} = 0$ for all $i \neq j$, and $H_{n+\alpha} = 0$ for all $\alpha > 1$. Hence $\bar{e}_{n+1}$ has the same direction as ξ and $\bar{L}_{n+1} = HI_n$, which means that x is a pseudo-umbilical point. □

REMARK 3.1. Chen [C5] has obtained Lemma 3.1 for $n = 2$.

Using Proposition 2.2, Lemma 2.2, and Lemma 3.1, we get the main result of this paper.

THEOREM 3.1. *Let M^n be a closed submanifold in E^{n+p} with $R \geq 0$. Then*

$$\int_{M^n} H^n \, dV \geq 2\kappa_n c_{n-1} + \left\{1 - 2\kappa_n\left(\frac{c_{n-1}}{c_n}\right)\right\} \int_{M^n} R^{n/2} \, dV,$$

and the equality holds when and only when M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+p}.

Let us conclude this section by giving another application of Proposition 2.2.

PROPOSITION 3.1. *Let M^n be a closed submanifold in E^{n+p} with $R \geq 0$. Suppose that $\{\lambda_\alpha\}$ are the αth scalar curvatures of M^n. Then*

$$\int_{M^n} H^n \, dV \geq \frac{\kappa_n c_{n-1}}{c_{n+p-1}} \int_{M^n} K_2^*(x) \, dV + \left\{1 - 2\kappa_n\left(\frac{c_{n-1}}{c_n}\right)\right\} \int_{M^n} R^{n/2} \, dV, \quad (3.2)$$

where $\kappa_n = (\sqrt{\pi}/4)\Gamma\left(\frac{n}{4}\right)\big/\Gamma\left(\frac{n+2}{4}\right)$. Moreover, the equality in (3.2) holds if and only if either (a) M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+p} or (b) M^n is pseudo-umbilical with $R = 0$ and $\lambda_\alpha = 0$ $(2 \leq \alpha \leq p-1)$.

Proof. Note that (3.2) follows from (2.14) and Lemma 3.1. Suppose that the equality in (3.2) holds. It follows from Lemma 3.1 that M^n is pseudo-umbilical.

Consider $M_0 = \{\, x \in M^n \mid R(x) > 0 \,\}$. From the proof of Proposition 2.2, we have $\lambda_2 = \cdots = \lambda_p = 0$ in M_0. Thus M_0 is totally umbilical. From a result in [C6, p. 50], we have that every connected component of M_0 is of constant curvature. Hence R is constant in M_0. Therefore $M_0 = M^n$ or $\emptyset$ since R is continuous on M^n.

If $M_0 = M^n$, then M^n is of constant curvature. Thus M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+p}. If $M_0 = \emptyset$, then $R = 0$ on M^n. Moreover, it follows from Proposition 2.2 that $\lambda_\alpha = 0$ $(2 \leq \alpha \leq p-1)$ on M^n.

The converse is clear. This completes the proof of Proposition 3.1. □

Let M^2 be a closed surface in E^{2+p}. Then the normalized scalar curvature R is just the Gauss curvature and $\mathrm{TA}_2(f)$ is precisely the total absolute curvature. Using Proposition 3.1 and Lemma 2.1, we can prove the following corollaries.

COROLLARY 3.1. *Let M^2 be a closed surface in E^{2+p} with nonnegative Gauss curvature. If $\int_{M^2} H^2 \, dV \leq (2+\pi)\pi$, then M^2 is homeomorphic to a 2-sphere.*

COROLLARY 3.2 [C8]. *Let M^2 be a closed flat surface in a Euclidean space E^{2+p} with $p \geq 2$. Then $\int_{M^2} H^2\,dV \geq 2\pi^2$. The equality holds when and only when M^2 is imbedded as a Clifford torus $T^2 = S^1(a) \times S^1(a) \subset E^4 \subset E^{2+p}$, where $S^1(a)$ is a plane circle of radius a.*

Proof of Corollary 3.1 and 3.2. From Proposition 3.1, we have

$$\int_{M^2} H^2\,dV \geq \frac{\pi^2}{2} \cdot \left(\frac{1}{c_{p+1}} \int_{M^2} K_2^*\,dV\right) + \left(1 - \frac{\pi}{4}\right) \int_{M^2} R\,dV, \tag{3.3}$$

where the equality holds when and only when either M^2 is imbedded as a 2-sphere or $R = 0$ on M^2. On the other hand, from Lemma 2.1 and Gauss–Bonnet formula, we have

$$\frac{1}{c_{p+1}} \int_{M^2} K_2^*\,dV \geq \mu(M^2) \geq \beta(M^2), \qquad \int_{M^2} R\,dV = 2\pi\chi(M^2). \tag{3.4}$$

If R is not identically equal to zero, then M^2 is homeomorphic to a real projective plane RP^2 or a unit sphere S^2. If M^2 is homeomorphic to RP^2, then $\chi(M^2) = 1$ and $\mu(M^2) \geq 3$. From (3.3) and (3.4), we have

$$\int_{M^2} H^2\,dV > (\pi + 2)\pi. \tag{3.5}$$

If $R = 0$ on M^2, then M^2 is homeomorphic to a torus T^2 or a Klein bottle K^2. In this case, $\chi(M^2) = 0$ and $\mu(M^2) \geq 4$. From (3.3) and (3.4), we have

$$\int_{M^2} H^2\,dV \geq 2\pi^2. \tag{3.6}$$

Therefore, Corollary 3.1 follows from (3.5) and (3.6).

Let us prove Corollary 3.2. The first part of Corollary 3.2 follows from (3.6). Suppose that the equality in (3.6) holds. It follows from Lemma 3.1 that M^2 is pseudo-umbilical. Using a result of Chen [C7], we get the second part of Corollary 3.2. The converse is trivial. This completes the proof. □

REMARK 3.2. Chen has proved Corollary 3.1 for $p = 2$ in [C5] and for M^2 being pseudo-umbilical in [C7].

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