Quasiconformally Homogeneous Compacta in the Complex Plane

Paul MacManus, Raimo Näkki, & Bruce Palka

To F. W. Gehring on the occasion of his 70th birthday

1. Introduction

This paper summarizes some of the findings of the co-authors in their investigation of sets E in the extended complex plane $\hat{\mathbb{C}}$ that exhibit a high degree of symmetry with respect to the action of the quasiconformal group. Our focus is exclusively on compact subsets E of $\hat{\mathbb{C}}$. A situation roughly dual to this one, the case of quasiconformally homogeneous domains, was studied in [GP] and [Sa].

We begin by establishing some convenient notation and terminology. The symbol $\mathcal{T}$ stands for the group of sense-preserving homeomorphisms of $\hat{\mathbb{C}}$ to itself; $\mathcal{Q}$ signifies the subgroup of $\mathcal{T}$ that comprises all the quasiconformal self-mappings of $\hat{\mathbb{C}}$; for $1 \leq K < \infty$, $\mathcal{Q}_K$ denotes the family of mappings in $\mathcal{Q}$ that are K-quasiconformal. (N.B. We observe accepted convention in requiring as part of the definition of a plane quasiconformal mapping that it be sense-preserving, although orientation will not be a serious concern in what follows.) The family $\mathcal{Q}_1$, be it noted, is nothing other than the classical Möbius group, the group of linear fractional transformations of $\hat{\mathbb{C}}$. By contrast, when $K > 1$ the family $\mathcal{Q}_K$ is not closed under composition and so does not constitute a group. For each nonempty subset E of $\hat{\mathbb{C}}$ we write

$$\mathcal{T}(E) = \{ f \in \mathcal{T} : f(E) = E \}, \quad \mathcal{Q}(E) = \mathcal{Q} \cap \mathcal{T}(E), \quad \mathcal{Q}_K(E) = \mathcal{Q}_K \cap \mathcal{T}(E).$$

Thus $\mathcal{T} = \mathcal{T}(\hat{\mathbb{C}})$, $\mathcal{Q} = \mathcal{Q}(\hat{\mathbb{C}})$ and $\mathcal{Q}_K = \mathcal{Q}_K(\hat{\mathbb{C}})$.

A nonempty subset E of $\hat{\mathbb{C}}$ is said to be *quasiconformally homogeneous* (resp., *K-quasiconformally homogeneous*) if the action on E of the group $\mathcal{Q}(E)$ (resp., the family $\mathcal{Q}_K(E)$) is transitive: for each pair of points a and b of E there exists a mapping f in $\mathcal{Q}(E)$ (resp., in $\mathcal{Q}_K(E)$) such that $f(a) = b$. Since $\mathcal{Q}_K(E)$ is not a group when $K > 1$, this definition does entail a slight departure from the standard meaning of "action." The expression "conformally homogeneous" will be employed as a preferred synonym for "1-quasiconformally homogeneous." We define the *index of quasiconformal homogeneity* $\mathcal{K}(E)$ of a quasiconformally homogeneous set E in $\hat{\mathbb{C}}$ by

Received January 2, 1997. Revision received August 18, 1997.
The work of the first author was partially supported by NSF grant 9305792.
Michigan Math. J. 45 (1998).

$$\mathcal{K}(E) = \inf\{K : E \text{ is } K\text{-quasiconformally homogeneous}\}.$$

Then $1 \le \mathcal{K}(E) \le \infty$. We stress that $\mathcal{K}(E)$ is assigned meaning only for sets E that are assumed from the outset to be quasiconformally homogeneous. We also point out that the definition of $\mathcal{K}(E)$ leaves open the possibility of $\mathcal{K}(E)$ being finite yet E not being K-quasiconformally homogeneous for $K = \mathcal{K}(E)$. For want of a better name we label a nonempty set E in $\hat{\mathbb{C}}$ *topologically homogeneous* when the group $\mathcal{T}(E)$ acts transitively on E. We alert the reader to the fact that this definition is at variance with the common usage of the term "homogeneous" in topology, where homogeneity for a plane set E usually makes reference to the action on E of its full internal homeomorphism group, not merely to the action on E of $\mathcal{T}(E)$.

We use $S(z_0, r)$ and $B(z_0, r)$ to denote the circle and open disk in $\mathbb{C}$, respectively, that are centered at z_0 and have radius r, abbreviating $S(0, 1)$ to S and $B(0, 1)$ to B. We remind the reader that a Jordan curve J in $\hat{\mathbb{C}}$ is termed a *quasicircle* (resp., *K-quasicircle*) under the condition that $J = f(S)$ for some f from $\mathcal{Q}$ (resp., from $\mathcal{Q}_K$). In particular, J is a 1-quasicircle if and only if J is a *circle in* $\hat{\mathbb{C}}$, meaning that J is either a true Euclidean circle in $\mathbb{C}$ or a set of the type $J = L \cup \{\infty\}$, where L is a Euclidean line in $\mathbb{C}$. Similarly, a domain D in $\hat{\mathbb{C}}$ is designated a *quasidisk* (resp., *K-quasidisk*) provided $D = f(B)$ for some f from $\mathcal{Q}$ (resp., $\mathcal{Q}_K$). Obviously D is a quasidisk precisely when ∂D is a quasicircle, an analogous statement expressing the relation of K-quasidisk to K-quasicircle. A 1-quasidisk is therefore an *open disk in* $\hat{\mathbb{C}}$: under the umbrella of this term are included open Euclidean disks in $\mathbb{C}$, open Euclidean half-planes in $\mathbb{C}$, and the complements in $\hat{\mathbb{C}}$ of closed Euclidean disks in $\mathbb{C}$.

Let E be a K-quasicircle or a K-quasidisk in $\hat{\mathbb{C}}$. With the aid of elementary properties of Möbius transformations, it is a simple matter to check that $\mathcal{K}(E) \le K^2$. Indeed, if E is a K-quasicircle then the family $\mathcal{Q}_{K^2}(E)$ contains a group whose action on E is triply transitive. The purpose of this paper is to explore implications in the opposite direction, that is, to determine features that are imposed upon a compact subset E of $\hat{\mathbb{C}}$ by virtue of homogeneity with respect to the action of $\mathcal{Q}(E)$ or $\mathcal{Q}_K(E)$. Our point of departure is a beautiful theorem discovered by Erkama [E] (see also [BE]): *a Jordan curve J in $\hat{\mathbb{C}}$ is quasiconformally homogeneous if and only if J is a quasicircle.* In order to place the discussion of quasiconformally homogeneous compacta into context, we briefly review the situation for the well-understood counterparts of such sets in the topological and conformal categories.

ACKNOWLEDGMENT. The authors would like to thank Pertti Mattila for a number of constructive and illuminating suggestions.

2. The Topological and Conformal Cases

The topologically homogeneous compacta in the extended complex plane are completely classified by the following theorem.

THEOREM 2.1. *A compact subset E of $\hat{\mathbb{C}}$ is topologically homogeneous if and only if one of the following is true*: (i) $E = \hat{\mathbb{C}}$; (ii) *E is a finite set of points*; (iii) *E is the union of a finite collection of Jordan curves that constitute the boundary components of a domain in $\hat{\mathbb{C}}$*; (iv) *E is a Cantor set*.

It has long been known that any subset E of $\hat{\mathbb{C}}$ that falls into one of the categories listed in this theorem is topologically homogeneous. The fact that this list is actually exhaustive follows without difficulty from the Baire category theorem and a result of Burgess [Bu, Thm. 3], which identifies Jordan curves as the only continua E in $\hat{\mathbb{C}}$ with the property that $\mathcal{T}(E)$ acts transitively on E.

The expected analog of the Burgess theorem in the setting of conformal homogeneity is common knowledge to mathematicians conversant with Lie groups and their homogeneous spaces—for instance, it is a straightforward consequence of the classification of Lie subgroups of the Möbius group, say as found in [G]—but this pretty result remains surprisingly obscure among geometric function theorists.

THEOREM 2.2. *A continuum E in $\hat{\mathbb{C}}$ is conformally homogeneous if and only if E is a circle in $\hat{\mathbb{C}}$.*

By combining Theorems 2.1 and 2.2 with Theorem 9.18 in [Sp], one arrives easily at a partial classification of the conformally homogeneous compact subsets of $\hat{\mathbb{C}}$.

THEOREM 2.3. *If a compact set E in $\hat{\mathbb{C}}$ is conformally homogeneous, then one of the following is true*: (i) $E = \hat{\mathbb{C}}$; (ii) *E is a finite set of points*; (iii) *E is a circle in $\hat{\mathbb{C}}$*; (iv) *E is the disjoint union of two circles in $\hat{\mathbb{C}}$.*

Not all finite subsets of $\hat{\mathbb{C}}$ are homogeneous with respect to the action of the Möbius group, of course, so elaboration on case (ii) of Theorem 2.3 is necessary if the classification of conformally homogeneous compacta in $\hat{\mathbb{C}}$ is to be completed. (It is not hard to check that any set of type (i), (iii), or (iv) in Theorem 2.3 is conformally homogeneous.) A bit of notation and terminology will facilitate the statement of the desired result, the proof of which is a straightforward application of the ideas discussed in [B2, pp. 84–86]. For each positive integer n, U_n indicates the set of complex nth-roots of unity. A regular polyhedron P inscribed in the unit sphere S^2 in $\mathbb{R}^3$ is said to be *in standard position* provided that $(0, 0, 1)$ is a vertex of P and that some vertex of P adjacent to $(0, 0, 1)$ corresponds under stereographic projection from $(0, 0, 1)$ to a point of the positive real axis in the complex plane.

THEOREM 2.4. *Let E be a subset of $\hat{\mathbb{C}}$ of finite cardinality $n \geq 1$. Then E is conformally homogeneous if and only if E is Möbius equivalent to a subset E_0 of $\hat{\mathbb{C}}$ that belongs to one of the following categories*: (i) *$n \leq 3$ and E_0 is a subset of* $\{0, 1, \infty\}$; (ii) *$n = 4$ and $E_0 = \{0, 1, \infty, \lambda\}$ for some real number λ different from* 0 *and* 1; (iii) *$n \geq 5$ and $E_0 = U_n$*; (iv) *$n \geq 6$ is even and $E_0 = U_m \cup cU_m$, where $m = n/2$ and c is a nonzero complex number such that $c^m \neq 1$*; (v) $n = 6$, 8, 12, *or* 20 *and* E_0 *is the set of points that correspond under stereographic projection to the* n *vertices of a regular polyhedron inscribed in standard position in the sphere* S^2.

3. The Quasiconformal Case

We open the discussion with the observation that any quasiconformally homogeneous compact subset of $\hat{\mathbb{C}}$ automatically has a finite index of quasiconformal homogeneity.

THEOREM 3.1. *If a compact set* E *in* $\hat{\mathbb{C}}$ *is quasiconformally homogeneous, then it must be the case that* $\mathcal{K}(E) < \infty$.

Proof. Theorem 2.1 delineates the possible topological structures that E can exhibit. Since the present theorem is plainly true when $E = \hat{\mathbb{C}}$ or E is a finite set of points, we need only concern ourselves with alternatives (iii) and (iv) in Theorem 2.1. We assume, as we may, that E lies in the finite plane. Moreover, in the situation where E is a union of Jordan curves, we are free to suppose that ∞ lies in the domain whose boundary components are provided by the curves in question and, if E is itself a Jordan curve, that the origin belongs to the bounded component of E^c, the complement of E in $\hat{\mathbb{C}}$. When E is a Jordan curve, we use $\mathcal{F}$ to signify the group of mappings in $\mathcal{Q}(E)$ that leave the set $\{0, \infty\}$ invariant; in all other cases, we take $\mathcal{F}$ to mean the subgroup of $\mathcal{Q}(E)$ consisting of those mappings that fix the point ∞. In each instance, $\mathcal{F}$ acts transitively on E, a fact readily confirmed with the aid of [N, Lemma 1.14]. For $n = 1, 2, \ldots$ we define $\mathcal{F}_n = \{ f \in \mathcal{F} : K(f) \le n \}$, where $K(f)$ designates the maximal dilatation of a quasiconformal mapping f.

Now fix a point z_0 of E and let $E_n = \mathcal{F}_n(z_0)$, the orbit of z_0 under the action of $\mathcal{F}_n$. Then $E = \bigcup_{n=1}^{\infty} E_n$ and, in conjunction with well-known compactness and convergence properties of quasiconformal mappings (see [V1]), the normalizations imposed on members of $\mathcal{F}$ ensure that each of the sets E_n is closed. By the Baire category theorem, at least one of these sets, say E_N, has an interior point w_0 in the relative topology of E. Fix an open disk $U = B(w_0, r)$ such that $U \cap E$ is contained in E_N. The collection of sets $f(U)$ with f in $\mathcal{Q}(E)$ is an open covering of E, so we can find a finite number of mappings $f_1, f_2, \ldots, f_p$ in $\mathcal{Q}(E)$ such that $f_1(U), f_2(U), \ldots, f_p(U)$ cover E. It follows that any point z of E can be expressed as $z = f_j \circ f(z_0)$ for some j and some f in $\mathcal{F}_N$, from which we infer that $\mathcal{K}(E) \le K^2 N^2$ with $K = \max K(f_j)$. $\square$

The kind of restriction imposed on a compact set as a result of quasiconformal homogeneity is typified by the following theorem, wherein the notation $\mathcal{H}$-dim(E) indicates the Hausdorff dimension of a subset E of $\hat{\mathbb{C}}$ with respect to the chordal metric. Because Möbius transformations are bi-Lipschitz mappings in the chordal metric, $\mathcal{H}$-dim(E) is a Möbius invariant. Of course, if E lies in the finite plane then $\mathcal{H}$-dim(E) coincides with the usual Hausdorff dimension of E relative to the Euclidean metric.

THEOREM 3.2. *If a compact proper subset E of $\hat{\mathbb{C}}$ is K-quasiconformally homogeneous, then $\mathcal{H}$-dim$(E) \le d < 2$, where d depends only on K.*

Proof. We may assume that E is a set in the finite plane. We first show that E is locally α-porous at each of its points, where $\alpha = (1/2)\exp(-2\pi K)$. (Recall: if $0 < \alpha \le 1/2$ then a subset A of $\mathbb{C}$ is *locally α-porous at the point z of A* provided there exists a number $\rho = \rho(z) > 0$ such that, for each r in the interval $(0, \rho]$, the set $B(z, r) \setminus A$ contains some open disk of radius αr.)

Let z_0 be a point of E for which $\operatorname{Re} z_0 = \max\{\operatorname{Re} z : z \in E\}$. (The set E obviously satisfies the stated porosity requirement at z_0—in fact, E is locally $(1/2)$-porous at z_0, and one can take $\rho(z_0)$ to be any positive number.) Consider an arbitrary point z of E. By assumption, the family $\mathcal{Q}_K(E)$ includes a mapping f such that $f(z_0) = z$. It is known that the linear dilatation H_f of a K-quasiconformal self-mapping f of $\hat{\mathbb{C}}$ is bounded above by $H = e^{\pi K}$ (see [LV, Thm. II.9.2]). We remind the reader that, for w different from ∞ and $f^{-1}(\infty)$, $H_f(w)$ is defined by

$$H_f(w) = \limsup_{r\to 0} \frac{L_f(w, r)}{l_f(w, r)},$$

in which

$$L_f(w, r) = \max_{|h|=r} |f(w + h) - f(w)|, \qquad l_f(w, r) = \min_{|h|=r} |f(w + h) - f(w)|$$

for $0 < r < |w - f^{-1}(\infty)|$. Take $\tau > 0$ small enough that $f(\infty)$ does not lie in the closure of $B(z, 4\tau)$. The proof of the theorem in [LV] just cited actually demonstrates that

$$\frac{L_f(w, s)}{l_f(w, s)} \le H$$

whenever $|w - z_0| \le \sigma$ and $0 < s \le \sigma$, with $\sigma = (1/2)\operatorname{dist}(z_0, f^{-1}[S(z, \tau)])$. We now choose $\rho = \rho(z)$ in $(0, \tau)$ so that the distance from z_0 to the Jordan curve $f^{-1}[S(z, \rho)]$ is smaller than σ.

Let $0 < r \le \rho$ and let $w_0 = z_0 + (s/2)$, where s is the distance from z_0 to $f^{-1}[S(z, r)]$. Then $0 < s \le \sigma$. The set $A = f[S(w_0, s/2)]$ contains $z = f(z_0)$ as well as some point of $f[S(z_0, s)]$, so A has diameter $d(A)$ no smaller than $l_f(z_0, s)$. But

$$\frac{r}{l_f(z_0, s)} = \frac{L_f(z_0, s)}{l_f(z_0, s)} \le H,$$

from which we conclude that

$$L_f(w_0, s/2) \ge \frac{d(A)}{2} \ge \frac{l_f(z_0, s)}{2} \ge \frac{r}{2H}.$$

Since $|w_0 - z_0| \le \sigma$, this leads to

$$l_f(w_0, s/2) \geq \frac{L_f(w_0, s/2)}{H} \geq \frac{r}{2H^2},$$

ensuring that $B(z, r) \setminus E$ contains the disk $B[f(w_0), r/(2H^2)]$. As r in $(0, \rho]$ was arbitrary, this confirms the local α-porosity of E at z for $\alpha = 1/(2H^2) = (1/2)\exp(-2\pi K)$.

Finally, a simple geometric argument (see e.g. [S, p. 353]) reveals that $\mathcal{H}$-dim$(E) \leq d < 2$, where d depends only on α—hence, depends only on K. □

The next result dramatizes the extent to which information is refined through passage from the realm of topological homogeneity to that of quasiconformal homogeneity.

THEOREM 3.3. *If a compact subset E of $\hat{\mathbb{C}}$ is quasiconformally homogeneous, then one of the following is true*: (i) $E = \hat{\mathbb{C}}$; (ii) *E is a finite set of points*; (iii) *E is the union of a finite collection of quasicircles that constitute the boundary components of a domain in $\hat{\mathbb{C}}$*; (iv) *E is a Cantor set with* $\mathcal{H}$-dim$(E) \leq d < 2$, *where d depends only on $\mathcal{K}(E)$.*

Proof. Assume that E is described by neither (i) nor (ii). On the basis of Theorem 2.1 we can assert that either E is the union of finitely many Jordan curves which are the boundary components of a domain in $\hat{\mathbb{C}}$ or E is a Cantor set. In the former case, each component of E is quasiconformally homogeneous and, in view of the main result in [E], is thus a quasicircle. Theorem 3.2 justifies the statement dealing with the Cantor set alternative. □

Again it is a straightforward matter to confirm that any set E of type (i), (ii), or (iii) is quasiconformally homogeneous. Sets of type (iv) are a different matter entirely: that quasiconformally homogeneous Cantor sets exist in $\hat{\mathbb{C}}$ is by no means self-evident. To show that category (iv) is not vacuous, we shall construct such a Cantor set whose Hausdorff dimension is any prescribed number in $[0, 2)$. To do this we employ a standard procedure for constructing self-similar Cantor sets, into which we introduce an element of quasiconformal "mixing." We refer to the mechanism that underlies the mixing process as "conformal exchange." To be precise, consider a domain G in $\hat{\mathbb{C}}$ and disjoint subdomains D_1 and D_2 of G. We call a quasiconformal self-mapping σ of $\hat{\mathbb{C}}$ a *conformal exchange of D_1 and D_2 relative to G* provided that σ fixes each point of G^c while mapping D_1 conformally onto D_2 and D_2 conformally onto D_1. If σ is such a mapping and f is a Möbius transformation, then $f \circ \sigma \circ f^{-1}$ is clearly a conformal exchange of $D_1' = f(D_1)$ and $D_2' = f(D_2)$ relative to $G' = f(G)$. In conjunction with this observation, the following elementary lemma will provide all of the exchange maps that we require for this paper.

LEMMA 3.4. *Let G be a quasidisk in $\hat{\mathbb{C}}$ that is symmetric with respect to both the real and imaginary axes and has for its intersection with the former the interval $I = (-1, 1)$. Then $G_1 = \{z \in G : \operatorname{Im} z > 0\}$ and $G_2 = \{z \in G : \operatorname{Im} z < 0\}$ are*

Jordan domains and $G = G_1 \cup I \cup G_2$. If D_1 is a subdomain of G_1 that is symmetric with respect to the imaginary axis and is relatively compact in $G_1 \cup I$ and if $D_2 = \{\bar{z} : z \in D_1\}$, then there exists a quasiconformal self-mapping σ of $\hat{\mathbb{C}}$ such that $\sigma(z) = z$ for every z in G^c and $\sigma(z) = -z$ for every z in $D_1 \cup D_2$. In particular, σ maps D_1 conformally onto D_2 and vice versa, so σ effects a conformal exchange of D_1 and D_2 relative to G.

Proof. Let g_1 be the unique homeomorphism of $\bar{G}_1$ onto the closure of the half-disk $G_1' = \{z \in B : \operatorname{Im} z > 0\}$ that maps G_1 conformally to G_1' and fixes the points -1, 0, and 1. Extend g_1 by reflection to a homeomorphism g of $\bar{G}$ onto $\bar{B}$. Then g maps G conformally onto B. Moreover, the normalization of g_1 and the symmetry assumptions ensure that $g(-z) = -g(z)$ for every z in $\bar{G}$. Fix a quasiconformal extension h of g to $\hat{\mathbb{C}}$. Next, noting that $g(\bar{D}_1 \cup \bar{D}_2)$ is a compact subset of B, choose a quasiconformal mapping σ_0 of $\hat{\mathbb{C}}$ such that $\sigma_0(z) = z$ for each z in B^c and $\sigma_0(z) = -z$ whenever $|z| \le r = \max\{|z| : z \in g(\bar{D}_1 \cup \bar{D}_2)\}$. (For example, take $\sigma_0(z) = ze^{i\pi\varphi(|z|)}$, where $\varphi\colon \mathbb{R} \to [0, 1]$ is a C^∞-function such that $\varphi(t) = 1$ when $t \le r$, $\varphi(t) = 0$ when $t \ge 1$, and $\varphi'(t) < 0$ when $r < t < 1$.) Then $\sigma = h^{-1} \circ \sigma_0 \circ h$ is a mapping that fulfills all the stipulated requirements. □

THE CANTOR SET E_s. Fix a number s satisfying $0 < s < 1/2$. Write

$$Q = \{z \in \mathbb{C} : |\operatorname{Re} z| \le 1,\ \ |\operatorname{Im} z| \le 1\},$$

and for $1 \le k \le 4$ denote by $Q_k = Q_k(s)$ the closed square of side-length $2s$ in the kth quadrant H_k of the complex plane that is concentric with and has the same orientation as the square $Q \cap H_k$. If c_k is the center of Q_k, then Q_k is the image of Q under the similarity transformation $g_k(z) = 2sz + c_k$. For $n \ge 1$ and for $1 \le k_1, k_2, \ldots, k_n \le 4$, we set $Q_{k_1,k_2,\ldots,k_n} = g_{k_1} \circ g_{k_2} \circ \cdots \circ g_{k_n}(Q)$ and let $E_{s,n}$ signify the union of the 4^n disjoint closed squares of side-length $(2s)^n$ thus obtained. Then $E_s = \bigcap_{n=1}^\infty E_{s,n}$ is a Cantor set whose Hausdorff dimension is given by

$$\mathcal{H}\text{-dim}(E_s) = \frac{2\log\frac{1}{2}}{\log s}$$

(see e.g. [B1]), which ranges over the interval $(0, 2)$ as s varies over $(0, 1/2)$. By allowing the size of the squares that arise in this process to shrink at a more rapid rate, one can construct even “smaller” Cantor sets E, sets with $\mathcal{H}\text{-dim}(E) = 0$ or even with $\operatorname{cap}(E) = 0$, where “cap” indicates either logarithmic or conformal capacity. (In the plane, the collections of null sets for the two capacities are identical.) In particular, we denote by E_0 the Cantor set obtained for $s = 1/4$ when the 4^n squares that make up $E_{1/4,n}$ are replaced by squares that have the same centers and orientation as before but with side-length $2^{-8^{n-1}}$ instead of 2^{-n}. Denoting the union of these squares by $E_{0,n}$, we have $E_0 = \bigcap_{n=1}^\infty E_{0,n}$. It is not difficult to verify that E_0 has conformal capacity zero (which implies that $\mathcal{H}\text{-dim}(E_0) = 0$ as well). Each point z of E_s can be identified by a unique “address” $\omega(z) = (k_1, k_2, \ldots)$, an infinite sequence from the set $\{1, 2, 3, 4\}$, where k_j is the number of the square in the initial configuration $E_{s,1}$ that corresponds to the square into which z falls when passing from $E_{s,n-1}$ to $E_{s,n}$. We use $\omega_n(z)$ as an abbreviation for the "partial address" $(k_1, k_2, \dots, k_n)$ of z.

EXAMPLE 3.5. *For each s in $[0, 1/2)$, the Cantor set E_s is quasiconformally homogeneous and, when $0 < s < 1/2$, is K-quasiconformally homogeneous for K depending only on s.*

Proof. The argument is a variation on the proof of [GV, Thm. 5]. We initially consider s in $(0, 1/2)$ and fix for $1 \le l < k \le 4$ a quasidisk G_{kl} in Q that contains Q_k and Q_l, as indicated in Figure 1. (The situation when Q_k and Q_l are horizontally aligned is the obvious analog of the one for the case of vertical alignment.)

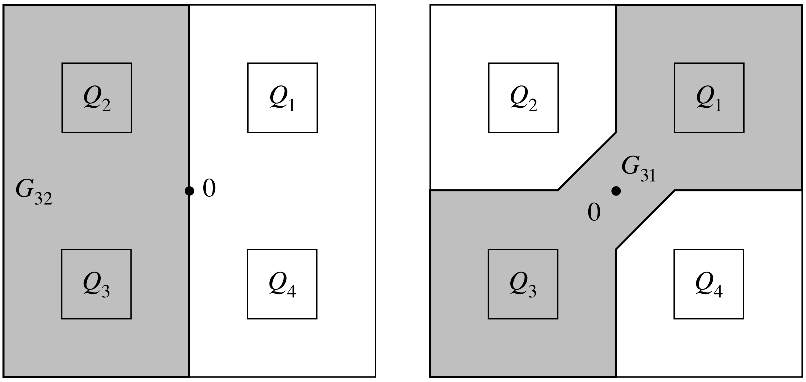

Figure 1

Because the configuration of Q_k, Q_l, and G_{kl} has two orthogonal lines of symmetry, it is the image under a similarity transformation h of a configuration meeting the hypotheses of Lemma 3.4. If we fix a mapping σ that satisfies the conclusion of Lemma 3.4 for said configuration, then $\sigma_{kl} = h \circ \sigma \circ h^{-1}$ furnishes a conformal exchange of the interiors of Q_k and Q_l relative to G_{kl}. We set $\sigma_{kl} = \sigma_{lk}$ for $1 \le k < l \le 4$, take $\sigma_{kk} = \text{id}$ for $1 \le k \le 4$, and define

$$K = K_s = \max\{ K(\sigma_{kl}) : 1 \le k, l \le 4 \}.$$

Now let a and b be points of E_s, say with addresses $\omega(a) = (k_1, k_2, \dots)$ and $\omega(b) = (l_1, l_2, \dots)$. We recursively define a sequence $\langle f_n \rangle$ of mappings from $\mathcal{Q}_K$ by $f_1 = \sigma_{k_1 l_1}$ and

$$f_{n+1} = f_n \circ g_{k_1} \circ g_{k_2} \circ \cdots \circ g_{k_n} \circ \sigma_{k_{n+1} l_{n+1}} \circ (g_{k_1} \circ g_{k_2} \circ \cdots \circ g_{k_n})^{-1}$$

for $n \ge 1$. Then f_n maps $E_{s,n}$ onto itself, transforming the interior of the square $Q_n = Q_{\omega_n(a)}$ conformally to the interior of $Q_n' = Q_{\omega_n(b)}$, and $f_{n+1} = f_n$ in Q_n^c. Since $K(f_1) = K(\sigma_{k_1 l_1}) \le K$ and

$$K(f_{n+1}) \le \max\{K(f_n), K(\sigma_{k_{n+1}l_{n+1}})\} \le \max\{K(f_n), K\},$$

it follows by induction that f_n belongs to $\mathcal{Q}_K$ for each n. The sequence $\langle f_n \rangle$ is drawn from the normal family $\{ f \in \mathcal{Q}_K : f(z) = z \text{ for each } z \in Q^c \}$ and plainly converges pointwise on E_s^c, a dense subset of $\hat{\mathbb{C}}$. Standard convergence theorems for quasiconformal mappings thus ensure that $f_n \to f$ uniformly on $\hat{\mathbb{C}}$ (with respect to the chordal metric), where f is a member of $\mathcal{Q}_K$. Because $f_m(E_{s,n}) = E_{s,n}$ and $f_m(Q_n) = Q_n'$ whenever $m \ge n$, we conclude that f preserves $E_{s,n}$ for each n, mapping Q_n to Q_n', from which we then infer that $f(E_s) = E_s$ and $f(a) = b$. But a and b were arbitrary points of E_s, so $\mathcal{K}(E_s) \le K$.

Finally, the case $s = 0$ does not really demand a separate discussion. Indeed, our construction is such that the mapping f_n introduced to treat a pair of points a and b from $E_{1/4}$ leaves invariant the subset $E_{0,n}$ of $E_{1/4,n}$ and maps the square in $E_{0,n}$ which contains a to the one which contains b, whenever a and b are elements of E_0. Thus, in this situation, the limit mapping f actually belongs to $\mathcal{Q}_K(E_0)$ for $K = K_{1/4}$. □

A second example shows that the existence of a K-quasiconformally homogeneous Cantor set for $K > 1$ does not place any restriction on K.

EXAMPLE 3.6. *For each $K > 1$ there exists a K-quasiconformally homogeneous linear Cantor set E in the complex plane.*

Proof. In this instance we tailor a Cantor set to fit a prescribed conformal exchange. For $t > 0$ the mapping $f_t \colon B \to B$ defined by $f_t(z) = z|z|^{2\pi i t}$ for $z \ne 0$ and $f_t(0) = 0$ is K_t-quasiconformal, where K_t satisfies

$$\frac{K_t - 1}{K_t + 1} = \frac{2\pi t}{\sqrt{1 + 4\pi^2 t^2}}.$$

In particular, K_t increases from 1 to ∞ as t increases from 0 to ∞. Given $K > 1$, let $t > 0$ be such that $K_t = K$, set $r = e^{-1/(2t)}$, and define $\sigma : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by $\sigma(z) = z$ if $|z| \ge 1$, by $\sigma(z) = f_t(z)$ if $r < |z| < 1$, and by $\sigma(z) = -z$ if $|z| \le r$. Then σ is a K-quasiconformal self-mapping of $\hat{\mathbb{C}}$. Moreover, σ provides a conformal exchange of the open disks $D_1 = B(z_0, \rho)$ and $D_2 = B(-z_0, \rho)$ relative to B, where $z_0 = (r + r^2)/2$ and $\rho = (r - r^2)/2$. If Cantor's "middle interval" construction is performed, beginning with $E_0 = [-r, r]$ and obtaining E_{n+1} from E_n by removing from each component C of E_n the open middle interval with length r times that of C, then $E = \bigcap_{n=1}^{\infty} E_n$ is a self-similar Cantor set whose construction meshes with the mapping σ in such a way as to guarantee that E is K-quasiconformally homogeneous. □

Taking $t = 1/\log 9$ in Example 3.6 leads to a Cantor set that is the image under a similarity transformation of Cantor's classical "middle third" set C. Since K_t is approximately 34.65 for this value of t, Example 3.6 yields the explicit bound $\mathcal{K}(C) \le 35$ for the granddaddy of all Cantor sets. It is worth mentioning that the

Cantor sets exhibited in Examples 3.5 and 3.6 are actually homogeneous with respect to families of uniformly bi-Lipschitz self-mappings of $\mathbb{C}$, as closer scrutiny of their constructions would reveal. There are no conformally homogeneous Cantor sets in $\hat{\mathbb{C}}$, of course, linear or otherwise. It is still not known whether Cantor sets E exist for which $\mathcal{K}(E) = 1$, but it is clear that the specific method used to generate the examples here cannot produce such a set, since all the mappings that arise in this construction fix the point ∞. An easy normal family argument reveals that a Cantor set E in $\hat{\mathbb{C}}$ on which the family $\mathcal{Q}_K(E \cup \{z_0\})$ acts transitively for each $K > \mathcal{K}(E)$, where z_0 is a point of E^c, must be K-quasiconformally homogeneous for $K = \mathcal{K}(E)$. Cantor sets E in $\hat{\mathbb{C}}$ that arise as the limit sets of certain Kleinian groups (finitely generated Schottky groups, for example) also have $\mathcal{K}(E) < \infty$—indeed, exhibit a considerably stronger form of quasiconformal homogeneity than the one under discussion here. Details will appear in [MNP].

The fact that a quasiconformally homogeneous Cantor set E is linear has some noteworthy implications for E, as the following result illustrates.

THEOREM 3.7. *If a quasiconformally homogeneous Cantor set E lies on a circle in $\hat{\mathbb{C}}$ then $\mathcal{H}\text{-dim}(E) \le d < 1$, where d depends only on $\mathcal{K}(E)$. Furthermore, $\mathcal{H}\text{-dim}(E) \to 0$ as $\mathcal{K}(E) \to 1$.*

Proof. We may assume that E lies in $\mathbb{R}$ and that 0 is the smallest element of E. We show that E is locally α-porous at each of its points for some α in $(0, 1/2]$ that depends only on $K = 2\mathcal{K}(E)$, where porosity is now taken to mean porosity with respect to the real line—open intervals replace the open disks of the 2-dimensional porosity condition encountered in the proof of Theorem 3.2. Fixing a point x_0 of E, we choose a mapping f in $\mathcal{Q}_K(E)$ that maps x_0 to the origin. Set $c = f^{-1}(\infty)$ and $\rho = (1/2)\operatorname{dist}(c, E) > 0$. Let $0 < \alpha < 1/2$ be such that E is not locally α-porous at x_0 with $\rho(x_0) = \rho$. We shall derive a positive lower bound for α that depends only on K.

Because E fails to be locally α-porous at x_0 with $\rho(x_0) = \rho$, there exists an r in $(0, \rho]$ such that the interval $(x_0 - r, x_0 + r)$ contains no open subinterval of length $2\alpha r$ that is disjoint from E. It follows that we can select finite sets of points $a_1 < a_2 < \cdots < a_p = x_0$ in $E \cap (x_0 - r, x_0]$ and $x_0 = b_q < \cdots < b_2 < b_1$ in $E \cap [x_0, x_0 + r)$ with $p \ge 2$, $q \ge 2$, and

$$a_{j+1} - a_j < 2\alpha r, \qquad b_k - b_{k+1} < 2\alpha r$$

for $j = 1, 2, \dots, p-1$ and $k = 1, 2, \dots, q-1$. Except for a_p and b_q, the points a_j and b_k are mapped by f into $(0, \infty)$. We are free to suppose that $f(a_1) < f(b_1)$, the opposite case being handled similarly. Because $f(b_q) = f(x_0) = 0 < f(a_1)$, it follows that $f(a_1)$ must lie in at least one of the open intervals I_k in $\mathbb{R}$ whose endpoints are $f(b_k)$ and $f(b_{k+1})$. Fix such a k and assume that

$$|f(a_1) - f(b_{k+1})| \le |f(a_1) - f(b_k)|$$

(again, the other case has an analogous treatment). Then

$$\frac{|f(a_1) - f(b_{k+1})|}{|f(b_k) - f(b_{k+1})|} \le \frac{1}{2},$$

while

$$\frac{|a_1 - b_{k+1}|}{|b_k - b_{k+1}|} \ge \frac{|a_1 - x_0|}{2\alpha r} > \frac{r - 2\alpha r}{2\alpha r} = \frac{1 - 2\alpha}{2\alpha}.$$

Now the restriction of f to the disk $B(x_0, r)$ is an η_K-quasisymmetric embedding of $B(x_0, r)$ into the complex plane, where the homeomorphism $\eta_K : [0, \infty) \to [0, \infty)$ depends entirely on K (see [V2, Thm. 2.4]). This implies that f^{-1} is η^*-quasisymmetric in the domain $f[B(x_0, r)]$, with $\eta^*(t) = [\eta_K^{-1}(t^{-1})]^{-1}$ for $t > 0$ [TV1, Thm. 2.2]. In particular,

$$\frac{1 - 2\alpha}{2\alpha} < \frac{|a_1 - b_{k+1}|}{|b_k - b_{k+1}|} \le \eta^*\left(\frac{|f(a_1) - f(b_{k+1})|}{|f(b_k) - f(b_{k+1})|}\right) \le \eta^*\left(\frac{1}{2}\right),$$

which leads to the inequality

$$\alpha > \frac{1}{2 + 2\eta^*(1/2)}.$$

We conclude that E is locally α-porous at x_0 for $\alpha = [2 + 2\eta^*(1/2)]^{-1}$ and $\rho(x_0) = \rho$. Since x_0 was an arbitrary point of E, [S, Thm. 3.8.1] guarantees that $\mathcal{H}$-dim$(E) \le d$, where

$$d = \frac{\log 2}{\log\left(\dfrac{2 - 2\alpha}{1 - 2\alpha}\right)} < 1$$

is a number that depends solely on $\mathcal{K}(E)$.

To prove the final assertion we shall demonstrate that, when E is K-quasiconformally homogeneous with K sufficiently close to 1, the preceding argument can be reworked to show that E satisfies a local α-porosity condition with α as close to $1/2$ as desired. The key idea here is supplied by [TV2, Thm. 2.6], which implies that for each $K \ge 1$ there exists a homeomorphism $\eta_K : [0, \infty) \to [0, \infty)$ endowed with the following two properties: every K-quasiconformal self-mapping of the finite complex plane is η_K-quasisymmetric, and

$$\lim_{K \to 1} \eta_K(t) = \eta_1(t) = t$$

uniformly on compact subsets of $(0, \infty)$. (For explicit bounds on $\eta_K(t)$ the reader is referred to [VVW, p. 125].) Suppose that α in $(0, 1/2)$ is specified. Fix ε in $(0, 1)$ such that

$$\frac{(1 + \varepsilon)^2}{1 - \varepsilon} < 2 - 2\alpha$$

and then fix K_0 for which

$$\eta_K(t) \le (1 + \varepsilon)t$$

whenever $1 \le K \le K_0$ and $(1+\varepsilon)(1-2\alpha)/[2(1-\varepsilon)] \le t \le (1+\varepsilon)/[(2-2\alpha) \times (1 - \varepsilon)]$.

We return to the argument in the first part of this proof, now assuming that E is K-quasiconformally homogeneous with $1 < K \le K_0$. In the present situation we claim that E is locally α-porous at each of its points. Supposing this not to be the case, we consider a point x_0 of E at which the local α-porosity condition fails. Proceeding as before, we select a mapping f from $\mathcal{Q}_K(E)$ for which $f(x_0) = 0$. Next, let g be a Möbius transformation such that $g(x_0) = x_0$, $g'(x_0) = 1$, and $g(c) = \infty$, where again $c = f^{-1}(\infty)$. We can choose $r > 0$ such that c does not lie in $B(x_0, r)$ and such that

$$1 - \varepsilon \le \frac{|g(z) - g(w)|}{|z - w|} \le 1 + \varepsilon$$

for each pair of distinct points z and w in $B(x_0, r)$. Because the local α-porosity condition is assumed not to hold at x_0, we may further require of r that $(x_0 - r, x_0 + r) \setminus E$ include no open interval of length $2\alpha r$. We can thus choose points a and b in $(x_0 - r, x_0 + r) \cap E$ for which $a < x_0 < b$ and

$$x_0 - a > r - 2\alpha r, \qquad b - x_0 > r - 2\alpha r.$$

Then $f(a) > 0$ and $f(b) > 0$. We furnish details for the case $f(b) < f(a)$; the other case is similar. We have

$$\frac{|f(a) - f(b)|}{|f(a) - f(x_0)|} = \frac{f(a) - f(b)}{f(a)} < 1,$$

while

$$\frac{|a - b|}{|a - x_0|} = \frac{b - a}{x_0 - a} = 1 + \frac{b - x_0}{x_0 - a} > 1 + \frac{r - 2\alpha r}{r} = 2 - 2\alpha.$$

It follows that

$$\frac{1 - 2\alpha}{2} < \frac{|a - x_0|}{|a - b|} < \frac{1}{2 - 2\alpha}.$$

Now $h = f \circ g^{-1}$ is a K-quasiconformal mapping of $\mathbb{C}$ onto itself, so by the selections of K_0, a, and b we have

$$\begin{aligned} 1 < \frac{|f(a) - f(x_0)|}{|f(a) - f(b)|} &= \frac{|h[g(a)] - h[g(x_0)]|}{|h[g(a)] - h[g(b)]|} \le \eta_K\left(\frac{|g(a) - g(x_0)|}{|g(a) - g(b)|}\right) \\ &\le \eta_K\left(\frac{(1+\varepsilon)|a - x_0|}{(1-\varepsilon)|a - b|}\right) \le \frac{(1+\varepsilon)^2}{(1-\varepsilon)} \cdot \frac{|a - x_0|}{|a - b|} < \frac{(1+\varepsilon)^2}{(1-\varepsilon)(2 - 2\alpha)}. \end{aligned}$$

As $(1+\varepsilon)^2(1-\varepsilon)^{-1} < 2 - 2\alpha$, this leads to

$$1 < \frac{(1+\varepsilon)^2}{(1-\varepsilon)(2 - 2\alpha)} < 1,$$

a clear contradiction. Accordingly, E must be locally α-porous at each of its points. If E_n signifies the set of all x in E at which E is locally α-porous with $\rho(x) = 1/n$, then [S, Thm. 3.8.1] implies that

$$\mathcal{H}\text{-dim}(E) \le \sup_n \mathcal{H}\text{-dim}(E_n) \le \frac{\log 2}{\log\left(\dfrac{2-2\alpha}{1-2\alpha}\right)}$$

whenever $1 \le \mathcal{K}(E) < K_0$. Since α in $(0, 1/2)$ was arbitrary, we have $\mathcal{H}$-dim$(E) \to 0$ as $\mathcal{K}(E) \to 1$. □

(N.B. There do exist linear Cantor sets E in $\mathbb{C}$ having $\mathcal{H}$-dim$(E) < 1$ that are not quasiconformally homogeneous. As a simple example we cite $E = E' \cup E''$, where E' is a Cantor subset of $(-\infty, 0)$ for which $\mathcal{H}$-dim$(E') = 0$ and E'' is a Cantor set in $(0, \infty)$ with the feature that $0 < \mathcal{H}$-dim$(E'' \cap U) < 1$ for every open set U in $\mathbb{C}$ with $U \cap E'' \neq \emptyset$. Properties of quasiconformal mappings—see [GV]—imply that no member f of $\mathcal{Q}(E)$ can move a point of E' to a point of E''.)

It would ultimately be nice to characterize the quasiconformally homogeneous Cantor sets E in $\hat{\mathbb{C}}$. Indeed, a characterization of the linear Cantor sets of this type would already be a welcome development. The outlook for finding one is not encouraging: recent work on the bi-Lipschitz equivalence of Cantor sets by Falconer and Marsh [FM] (see [CP] as well) suggests that, even in the linear case, any characterization is likely to be quite subtle, involving delicate algebraic invariants.

A result of Astala [A] leads to the following generalization of Theorem 3.7.

Corollary 3.8. *If a quasiconformally homogeneous Cantor set E lies on a K-quasicircle in $\hat{\mathbb{C}}$, then* $\mathcal{H}$-dim$(E) \le d < 2K/(K+1)$, *where d depends only on $\mathcal{K}(E)$ and K. Moreover,* $\mathcal{H}$-dim$(E) \to 0$ *when both* $\mathcal{K}(E) \to 1$ *and* $K \to 1$.

Proof. Suppose that E is a subset of $f(S)$ for a mapping f from the family $\mathcal{Q}_K$. Then $E' = f^{-1}(E)$ is a quasiconformally homogeneous Cantor subset of S for which the estimate $\mathcal{K}(E') \le K^2\mathcal{K}(E)$ holds. By Theorem 3.7, $\mathcal{H}$-dim$(E') \le d' < 1$, where d' depends only on $\mathcal{K}(E')$—hence, only on $\mathcal{K}(E)$ and K—and $\mathcal{H}$-dim$(E') \to 0$ when both $\mathcal{K}(E) \to 1$ and $K \to 1$. Corollary 1.3 in [A] implies that

$$\mathcal{H}\text{-dim}(E) \le \frac{2K\,\mathcal{H}\text{-dim}(E')}{2+(K-1)\,\mathcal{H}\text{-dim}(E')} \le d = \frac{2Kd'}{2+(K-1)d'} < \frac{2K}{K+1}.$$

The number d depends only on $\mathcal{K}(E)$ and K. The foregoing inequality also shows that $\mathcal{H}$-dim$(E) \to 0$ as $\mathcal{K}(E) \to 1$ and $K \to 1$. □

We close this paper by remarking that the options in Theorem 3.3 become considerably more limited when $\mathcal{Q}_K(E)$ contains a group that acts transitively on E.

Theorem 3.9. *If E is a compact subset of $\hat{\mathbb{C}}$ such that $\mathcal{Q}_K(E)$ contains a group Γ that acts transitively on E, then one of the following is true*: (i) $E = \hat{\mathbb{C}}$; (ii) E *is a finite set of points*; (iii) E *is a $K^{2+\sqrt{2}}$-quasicircle*; (iv) E *is the disjoint union of two $K^{2+\sqrt{2}}$-quasicircles.*

Proof. It follows from a well-known result of Sullivan and Tukia (see [Su; T]) that $\Gamma = g \circ \Gamma' \circ g^{-1}$, where Γ' is a group of Möbius transformations and g is a member of $\mathcal{Q}_{K'}$ for $K' = K^{(2+\sqrt{2})/2}$. Thus $E' = g(E)$ is a conformally homogeneous compact set, and the indicated classification is an immediate consequence of Theorem 2.3. □

References

[A] K. Astala, *Area distortion of quasiconformal mappings,* Acta. Math. 173 (1994), 37–60.

[B1] A. F. Beardon, *On the Hausdorff dimension of general Cantor sets,* Proc. Cambridge Philos. Soc. 61 (1965), 679–694.

[B2] ———, *The geometry of discrete groups,* Springer, Berlin, 1983.

[BE] B. Brechner and T. Erkama, *On topologically and quasiconformally homogeneous continua,* Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1979), 207–208.

[Bu] C. E. Burgess, *Some conditions under which a homogeneous continuum is a simple closed curve,* Proc. Amer. Math. Soc. 10 (1959), 613–615.

[CP] D. Cooper and T. Pignataro, *On the shape of Cantor sets,* J. Differential Geom. 28 (1988), 203–221.

[E] T. Erkama, *Quasiconformally homogeneous curves,* Michigan Math. J. 24 (1977), 157–159.

[FM] K. J. Falconer and D. T. Marsh, *On the Lipschitz equivalence of Cantor sets,* Mathematika 39 (1992), 223–233.

[GP] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains,* J. Analyse Math. 30 (1976), 172–199.

[GV] F. W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings,* J. London Math. Soc. (2) 6 (1973), 504–512.

[G] L. Greenberg, *Discrete subgroups of the Lorentz group,* Math. Scand. 10 (1962), 85–107.

[LV] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane,* 2nd ed., Springer, Berlin, 1973.

[MNP] P. MacManus, R. Näkki, and B. Palka, *Quasiconformally bi-homogeneous compacta in the complex plane,* Proc. London Math. Soc. (to appear).

[N] R. Näkki, *Boundary behavior of quasiconformal mappings in n-space,* Ann. Acad. Sci. Fenn. Ser. A I Math. 484 (1970), 1–50.

[S] A. Salli, *On the Minkowski dimension of strongly porous fractal sets in $\mathbb{R}^n$,* Proc. London Math. Soc. (3) 62 (1991), 353–372.

[Sa] J. Sarvas, *Boundary of a homogeneous Jordan domain,* Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 511–514.

[Sp] G. Springer, *Introduction to Riemann surfaces,* Addison-Wesley, Reading, MA, 1957.

[Su] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions,* Riemann surfaces and related topics: Proceedings of the 1978 Stonybrook conference, Ann. of Math. Stud., 97, pp. 465–496, Princeton Univ. Press, Princeton, NJ, 1981.

[T] P. Tukia, *On two-dimensional quasiconformal groups,* Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 73–78.

[TV1] P. Tukia and J. Väisälä, *Quasisymmetric embeddings of metric spaces,* Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 97–114.

[TV2] ———, *Extension of imbeddings close to isometries or similarities,* Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984), 153–175.

[V1] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings,* Lecture Notes in Math., 229, Springer, Berlin, 1971.

[V2] ———, *Quasisymmetric embeddings in euclidean spaces,* Trans. Amer. Math. Soc. 264 (1981), 191–204.

[VVW] J. Väisälä, M. Vuorinen, and H. Wallin, *Thick sets and quasisymmetric maps,* Nagoya Math. J. 135 (1994), 121–148.

P. MacManus
Department of Mathematics
University of Edinburgh
Edinburgh EH9 3JZ
Scotland

R. Näkki
Department of Mathematics
University of Jyväskylä
P.O. Box 35
SF-40351
Finland

B. Palka
Department of Mathematics
University of Texas at Austin
Austin, TX 78712