Inner Functions in the Hyperbolic Little Bloch Class

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1. Introduction

The hyperbolic derivative of an analytic self-map $\varphi\colon D \to D$ of the unit disk is given by $|\varphi'|/(1-|\varphi|^2)$. To explain the terminology, we note that integrating $|\varphi'|/(1-|\varphi|^2)$ over a rectifiable curve γ in D gives the hyperbolic arclength of $\varphi(\gamma)$. This notion of derivative has been used by Yamashita to study hyperbolic versions of the classical Hardy and Dirichlet spaces; see [Y1] and [Y2]. More recently, in [MM] and [SZ], hyperbolic derivatives have been shown to be pertinent to the study of composition operators on certain subspaces of $H(D)$, the space of analytic functions on D. An analytic self-map φ of D induces a linear operator $C_\varphi\colon H(D) \to H(D)$ defined by $C_\varphi f = f \circ \varphi$. This operator is called the *composition operator* induced by φ.

Recall that an analytic function f on D is said to belong to the Bloch space $\mathcal{B}$ provided that $(1-|z|^2)|f'(z)|$ is uniformly bounded for $z \in D$. Similarly, $f \in \mathcal{B}_0$, the little Bloch space, if $(1-|z|^2)|f'(z)| \to 0$ uniformly as $|z| \to 1$. The hyperbolic Bloch class $\mathcal{B}^h$ is defined by using the hyperbolic derivative in place of the ordinary derivative in the definition of the Bloch space. That is, $\varphi \in \mathcal{B}^h$ if $\varphi\colon D \to D$ is analytic and

$$\sup_{z\in D} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

Similarly, we say $\varphi \in \mathcal{B}_0^h$, the hyperbolic little Bloch class, if $\varphi \in \mathcal{B}^h$ and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

Note that these are not linear spaces, since φ is required to be a self-map of D. It is an easy consequence of the Schwarz–Pick lemma that every analytic self-map of D belongs to $\mathcal{B}^h$, and in fact the supremum above is at most 1; see [G, p. 2]. Membership in the hyperbolic little Bloch class, on the other hand, is nontrivial.

It is easy to see that $C_\varphi\colon \mathcal{B} \to \mathcal{B}$ is bounded for every analytic self-map φ of D, while $C_\varphi\colon \mathcal{B}_0 \to \mathcal{B}_0$ is bounded if and only if $\varphi \in B_0$. It is a recent result of Madigan and Matheson that $C_\varphi\colon \mathcal{B}_0 \to \mathcal{B}_0$ is compact if and only if $\varphi \in \mathcal{B}_0^h$;

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see [MM, Thm. 1]. Membership of φ in $\mathcal{B}_0^h$ has a simple geometric interpretation when φ is univalent, since $(1-|z|^2)|\varphi'(z)|$ is comparable to the distance from $\varphi(z)$ to $\partial\varphi(D)$. This results in C_φ having very strong properties. In particular, the author showed that if φ is univalent and in $\mathcal{B}_0^h$, then $C_\varphi\colon L_a^p \to H^q$ is compact for all $0 < p < q < \infty$; see [Sm, Thm. 6.4]. Here L_a^p and H^q are the classical Bergman and Hardy spaces. This paper resulted from an effort to understand $\mathcal{B}_0^h$ when the univalence assumption is not made. Our main result, Theorem 1.2, shows that $\mathcal{B}_0^h$ contains inner functions. Thus $\varphi \in B_0^h$ does not even imply that C_φ is compact on H^2, since an inner function can not induce such an operator.

We introduce the notation

$$\tau_\varphi(z) = \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2},$$

so that $\varphi \in \mathcal{B}_0^h$ if and only if $\lim_{|z|\to 1} \tau_\varphi(z) = 0$. Our first result shows that there is a restriction on the average rate at which τ_φ can go to zero when φ is an inner function.

1.1. THEOREM. *If φ is an inner function, then*

$$\int_D \frac{\tau_\varphi(z)^2}{1-|z|^2}\, dA(z) = \infty.$$

Although it is clear that finite Blaschke products belong to $\mathcal{B}_0$, it is not obvious that B_0 contains other inner functions as well. Several constructions of such functions have appeared in the literature recently; see [Sa; St; B1; B2]. On the other hand, it is not obvious that B_0^h contains any inner functions at all. In particular, it is easy to see that if φ is an inner function in $\mathcal{B}_0^h$, then the whole unit circle is in the singular set for φ; that is, φ does not have an analytic continuation across any arc in ∂D. Thus $\mathcal{B}_0^h$ contains no finite Blaschke products. Our main result is that there are inner functions in $\mathcal{B}_0^h$.

1.2. THEOREM. *Let η be a nonnegative increasing function such that*

$$\int_0^1 \frac{\eta(t)^2}{t}\, dt = \infty \quad \textit{and} \quad \eta(4t) \le 2\eta(t), \quad 0 < t < t_0,$$

for some $t_0 > 0$. Then there exist an inner function φ and a constant C such that

$$\tau_\varphi(z) \le C\eta(1-|z|^2). \tag{1.1}$$

From Theorem 1.1, we see that Theorem 1.2 is, subject to the regularity assumption on η, the best possible result of this kind. A typical function satisfying the hypotheses of Theorem 1.2 is $\eta(t) = |\log t|^{-1/2}$. The result remains valid when the regularity assumption that $\eta(4t) \le 2\eta(t)$ is replaced by $\eta(4t) \le (4-\varepsilon)\eta(t)$ for some $\varepsilon > 0$. This is equivalent to assuming that η is of upper type less than one; see [J]. It is for clarity of presentation that the simpler form of this regularity condition is used here. Since containment in $\mathcal{B}_0^h$ characterizes compact composition operators on $\mathcal{B}_0$, we obtain the following corollary to Theorem 1.2.

1.3. COROLLARY. *There exists an inner function φ such that $C_\varphi\colon \mathcal{B}_0 \to \mathcal{B}_0$ is compact.*

The function constructed to prove Theorem 1.2 will be a singular inner function φ. A Möbius map σ from D onto D satisfies the identity $1-|\sigma(z)|^2 = (1-|z|^2)|\sigma'(z)|$, and from this it is easy to check that $\tau_\varphi = \tau_{\sigma\circ\varphi}$. It is well known that σ can be chosen so that $\sigma \circ \varphi$ is a Blaschke product, so there are Blaschke products that satisfy (1.1). It would be interesting to have a description of the zero sets of Blaschke products in $\mathcal{B}_0^h$, such as that given by Bishop in [B1] for $\mathcal{B}_0$. The singular set of each such Blaschke product is the full unit circle, as noted above, and so every point on the unit circle is a limit of its zeros.

To see what is involved in the construction of the required singular inner function, let $\varphi(z) = \exp(-F(z))$, where

$$F(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)$$

and μ is a positive singular measure. Then

$$\tau_\varphi(z) = \frac{(1-|z|^2)|F'(z)|\exp(-\Re F(z))}{1-\exp(-2\Re F(z))} \le \frac{(1-|z|^2)|F'(z)|}{\Re F(z)}, \tag{1.2}$$

since $x \le e^x(1-e^{-2x})$ when $0 \le x$. For φ to belong to $\mathcal{B}_0$, it is only required that the numerator of this estimate for $\tau_\varphi(z)$ tends to 0 as $|z| \to 1$, that is, $F \in \mathcal{B}_0$. This is how an inner function in $\mathcal{B}_0$ was shown to exist by Sarason [Sa], who observed that $F \in \mathcal{B}_0$ if the indefinite integral f of μ belongs to the Zygmund class λ_*. Recall that a continuous function f is said to belong to λ_* if the second differences $\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h)$ are uniformly $o(h)$ as $h \to 0$. The construction was then completed by citing constructions of Kahane [K], Piranian [P], and Shapiro [Sha] of increasing singular functions in λ_*. In the present situation, an appropriate lower bound for the denominator of the estimate for τ_φ in (1.2) is also required. Such a lower bound will result from a lower bound for the first differences $\Delta_h f(x) = f(x+h) - f(x)$ of f. We therefore need a construction of an increasing singular function that produces appropriate estimates for $\Delta_h f$ and $\Delta_h^2 f$, from below and above, respectively. The constructions of Kahane, Piranian, and Shapiro cited here do not provide the required lower bounds for $\Delta_h f$. However, their methods can be adapted to produce the required function. The formulation of the resulting theorem and the construction will be given in Section 3. It should be remarked that the assumptions made on η in Theorem 1.2 are essentially best possible for the existence of a monotone singular function f satisfying $|\Delta_h^2 f| \le Ch\eta(h)$; see [K] and [Sha].

It also is of interest to express the estimate (1.2) for τ_φ in terms of μ. Noting that $\Re F(z)$ is just the Poisson–Stieltjes integral of μ, it is easy to verify that

$$\frac{(1-|z|^2)|F'(z)|}{\Re F(z)} = \left| \int_0^{2\pi} \frac{2e^{it}}{(e^{it}-z)^2}\, d\mu(t) \right| \bigg/ \int_0^{2\pi} \frac{1}{|e^{it}-z|^2}\, d\mu(t).$$

Thus the positive singular measure μ we construct will have the property that sufficient cancellation occurs in the numerator for this ratio to tend to 0 as $|z| \to 1$.

The proof of Theorem 1.1 will be given in Section 2, and the construction of f is in Section 3. We begin Section 4 by proving a theorem that shows how the estimate we get for $\Delta_h^2 f$ gives an estimate for the growth of the second derivative of the Herglotz integral of f. This is then applied to prove Theorem 1.2.

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2. Proof of Theorem 1.1

The proof of Theorem 1.1 uses the following non-univalent change-of-variable formula; see [Sh, p. 398]. If g is a measurable function on D and $\varphi\colon D \to D$ is analytic, then

$$\int_D g \circ \varphi(z)|\varphi'(z)|^2 \log\left(\frac{1}{|z|}\right) dA = \int_D gN_\varphi \, dA.$$

Here dA is area measure on D and N_φ is the Nevanlinna counting function, defined by

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}\{w\}} \log\left(\frac{1}{|z|}\right), \quad w \in D \setminus \{\varphi(0)\}.$$

We also recall Littlewood's inequality, which asserts that if $\varphi\colon D \to D$ is analytic then

$$N_\varphi(w) \le \log\left|\frac{1-\bar{w}\varphi(0)}{w-\varphi(0)}\right|$$

for all $w \in D \setminus \{\varphi(0)\}$. Moreover, if φ is an inner function, then equality holds for all w outside a set of logarithmic capacity 0; see [L] or [Sh].

Proof of Theorem 1.1. The change-of-variable formula shows that

$$\begin{aligned}\int_D \frac{\tau_\varphi(z)^2}{(1-|z|^2)^2} \log\frac{1}{|z|}\, dA(z) &= \int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \log\frac{1}{|z|}\, dA(z)\\ &= \int_D \frac{N_\varphi(w)}{(1-|w|^2)^2}\, dA(w).\end{aligned}$$

Because φ is an inner function, equality holds outside a set of area measure 0 in Littlewood's inequality. Thus $N_\varphi(w)$ is comparable to $1-|w|^2$ off this set, and so this last integral diverges. Hence the first integral above also diverges, which finishes the proof since $\log(1/|z|)$ is comparable to $1-|z|^2$ for $1/2 < |z| < 1$. $\square$

3. Construction of the Increasing Singular Function

In this section we construct the increasing singular function f that has the good estimates for both $\Delta_h f$ and $\Delta_h^2 f$ described in Section 1. The construction should be compared to those by Kahane and Shapiro (in [K] and [Sha]) of monotone singular functions in the Zygmund class λ^*. While these constructions do produce the estimate for $\Delta_h^2 f$ that we need, they do not provide the required estimate for $\Delta_h f$. Our construction uses ideas from both [K] and [Sha]. We begin with two elementary lemmas.

3.1. LEMMA. *Suppose that $\{b_j\}$ is a sequence of positive real numbers such that*

$$b_j \leq 3b_{j+1}, \quad \textit{all } j \geq J,$$

for some integer J. Then there is a constant C such that

$$\sum_{j=1}^{n} b_j 4^j \leq C b_n 4^n, \quad \textit{all } n \geq 1.$$

Proof. Assume first that $n \geq J$. If $J \leq j \leq n$ then $b_j \leq 3^{n-j} b_n$, and so

$$\sum_{j=1}^{n} b_j 4^j \leq C + b_n \sum_{j=J}^{n} 3^{n-j} 4^j \leq C b_n 4^n.$$

After increasing the constant C, this estimate will hold for $n < J$ as well. □

3.2. LEMMA. *Suppose that the function η satisfies $\int_0^1 \eta(t)^2 t^{-1}\, dt = \infty$. Then there exists an increasing function ρ such that*

$$\lim_{t \to 0+} \rho(t) = 0, \quad \rho(4t) \leq \frac{3}{2}\rho(t), \quad \textit{and} \quad \int_0^1 \frac{[\eta(t)\rho(t)]^2}{t}\, dt = \infty.$$

Proof. Let $a_0 = 1$, and by induction choose a_k, $k \geq 1$, so that

$$0 < a_k \leq \frac{a_{k-1}}{4} \quad \text{and} \quad \int_{a_k}^{a_{k-1}} \frac{\eta(t)^2}{t}\, dt \geq 1.$$

Now define

$$\rho(t) = 1/\sqrt{k+1}, \quad a_{k+1} < t \leq a_k.$$

It is easy to verify that this function has the stated properties. □

3.3. THEOREM. *Let η be a nonnegative increasing function such that, for some $t_0 > 0$,*

$$\int_0^1 \frac{\eta(t)^2}{t}\, dt = \infty \quad \textit{and} \quad \eta(4t) \leq 2\eta(t), \quad 0 < t < t_0.$$

Let ρ be the associated function from Lemma 3.2. Then there exists an increasing singular function f defined on $[0, 2\pi]$ and a positive constant C such that, for $h > 0$,

$$|f(x+h) - 2f(x) + f(x-h)| \leq C\eta(h)\rho(h)h \tag{3.1}$$

and

$$f(x+h) - f(x) \geq C^{-1}\rho(h)h, \tag{3.2}$$

provided $[x-h, x+h] \subset [0, 2\pi]$.

Proof. Following Shapiro, the required function f will be presented as a limit of functions $\{f_n\}$, where each function is constructed from its predecessor using the basic building block

$$g(x) = \frac{\sin x}{2} - \frac{\sin 2x}{4}.$$

Note that g, g', and g'' all vanish at 0 and 2π, and further that

$$|g(x)| \le \tfrac{3}{4}, \quad |g'(x)| \le 1, \quad |g''(x)| < \tfrac{3}{2}.$$

Let ρ be the function from Lemma 3.2 and define, for integers $n \ge 0$,

$$b_n = \eta(4^{-n})\rho(4^{-n}) \quad \text{and} \quad c_n = \frac{2\eta(1)}{\eta(4^{-n})}.$$

Note that if $4^{-n} < t_0$ then

$$b_n = \eta(4^{-n})\rho(4^{-n}) \le 2\eta(4^{-n-1})\tfrac{3}{2}\rho(4^{-n-1}) = 3b_{n+1}, \tag{3.3}$$

and also

$$\sum_{n=0}^{\infty} b_n^2 = \sum_{n=0}^{\infty}[\eta(4^{-n})\rho(4^{-n})]^2 \ge \frac{1}{\log 4}\sum_{n=0}^{\infty}\int_{4^{-n-1}}^{4^{-n}} \frac{[\eta(t)\rho(t)]^2}{t}\,dt = \infty. \tag{3.4}$$

By multiplying ρ by a constant, we may assume that $b_0 = 1$. Then, using that η and ρ are increasing and $\lim_{t\to 0+}\rho(t) = 0$, we have

$$1 \ge b_n \ge b_{n+1} \to 0, \quad b_n c_n \ge b_{n+1}c_{n+1} \to 0, \quad \text{and} \quad 2 = c_0 \le c_n. \tag{3.5}$$

Let $f_0(x) = x$ and, for $n \ge 1$, suppose $f_{n-1}(x)$ has been defined so that it is increasing and twice differentiable. Divide $[0, 2\pi]$ into 4^n equal intervals $\{ I_{n,k} : 1 \le k \le 4^n \}$ of length $\delta_n = 2\pi \cdot 4^{-n}$, and set $m_{n,k} = \min\{ f'_{n-1}(x) : x \in I_{n,k} \}$. For $x \in I_{n,k}$, we now define

$$f_n(x) = \begin{cases} f_{n-1}(x) & \text{if } m_{n,k} \le b_n c_n, \\ f_{n-1}(x) + b_n\psi_n(x) & \text{if } m_{n,k} > b_n c_n, \end{cases}$$

where $\psi_n(x) = 4^{-n}g(4^n x)$. Since $|\psi_n'(x)| \le 1$ and $c_n \ge 2$, it follows that f_n is increasing. Writing $u_n = f_n - f_{n-1}$, we see that

$$|u_n(x)| \le b_n 4^{-n} \quad \text{and} \quad |u_n''(x)| < 3b_n 4^n/2. \tag{3.6}$$

Thus $f_n = f_0 + \sum_{j=1}^n u_j$ converges uniformly to a nondecreasing function, which we denote by f. Also, by (3.3) we may apply Lemma 3.1 to obtain

$$|f''_{n-1}(x)| < \sum_{j=1}^{n-1} \frac{3}{2} b_j 4^j \le C b_n 4^n,$$

and so

$$m_{n,k} \le f'_{n-1}(x) \le m_{n,k} + C b_n 4^n \delta_n < m_{n,k} + C b_n \tag{3.7}$$

for $x \in I_{n,k}$. Throughout, C will denote a constant whose value may change from line to line but is independent of any parameters, such as n in the inequality of (3.7). To prove that f is singular, consider a point t for which $f'(t)$ exists and is positive. Write $d_{n,k}(G) = \big(G((k+1)\delta_n) - G(k\delta_n)\big)\delta_n^{-1}$ for the difference quotient of a function G over the interval $I_{n,k}$. Then $d_{n,k}(f) = d_{n,k}(f_{n-1})$, since u_j is zero at the endpoints of $I_{n,k}$ for all $j \geq n$. Thus, choosing $k(n)$ so that $t \in I_{n,k(n)}$,

$$f'(t) = \lim_{n\to\infty} d_{n,k(n)}(f) = \lim_{n\to\infty} d_{n,k(n)}(f_{n-1}) = \lim_{n\to\infty} f'_{n-1}(x_n),$$

where $x_n \in I_{n,k(n)}$ is chosen to satisfy $d_{n,k(n)}(f_{n-1}) = f'_{n-1}(x_n)$. By (3.7),

$$|f'_{n-1}(t) - f'_{n-1}(x_n)| < Cb_n,$$

and since $b_n \to 0$ from (3.5), it follows that $\lim f'_n(t) = f'(t)$. Also, since $f'(t) > 0$, it follows from (3.7) that $\liminf m_{n,k(n)} > 0$. Recalling the definition of f_n and that $b_n c_n \to 0$ from (3.5), we now see that $f_n(t) = f_{n-1}(t) + b_n\psi_n(t)$ for all n sufficiently large. Since $\lim f'_n(t) = f'(t)$ and $f'_n(t) - f'_{n-1}(t) = b_n\psi'_n(t)$ for all large n, the series

$$\sum b_n\psi'_n(t) = \sum 2^{-1} b_n(\cos(4^n t) - \cos(2\cdot 4^n t))$$

is convergent. But $\sum b_n^2 = \infty$ from (3.4), and so this lacunary trigonometric series diverges off a set of measure 0; see [Z, p. 203]. Thus $f'(x) = 0$ a.e. and f is singular.

We now turn to the proofs of (3.1) and (3.2). We write

$$\Delta_h G(x) = G(x+h) - G(x)$$

and

$$\Delta_h^2 G(x) = G(x+h) - 2G(x) + G(x-h)$$

for the first and second differences of a function G. Note that bounds for the second difference are $|\Delta_h^2 G(x)| \leq 4\sup|G|$ and, when G is twice differentiable, $|\Delta_h^2 G(x)| \leq h^2 \sup|G''|$. Thus, using (3.6),

$$\frac{|\Delta_h^2 f|}{h} = \lim_{n\to\infty} \frac{|\Delta_h^2 f_n|}{h} \leq \sum_{k=1}^{\infty} \frac{|\Delta_h^2 u_k|}{h} \leq \sum_{k=1}^{\infty} b_k \min\left(\frac{3}{2}h4^k, \frac{4}{h4^k}\right).$$

We choose p to satisfy $4^{-p-1} < h \leq 4^{-p}$ and estimate

$$h\sum_{k=1}^{p} 4^k b_k \leq 4^{-p} C b_p 4^p \leq C b_p,$$

from Lemma 3.1. Also,

$$\frac{1}{h}\sum_{k=p+1}^{\infty} \frac{b_k}{4^k} \leq 4^{p+1} b_{p+1} 4^{-p} \leq 4b_p,$$

since the sequence $\{b_k\}$ is decreasing. Thus

$$\frac{|\Delta_h^2 f|}{h} \leq Cb_p = C\eta(4^{-p})\rho(4^{-p}) \leq C\eta(h)\rho(h),$$

from the definition of b_k, the choice of p, and the regularity properties of η and ρ. This proves (3.1).

To get a lower bound for $\Delta_h f$, we first show by induction that

$$f_n'(x) \ge b_n c_n/2 \tag{3.8}$$

for all x and n. Since $f_0(x) = x$, $b_0 \le 1$, and $c_0 = 2$, this is true for $n = 0$. Now assume that (3.8) has been established for $n-1$, and consider $x \in I_{n,k}$. If $m_{n,k} \le b_n c_n$ then, by the induction hypothesis and (3.5), $f_n'(x) = f_{n-1}'(x) \ge b_{n-1}c_{n-1}/2 \ge b_n c_n/2$, as required. The other possibility is that $m_{n,k} > b_n c_n$, in which case

$$f_n'(x) = f_{n-1}'(x) + b_n \psi_n'(x) \ge m_{n,k} - b_n > b_n(c_n - 1) \ge b_n c_n/2,$$

since $c_n \ge 2$. Thus (3.8) holds. Now let $h \in (0,1)$ and x be given, and choose the integer q to satisfy $4^{-q-1} < h/(4\pi) \le 4^{-q}$. Recalling that u_j vanishes at the endpoints of $I_{q+1,k}$ for all $j \ge q+1$, from (3.8) we have

$$f((k+1)\delta_{q+1}) - f(k\delta_{q+1}) = f_q((k+1)\delta_{q+1}) - f_q(k\delta_{q+1}) \ge b_q c_q \delta_{q+1}/2.$$

Because $h > 2\delta_{q+1}$, there are at least $h/(3\delta_{q+1})$ terms in the sum below, and so

$$\begin{aligned}\frac{\Delta_h f(x)}{h} &\ge \sum_{I_{q+1,k}\subset[x,x+h]} \frac{f((k+1)\delta_{q+1}) - f(k\delta_{q+1})}{h}\\ &\ge \frac{b_q c_q}{6} = \frac{\eta(1)\rho(4^{-q})}{3} \ge \frac{3\eta(1)\rho(h)}{4}.\end{aligned}$$

Thus (3.2) holds and the proof is complete. □

4. Proof of Theorem 1.2

Let f be a function on $[0, 2\pi]$ with continuous periodic extension, and let

$$G(z) = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} f(t)\,dt$$

be the Herglotz integral of f. Before proving Theorem 1.2, we need a preliminary result showing how an estimate of $\Delta_h^2 f$ gives an estimate on the growth of G''. It is well known that $f \in \lambda_*$ (i.e., $|\Delta_h^2 f| = o(h)$ as $|h| \to 0$) if and only if $G' \in \mathcal{B}_0$, or equivalently $(1-|z|^2)|G''(z)| = o(1)$ as $|z| \to 1$; see [Z, p. 263]. Since our goal is to construct a $\varphi \in \mathcal{B}_0^h$, and since $(1-|z|^2)|G''(z)|$ will provide an estimate for just the numerator of $\tau_\varphi(z)$, we need an estimate for the rate at which this goes to zero. The following theorem gives the estimate we need. The proof employs the same ideas used in the classical case that $f \in \lambda_*$ (cf. [Z, p. 109]). Note that we assume in the theorem that f is continuous, since there are nonmeasurable functions f satisfying $\Delta_h^2 f \equiv 0$.

4.1. THEOREM. *Suppose that $\omega(t)$ is positive and nondecreasing for $t > 0$, and that $\omega(4t) \le 3\omega(t)$ for all t sufficiently small. If f is continuous and satisfies $|\Delta_h^2 f| \le \omega(h)h$ for $h > 0$, then there is a constant C such that*

$$(1-|z|^2)|G''(z)| \le C\omega(1-|z|^2).$$

Proof. Let $U(r,\theta) = \Re G(re^{i\theta})$, so that

$$U(r,\theta) = \int_{-\pi}^{\pi} f(t)P(r,t-\theta)\,dt,$$

where $P(r,t) = (1-r^2)|e^{it}-r|^{-2}$ is the Poisson kernel. Noting that $P''(r,t)$ is even in t and integrates to 0, where $P''(r,t)$ represents the second derivative with respect to t, we see that

$$U_{\theta\theta}(r,\theta) = \int_{-\pi}^{\pi} f(t)\frac{\partial^2}{\partial\theta^2}P(r,t-\theta)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \Delta_t^2 f(\theta)\,P''(r,t)\,dt.$$

The hypothesized bound for $\Delta_t^2 f$ therefore gives

$$|U_{\theta\theta}(r,\theta)| \le \int_0^{\pi} t\omega(t)|P''(r,t)|\,dt.$$

One can check that $P''(r,t) < 0$ for $0 < t < \tau$ and $P''(r,t) > 0$ for $\tau < t < \pi$, where $\tau = \tau(r)$ is asymptotic to $(1-r)/\sqrt{3}$ as $r \to 1$; see [Z, p. 109]. The assumption $\omega(4t) \le 3\omega(t)$ implies that

$$\sup_{\tau\le t\le\pi} \omega(t)t^{-1} \le \sup_{\tau\le t\le 4\tau} \omega(t)t^{-1} \le 3\omega(\tau)\tau^{-1}.$$

Using this and the assumption that $\omega(t)$ is increasing, we derive

$$|U_{\theta\theta}(r,\theta)| \le \omega(\tau)\int_0^{\tau} -tP''(r,t)\,dt + 3\frac{\omega(\tau)}{\tau}\int_{\tau}^{\pi} t^2P''(r,t)\,dt.$$

Integrating by parts, we have

$$-\int_0^{\tau} tP''(r,t)\,dt = -\tau P'(r,\tau) + P(r,\tau) - P(r,0) \le -\tau P'(r,\tau) \le \frac{C}{\tau}.$$

Similarly, $\int_{\tau}^{\pi} t^2P''(r,t)\,dt \le C$, and so

$$|U_{\theta\theta}(r,\theta)| \le C\omega(\tau)/\tau \le C\omega(1-r)/(1-r),$$

since τ is asymptotic to $(1-r)/\sqrt{3}$ and $\omega(t)$ is increasing.

Now set $\rho = (1+r)/2$, where $r = |z|$, and let $H = G_{\theta\theta}$. Then

$$H(z) = \int_{-\pi}^{\pi} \frac{\rho e^{it}+z}{\rho e^{it}-z}U_{\theta\theta}(\rho,t)\,dt.$$

Differentiating with respect to z yields

$$\begin{aligned}|H'(z)| &\le 2\int_{-\pi}^{\pi} \frac{|U_{\theta\theta}(\rho,\theta+t)|}{\rho^2-2\rho r\cos t+r^2}\,dt\\ &\le C\frac{\omega(1-\rho)}{(1-\rho)}\int_{-\pi}^{\pi}\frac{dt}{\rho^2-2\rho r\cos t+r^2}\,dt.\end{aligned}$$

This last integral is equal to $2\pi/(\rho^2-r^2)$ and so, using that $\omega(t)$ is increasing, we have the estimate

$$|H'(z)| \le C\omega(1-r)/(1-r)^2.$$

Integrating H' to get $H = G_{\theta\theta}$, we now see that

$$|G_{\theta\theta}(z)| \leq |G_{\theta\theta}(0)| + C\int_0^r \frac{\omega(1-t)}{(1-t)^2}\,dt \leq C\int_{1-r}^1 \frac{\omega(t)}{t^2}\,dt.$$

Choosing p so that $4^{-p-1} < 1-r \leq 4^{-p}$ and using Lemma 3.1,

$$\begin{aligned}\int_{1-r}^1 \frac{\omega(t)}{t^2}\,dt &\leq \sum_{k=0}^{p} \int_{4^{-k-1}}^{4^{-k}} \frac{\omega(t)}{t^2}\,dt \\ &\leq C\sum_{k=0}^{p} 4^k\omega(4^{-k}) \leq C4^p\omega(4^{-p}) \leq C\frac{\omega(1-r)}{(1-r)}.\end{aligned} \tag{4.1}$$

Putting these estimates together, we obtain

$$|G_{\theta\theta}(z)| \leq C\frac{\omega(1-r)}{(1-r)}.$$

A computation shows that $G_{\theta\theta}(z) = -zG'(z) - z^2G''(z)$. Also, $G' \in \mathcal{B}$, since the assumptions on ω imply it is bounded and so $f \in \Lambda_*$. Also $\omega(1) \leq 3^k\omega(4^{-k})$, which implies $1 \leq C\omega(t)t^{-1}$, and so

$$|G'(z)| \leq C\log\frac{1}{1-|z|^2} = C\int_{1-r^2}^1 \frac{dt}{t} \leq C\int_{1-r^2}^1 \frac{\omega(t)}{t^2}\,dt \leq C\frac{\omega(1-r)}{(1-r)},$$

where (4.1) was used to derive the last inequality. Thus

$$|G''(z)| \leq |z|^{-1}|G'(z)| + |z|^{-2}|G_{\theta\theta}(z)| \leq C\frac{\omega(1-r)}{(1-r)}$$

for $1/2 < |z| < 1$, and this completes the proof of Theorem 4.1. □

Proof of Theorem 1.2. Let μ be the positive singular measure on $[0, 2\pi]$ with indefinite integral equal to the singular function f from Theorem 3.3, and let

$$F(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\mu(t)$$

be the Herglotz integral of μ. Recall from (1.2) that the associated singular inner function $\varphi(z) = \exp(-F(z))$ then satisfies

$$\tau_\varphi(z) \leq \frac{(1-|z|^2)|F'(z)|}{\Re F(z)}. \tag{4.2}$$

Integration by parts shows that $F(z) = izG'(z) - 2\pi K$, where

$$G(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}(f(t)+Kt)\,dt.$$

We set $K = (2\pi)^{-1}(f(0) - f(2\pi))$, so that the periodic extension of $f(t)+Kt$ is continuous and Theorem 4.1 can be applied. Then $\Delta_h^2[f(t)+Kt] = \Delta_h^2 f(t)$, and so the bound $\Delta_h^2 f \leq C\rho(h)\eta(h)h$ from Theorem 3.3 along with Theorem 4.1 (applied with $\omega(t) = C\rho(t)\eta(t)$) gives an upper bound for $|G''(z)|$. Since $|G'(z)| \leq |G'(0)| + |z|\max\{\,|G''(w)| : |w| \leq |z|\,\}$, it follows that

$$\begin{aligned}(1-|z|^2)|F'(z)| &\le (1-|z|^2)|zG''(z)| + (1-|z|^2)|G'(z)| \\ &\le C\eta(1-|z|^2)\rho(1-|z|^2) \end{aligned} \tag{4.3}$$

for all $|z|$ sufficiently close to 1.

Let $z = |z|e^{i\theta}$ where $\theta \in [0, 2\pi)$, and assume first that $2\pi \notin [\theta, \theta + (1-|z|^2))$. The denominator of the upper bound for τ_φ in (4.2) is just the Poisson–Stieltjes integral of μ, and so an estimate for it is

$$\begin{aligned}\Re F(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\, d\mu(t) &\ge C^{-1}(1-|z|^2)^{-1}\mu[\theta, \theta+(1-|z|^2)] \\ &= C^{-1}(1-|z|^2)^{-1}\Delta_{(1-|z|^2)} f(\theta) \\ &\ge C^{-1}\rho(1-|z|^2), \end{aligned} \tag{4.4}$$

where the estimate for $\Delta_h f$ from Theorem 3.3 was used for the last inequality. If $2\pi \in [\theta, \theta + (1-|z|^2))$ then $\mu[\theta, \theta + (1-|z|^2)] = \Delta_{h_1} f(\theta) + \Delta_{h_2} f(0)$, where $h_1 + h_2 = 1 - |z|^2$. Thus $\max\{h_1, h_2\} \ge (1-|z|^2)/2$, and it follows that (4.4) holds in this case as well. Inequalities (4.2), (4.3), and (4.4) now combine to yield

$$\tau_\varphi(z) \le C\eta(1-|z|^2),$$

and the proof is complete. □

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