Controls on the Plus Construction

R. J. DAVERMAN* & F. C. TINSLEY

0. Introduction

Given a closed n-manifold M ($n \geq 5$), a finitely presented group G, and an epimorphism $\mu: \pi_1(M) \to G$ with perfect kernel, Quillen's plus construction [Q] provides a cobordism (W, M, N) where W is a compact manifold satisfying:

(1) $\partial W = M \cup N$;
(2) the inclusion $N \to W$ is a homotopy equivalence; and
(3) $\pi_1(N)$ is isomorphic to the quotient G ($\cong \pi_1(M)/\ker(\mu)$).

We obtain a more controlled version of this construction, namely, a closed map $p: W \to [-1, 1]$ satisfying the additional properties:

(4) $p^{-1}(t)$ is a manifold for each $t \in [-1, 1]$; and
(5) $M = p^{-1}(1)$ and $N = p^{-1}(-1)$.

We call a map $p: W \to [-1, 1]$ satisfying properties (4)–(5) a *crumpled lamination on* (W, M, N) and denote it by the 4-tuple (W, M, N, p).

Our earlier constructions required special hypotheses on $\ker(\mu)$ in order to construct crumpled laminations. Here, we show none are needed. In particular, our following main result asserts that any cobordism arising from a Quillen plus construction admits a crumpled lamination.

MAIN THEOREM (THEOREM 1.1). *For any closed n-manifold M ($n \geq 6$), finitely presented group G, and epimorphism $\mu: \pi_1(M) \to G$ with perfect kernel, there is a cobordism W admitting a crumpled lamination (W, M, N, p), where $p^{-1}([-1, 0]) \approx N \times [-1, 0]$, $p^{-1}((0, 1]) \approx M \times (0, 1]$, $\pi_1(N) \cong G$, the inclusion $N \to W$ is a homotopy equivalence, and the inclusion $M \to W$ induces the homomorphism μ on fundamental groups.*

Simple homotopy theory and prior results [DT2] then yield the following characterization.

MAIN COROLLARY (COROLLARY 1.3). *Let (W, M, N) be a compact cobordism with $n = \dim(M) \geq 6$. Then W admits a crumpled lamination*

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(W, M, N, p) if and only if there is a closed n-manifold $P \subset \operatorname{int} W$ with incl: *$P \to W$ a homotopy equivalence.*

Building on the work of Coram and Duvall [CD], Liem [L], and others, Daverman has pursued an extensive project to study upper semicontinuous decompositions of $(n+k)$-manifolds into objects having the shape of closed n-manifolds for $k \geq 1$ [D4]. Our (W, M, N, p) precisely delineates the case $k = 1$, for p induces an upper semicontinuous decomposition of the $(n+1)$-manifold W into n-manifolds via the partition $\{p^{-1}(t) \mid t \in [-1, 1]\}$. The adjective "crumpled" in "crumpled lamination" is suggestive of the fact that some $p^{-1}(t)$ must be wildly embedded in W if $\ker(\mu)$ is nontrivial [D3]. This situation, distinctly different from the cases $k > 1$ in Daverman's program, highlights the beautiful connections among theories of cobordisms, wild and flat embeddings, cell-like decompositions, and perfect groups.

During the late 1970s, Cannon capitalized on these intimate relationships in his quest for a recognition theorem for topological manifolds. His techniques produced a proof of the double suspension theorem [C3]. Before turning to the study of 3-manifolds, Cannon catalogued a few of his many insights in [C2]. Of particular consequence to us was his understanding of the relevance of perfect fundamental groups to the study of wild embeddings. Specifically, for any codimension-1 submanifold N of an $(n+1)$-manifold W, point $x \in N$, and neighborhood U of x in W, there is a neighborhood V of x in W such that $\ker(\mathrm{incl}_{\#})$ is perfect, for incl: $(V-N)_+ \to ((U-N)_+ \cup N)$, where $_+$ denotes one particular side of N. Inspired by a helpful suggestion from John Walsh to "grope" the end of $(V-N)_+$, we apply Cannon's techniques from the proof of the double suspension theorem in a novel way to establish a partial converse to the above observation. Viewed this way, our main theorem takes a perfect normal subgroup of a finitely presented group ($\ker(\mu) < \pi_1(M)$) and realizes $N \to W$ so that

$$\mathrm{incl}\colon M \to (W-N)_+ \to ((W-N)_+ \cup N)$$

has $\ker(\mathrm{incl}_{\#}) = \ker(\mu)$ at the fundamental group level.

Our earlier, partial results required extra hypotheses on the nature of $\ker(\mu)$. Until now, groups for which no finitely generated subgroup contains a nontrivial perfect group remained somewhat of a mystery. The simplest example is a manifold M with

$$\pi_1(M) = \langle y, x, u \mid y = [y, x], x = y^u \rangle.$$

If we take $\mu\colon \pi_1(M) \to \mathbf{Z}$ given by $\mu(u) = 1$ and $\mu(x) = \mu(y) = 0$, then $\ker(\mu)$ is perfect and equals the normal closure of y in $\pi_1(M)$. Furthermore, it can be shown that no finitely generated subgroup of $\ker(\mu)$ contains a nontrivial perfect subgroup, so our earlier constructions do not apply. However, the plus construction gives a cobordism (W, M, N) satisfying conditions (1)–(3). Here, we show that this cobordism and others like it admit crumpled laminations.

Moreover, the constructions from this and previous papers exhibit the precise relationship between attributes of $\ker(\mu)$ and the nature of the wildness of $N = p^{-1}(0)$ in W. We identify three categories of perfect normal subgroups and wildness corresponding to our three types of constructions of crumpled laminations. In Section 6 we provide examples to show that these categories are, in fact, distinct. Consequently, the new techniques of this paper are indispensable in constructing crumpled laminations in some cases.

A natural by-product of our methods is a plethora of new examples to enhance the rather extensive literature on wild and flat embeddings of codimension-1 manifolds. Localized, all constructions give new, wild embeddings of S^n in S^{n+1} for $n \geq 5$.

In Section 1 we prove the main theorem (Theorem 1.1) and its corollaries using shrinking theory (developed in Section 2) and the end theory (developed in Section 3). Section 4 contains results for the case involving crumpled laminations $p: W \to [0,1]$ in which $p^{-1}(1) \subset \operatorname{Int} W$ and $W \setminus p^{-1}(1)$ is connected. Section 5 characterizes the boundaryless manifolds U admitting crumpled laminations in the broad sense—namely, a closed map p of U to some interval J such that each $p^{-1}(t)$, $t \in J$, is a closed, connected, codimension-1 manifold. Finally, Section 6 classifies certain perfect normal subgroups of finitely presented groups.

1. Crumpled Laminations

Recall from the introduction that a *crumpled lamination on a cobordism* (W, M, N) is a continuous, surjective map $p: W \to [-1,1]$ such that $p^{-1}(t)$ is a manifold for each $t \in [-1,1]$ and $M \cup N = p^{-1}(\{-1,1\})$.

Theorem 1.1 (Main Theorem). *For any closed n-manifold M $(n \geq 6)$, finitely presented group G, and epimorphism $\mu: \pi_1(M) \to G$ with perfect kernel, there is a cobordism W that admits a crumpled lamination (W, M, N, p), where $p^{-1}([-1,0)) \approx N \times [-1,0]$, $p^{-1}((0,1]) \approx M \times (0,1]$, $\pi_1(N) \cong G$,* incl: *$N \to W$ is a homotopy equivalence, and* incl: *$M \to W$ induces the homomorphism μ at the fundamental group level.*

Proof. Name a bicollared, codimension-1 submanifold L in M for which the inclusion $L \to M$ induces an isomorphism of fundamental groups. To get it, fix a finite 2-complex K with $\pi_1(K) \cong \pi_1(M)$; use the fundamental group relationship and general position to embed K as a tame subset of M; then take L to be the boundary of a manifold mapping cylinder neighborhood of (the embedded) K. Denote by $L \times [-1,1]$ the image of a bicollar on L.

Because the image group G of the given epimorphism $\mu: \pi_1(M) \to G$ is finitely presented, $\ker(\mu)$ is the normal closure in $\pi_1(M)$ of a finite set $\{w_1, \ldots, w_k\}$, which we express as $\operatorname{ncl}(\{w_1, \ldots, w_k\}; \pi_1(M))$.

Properly, tamely, and disjointly embed gropes $\gamma_1, \ldots, \gamma_k$ in $L \times (0, \frac{1}{2}] \subset L \times (0,1]$ so that $\operatorname{incl}_\#(\pi_1(\gamma_i)) \subset \ker(\mu)$ (see [C2, pp. 860–863]) and, considered as a loop in M, $\partial\gamma_i$ represents $w_i \in \ker(\mu)$. Since each γ_i is tamely embedded, we may assume the γ_is have disjoint neighborhoods satisfying the hypotheses of Proposition 3.3. As in Section 3, we make the proper grope replacements in $L\times(0,1]$ to obtain a new n-manifold V'.

First, we record that $\partial V' = L\times 1$. Then, by Lemma 3.1,

$$\begin{aligned}\pi_1(V') &\cong \pi_1(L\times(0,1])/\mathrm{ncl}(\{\mathrm{incl}_\#(\pi_1(\gamma_i))\colon 1\le i\le k\};\pi_1(L\times(0,1]))\\ &\cong \pi_1(M)/\mathrm{ncl}(\{\mathrm{incl}_\#(\pi_1(\gamma_i))\colon 1\le i\le k\};\pi_1(M))\\ &\cong \pi_1(M)/\mathrm{ncl}(\{w_1,\ldots,w_k\};\pi_1(M))\\ &\cong \pi_1(M)/\ker(\mu)\\ &\cong G.\end{aligned}$$

In particular, $\pi_1(V')$ is finitely presented, so the conclusions of Proposition 3.3 apply and require that the end of V' be homeomorphic to $L'\times(0,\delta)$, with incl: $L'\times(0,\delta)\to V'$ a homotopy equivalence and with $\partial V'$ naturally equal to L. Denote by $\mathrm{Cl}(V')$ the manifold obtained by adding the end, $L'\times 0$, to $L'\times(0,\delta)\subset V'$. We will refer to $\mathrm{Cl}(V')\setminus V'$ simply by L'. Thus, incl: $L'\to V'$ is a homotopy equivalence; consequently, $\pi_1(L')\cong G$.

The main construction takes place in $M\times[-1,1]\supset(L\times[-1,1])\times[-1,1]$. We restrict to the unit disk

$$B^2=\{\langle x,y\rangle\in\mathbb{R}^2\colon x^2+y^2\le 1\}\subset[-1,1]\times[-1,1]$$

to impose (generalized) polar coordinates on $L\times B^2\subset L\times[-1,1]\times[-1,1]\subset M\times[-1,1]$; in other words, express points of $L\times B^2$ as $x\times r\times\boldsymbol{\theta}$, where $x\in L$, $0\le r\le 1$ ($x\times 0\times\boldsymbol{\theta}=x'\times 0\times\boldsymbol{\theta}'$, of course), $\boldsymbol{\theta}$ is a complex number of modulus 1 (shown in bold), and the polar subspace, $L\times[0,1]\times\{\mathbf{-1},\mathbf{1}\}$, naturally corresponds to

$$L\times[-1,1]\times 0\subset(L\times B^2)\cap(M\times 0)\subset M\times[-1,1].$$

From now on we will refer to the complex numbers of modulus 1 as S^1, the unit circle, and frequently we will rewrite $L\times B^2$ as $L\times[0,1]\times S^1$. Another description of these coordinates is as a *spin of* $L\times[0,1]$ *about* $L\times 0$—that is, as the identification space $L\times[0,1]\times S^1$ with each circle $\{x\}\times 0\times S^1$ identified to a point—abbreviated as $\mathrm{Spin}(L\times[0,1],L\times 0)$.

Next we form a manifold with boundary identical to

$$\partial(\mathrm{Spin}(L\times[0,1],L\times 0))$$

by spinning $\mathrm{Cl}(V')$ about this new boundary component L' ($\equiv\mathrm{Cl}(V')-V'$). We denote this more general spin structure by $\mathrm{Spin}(\mathrm{Cl}(V'),L')$ and obtain it from $\mathrm{Cl}(V')\times S^1$ by identifying the circles, $v\times S^1$, to points for all $v\in L'$. For brevity we let $Y'=\mathrm{Spin}(\mathrm{Cl}(V'),L')$; Y' is a manifold since $\mathrm{Cl}(V')$ has "end" homeomorphic to $L'\times[0,\delta]$ and thus $L'\equiv L'\times 0$ has a neighborhood in Y' homeomorphic to $\mathrm{Spin}(L'\times[0,\delta),L'\times 0)\approx L'\times\mathrm{Int}\,B^2$. Since $\partial V'$ was undisturbed throughout the grope replacement and the spinning, we may take

$$\partial Y'=\partial(\mathrm{Spin}(\mathrm{Cl}(V'),L'))=\partial(\mathrm{Spin}(L\times[0,1],L\times 0)=L\times 1\times S^1.$$

We designate the properly embedded submanifold of Y',

$$(\mathrm{Cl}(V')\times -\mathbf{1})\cup(\mathrm{Cl}(V')\times \mathbf{1}),$$

by Z'; it is naturally homeomorphic to the double of $\mathrm{Cl}(V')$ along L'.

The grope replacements in $L\times(0,1]$ lead to a proper embedding d: $\mathbf{C}\times(0,1]\to V'$, $\mathbf{C}$ a Cantor set, such that image$(d)\subset \mathrm{Int}\, V$. This Cantor set arises as the finite, disjoint union of Cantor sets $\mathbf{C}_i$, one for each grope P_i'. The secret to the grope replacement process for (say) P_i' involves a collar $e_i\colon \partial B_i\times[0,1]\to B_i$ on ∂B_i: given $\mathbf{C}_i\subset \partial B_i$ so that $\partial B_i-\mathbf{C}_i$ is homeomorphic to $\partial P_i'$ (as in Section 3), it follows that

$$B_i\setminus e_i((\partial B_i\times[0,1))\cup(\mathbf{C}_i\times 1))$$

is equivalent to P_i' [CBL, Cor. 3.3]. Furthermore, because each $\partial P_i'$ is collared in P_i', one can readily produce a homeomorphism

$$\kappa\colon L\times(0,1]\to V'\setminus \mathrm{image}(d).$$

Here the embedding d is defined for $c\in \mathbf{C}_i\subset \mathbf{C}$ as $d(c+t)=e_i(c\times t)$; the spin operation determines a proper embedding

$$D\colon \mathbf{C}\times(0,1]\times S^1\to V'\times S^1$$

given by $D(c\times s\times\theta)=(d(c\times s)\times\theta)$.

In what follows, any reference to ($M\times$set) means the topology induced by rectangular coordinates. Otherwise, we ordinarily intend polar or spin coordinates.

We create part of the desired cobordism W by swapping two spin structures. Remove $\mathrm{Int}(\mathrm{Spin}(L\times[0,1], L\times 0))$ from $M\times[-1,1]$ and sew in $Y'=\mathrm{Spin}(\mathrm{Cl}(V'), L')$ via the identity along boundaries. Call the resulting manifold W'. Set

$$N=((M\times 0)\setminus(L\times[0,1]\times\{-\mathbf{1},\mathbf{1}\}))\cup Z'\subset W'.$$

Then N is a codimension-1 submanifold of W' that separates W'. Denote by W_+ the closure of the component of $W'\setminus N$ containing $M\times 1$. Let $Y_+'=Y'\cap W_+$; then, $Y_+'\cap N=Z'$.

Denote by S_+^1 the upper half circle $\{\mathbf{a}+\mathbf{bi}\in S^1\colon \mathbf{b}\geq 0\}$. Because there exists a strong deformation retraction ϕ_t of $M\times[0,1]$ to $(M\times 0\cup L\times[0,1]\times S_+^1)$ with

$$\phi_t(M\times[0,\ 1]\setminus L\times[0,1]\times S_+^1)\subset \mathrm{Cl}(M\times[0,1]\setminus L\times[0,1]\times S_+^1),$$

W_+ strong deformation retracts to $N\cup Y_+'$ via essentially the same deformation retraction. Using that V' strong deformation retracts to L' and the spin structure on Y_+', one can specify a strong deformation retraction of Y_+' onto Z'. Thus, incl: $N\to W_+$ is a homotopy equivalence.

Attach $N\times[-1,0]$ to W_+ via the obvious gluing $N\times 0\to N\subset W_+$ to form W; observe that ∂W consists of copies $M\times 1$ and $N\times\{-1\}$ of M and N, respectively. In view of the preceding paragraph, obviously $(W, M\times 1, N\times\{-1\})$ is a cobordism for which incl: $N\times\{-1\}\to W$ a homotopy equivalence.

Now we apply Proposition 2.1 with $T = \mathrm{Cl}(V')$ and with L' exactly as given here. For d and D defined above, select a neighborhood U of

$$D(\mathbf{C} \times (0,1] \times \{\mathbf{i}\})$$

so that the closure of U in $Y = \mathrm{Spin}(T, L')$ is compact and so that, for each point $x \times r \times \boldsymbol{\theta}$ (spin coordinates) in $\mathrm{Cl}(U)$, either $r = 0$ or $\boldsymbol{\theta}$ has positive imaginary part. Let h and g be the homeomorphisms promised by the conclusion of Proposition 2.1. Since the half spin $T \times S^1_+ \subset \mathrm{Spin}(T, L')$ has a neighborhood that obviously embeds in W, the decomposition G of W consisting of points and nondegenerate elements $\{g(c \times t \times B^1) : c \in \mathbf{C},\ t \in (-1,1) \backslash \{0\}\}$ is shrinkable (locally W/G is a manifold, so [E] applies).

We produce the desired crumpled lamination on W by prescribing one on the topologically equivalent space W/G. Let $q/W \to W/G$ be the decomposition map. The composite

$$q \circ \mathrm{incl} : N \times [-1, 0] \to W \to W/G$$

embeds $N \times [-1, 0]$ in W/G, since each nondegenerate element of the decomposition intersects $N \times [-1, 0]$ in exactly one point of $N \times 0$. Note that $q(D(\mathbf{C} \times (0,1] \times S^1_+)) \subset q(N \times 0)$. For $t > 0$ let M_t denote

$$(M \times \{t\} \backslash \mathrm{Spin}(L \times [0,1], L \times 0)) \cup \kappa(L \times [t,1]) \times \{\boldsymbol{\theta}, \boldsymbol{\theta}'\} \cup \kappa(L \times \{t\}) \times \alpha(\boldsymbol{\theta}),$$

where $\boldsymbol{\theta} = (1-t^2)^{1/2} + \mathbf{i} \cdot t$, $\boldsymbol{\theta}' = -(1-t^2)^{1/2} + \mathbf{i} \cdot t$, and $\alpha(\boldsymbol{\theta})$ is the subarc of S^1_+ bounded by $\{\boldsymbol{\theta}, \boldsymbol{\theta}'\}$. Clearly $\{M_t\}$ partitions $M \times [0,1] \backslash D(\mathbf{C} \times (0,1] \times S^1_+)$. Define the crumpled lamination $p : W/G \to [-1, 1]$ as $p^{-1}(t) = q(M_t)$ for $0 < t \le 1$ and $p^{-1}(t) = q(N \times t)$ for $-1 \le t \le 0$. If not totally obvious, continuity of p can be certified using [D3, Cor. 5.3]. □

REMARK. Although Quillen's plus construction is valid for $n = 5$, application of Proposition 3.3 in the proof of Theorem 1.1 causes that dimension to escape our grasp.

Our first corollary affirmatively answers Question 5.4 of [DT2].

COROLLARY 1.2. *Suppose (W, M, N) is a compact $(n+1)$-dimensional cobordism $(n \ge 6)$ such that* incl: $N \to W$ *is a homotopy equivalence. Then W admits a crumpled lamination (W, M, N, p) such that $p^{-1}((0,1]) \approx M \times (0,1]$ and $p^{-1}([-1,0])$ is an h-cobordism (possibly nontrivial).*

Proof. By duality [DT2, Lemma 2.5], $\mathrm{incl} : M \to W$ induces an epimorphism $\mathrm{incl}_\# : \pi_1(M) \to \pi_1(W) \cong \pi_1(N)$ with perfect kernel. By Theorem 1.1, there is a crumpled lamination (W', M, N', p') with $p'^{-1}((0,1]) \approx M \times (0,1]$, $p'^{-1}([-1,0]) \approx N' \times [-1,0]$, $\pi_1(N') \cong \pi_1(M)/(\ker(\mathrm{incl}_\# : \pi_1(M) \to \pi_1(W))) \cong \pi_1(N)$, and $\mathrm{incl} : N' \to W$ a homotopy equivalence. By [DT2, Lemma 5.1], W is homeomorphic to

$$W' \cup_{N'} W''$$

where $W'' = (W'', N', N'')$ is an h-cobordism. Reparameterization gives Corollary 1.2. □

COROLLARY 1.3. *Suppose (W, M, N) is a compact $(n+1)$-dimensional cobordism $(n \geq 6)$. Then W admits a crumpled lamination (W, M, N, p) if and only if there is a closed n-manifold $P \subset \operatorname{Int} W$ with $\operatorname{incl}: P \to W$ a homotopy equivalence.*

Proof. Sufficiency follows from Theorem 1.1 of [DT3]. For necessity, we can assume that P is bicollared in W [AC] and that the inclusion of P into the closure of each component of $W \backslash P$ is a homotopy equivalence. Then Corollary 1.2 ensures that the closure of each component of $W \backslash P$ admits a crumpled lamination. □

2. Decomposition Theory

We will exploit a somewhat unusual phantom shrinking operation, "phantom" because certain standard convergence problems conveniently disappear. Our prototype involves the sine$(1/x)$ curve S in $\mathbb{R}^2$, with its limiting segment Λ. Given any open subset $V_j \supset S \backslash \Lambda$, one can produce a self-homeomorphism ψ_j of $\mathbb{R}^2$ fixed outside V_j and sending S to some $S_j \subset S$ within $1/j$ of Λ. Moreover, with a careful choice of $\{V_j;\ j = 1, 2, \ldots\}$, one can obtain a phantom shrinking of S—namely, a homeomorphism h of $\mathbb{R}^2 \backslash S$ onto $R^2 \backslash \Lambda$ where

$$h(x) = \lim_{j \to \infty} \psi_j \circ \psi_{j-1} \circ \cdots \circ \psi_1(x).$$

Careful choice of $\{V_i\}$ will ensure that to each $x \in \mathbb{R}^2 \backslash S$ corresponds an integer $K = K(x) > 0$ such that, for $k \geq K$ and points y sufficiently close to x,

$$\psi_{k+1} \circ \psi_k \circ \cdots \circ \psi_1(y) = \psi_k \circ \cdots \circ \psi_1(y).$$

Moreover, for $s \in S \backslash \Lambda$ and $\epsilon > 0$, all but finitely many of $\{\psi_j \circ \cdots \circ \psi_1(s):\ j = 1, 2, \ldots\}$ will lie within ϵ of Λ. However, the sequence need not converge, so we make no attempt to define $h(s)$; also, h is defined on Λ ($h(s \in \Lambda) = \lim_{j \to \infty} \psi_j \circ \psi_{j-1} \circ \cdots \circ \psi_1(s) = s$), but h^{-1} fails to be continuous along $\Lambda = h(\Lambda)$, so again we ignore such points.

The main result of this section is a technical shrinking theorem that is the key to constructing laminations. As before, S^1 is the unit circle in $\mathbb{R}^2$ with points of S^1 written in bold as complex numbers of modulus 1, and $S^1_+ \subset S^1$ is the upper half-circle. Throughout this section $\mathbf{C}$ will denote a Cantor set in S^{n-1}, T an n-manifold with boundary (possibly noncompact), L' a component of ∂T, Y the spin of T about L', and B^1 a standard interval. Given an embedding $e: S^{n-1} \times (0,1] \to \operatorname{Int} T$ such that $e(\mathbf{C} \times t)$ approaches L' as $t \to 0$, we use the spin structure on Y to spin $e(S^{n-1} \times (0,1]$ about L' and thus determine an embedding $D: \mathbf{C} \times (0,1] \times S^1 \to Y$ with

$$D(c \times t \times \boldsymbol{\theta}) = e(c \times t) \times \boldsymbol{\theta}.$$

Hence, for each $c \in \mathbf{C}$, $D(c \times t \times S^1) \to L'$ as $t \to 0$. We assume that D is fixed throughout.

PROPOSITION 2.1. *Given any neighborhood U of $D(\mathbf{C} \times (0,1] \times \{\mathbf{i}\})$, there is a phantom homeomorphism $h: Y \backslash (L' \cup D(\mathbf{C} \times (0,1] \times \{\mathbf{i}\})) \to Y \backslash L'$ satisfying:*

(1) *h is the identity over* $\operatorname{Int} T\times\{\pm 1\}$ *and over* $Y\setminus U$;

(2) *there exists a* $\mathbf{C}$*-preserving homeomorphism*

$$g\colon \mathbf{C}\times[(-1,1)\setminus\{0\}]\times B^1\to hD(\mathbf{C}\times(0,1]\times(\mathbf{S}^1_+\setminus\{\pm\mathbf{1},\mathbf{i}\}))$$

such that $g(\mathbf{C}\times t\times B^1)$ *approaches* $hD(\mathbf{C}\times 1\times\mathbf{1})$ *as* $t\to 1$, $g(\mathbf{C}\times t\times B^1)$ *approaches* $hD(\mathbf{C}\times 1\times -\mathbf{1})$ *as* $t\to -1$, *and* $\operatorname{diam} g(c\times t\times B^1)\to 0$ *as* $|t|\to 0$; *and*

(3) *the cell-like decomposition of Y consisting of points and the arcs*

$$\{g(c\times t\times B^1)\colon c\in\mathbf{C},\ t\in(-1,1)\setminus\{0\}\}$$

is upper semicontinuous and shrinkable.

Remarks. The $\mathbf{C}$-preserving homeomorphism g introduced in (2) partitions $hD(\mathbf{C}\times(0,1]\times(\mathbf{S}^1_+\setminus\{\pm\mathbf{1},\mathbf{i}\}))$ into the "vertical" arcs $g(c\times t\times B^1)$; the other features of (2) are designed to ensure that the partition of Y consisting of these arcs plus the singletons from the complement of their union is upper semicontinuous. In this context, upper semicontinuity of that partition is equivalent to closedness of the natural map q to the quotient space, and shrinkability means that q can be approximated, arbitrarily closely, by homeomorphisms; for our purposes, the chief benefit of the latter is its confirmation that the quotient space is topologically equivalent to Y.

We address one tameness issue and a related decomposition-shrinking issue before turning to the proof of Proposition 2.1. Given $X\subset Y$, one writes that X is *1-LCC in Y* (for *locally 1-co-connected in Y*) if, for each neighborhood $O\subset Y$ of $x\in X$, there is a neighborhood $O'\subset O$ of x such that each map $\partial B^2\to O'\setminus X$ can be extended to a map $B^2\to O\setminus X$. Given a compact, 1-dimensional subset X of an n-manifold Y, $n>5$, we call X *tame* (in Y) if it is 1-LCC in Y. The terminology is justified by work of Bryant [Br] showing that any two homotopic, tame embeddings of X in Y are isotopic, via a compactly supported isotopy of Y.

The result below is similar to Proposition 2 of [D1], which is merely stated there, not proved. We supply an argument here for the sake of completeness.

Lemma 2.2. *Consider a manifold M, a closed subset K of M such that* $2+\dim K\le\dim M$, *and a compact subset X of $K\times\mathbb{R}^k\subset M\times\mathbb{R}^k$ such that* $\dim X<k$. *Then X is 1-LCC in $M\times\mathbb{R}^k$.*

Proof. Given a typical neighborhood $U\times V\subset M\times\mathbb{R}^k$ of an arbitrary point $\langle y,z\rangle\in X$, where $V\subset\mathbb{R}^k$ is contractible, one can find $z'\in V$ with $\langle y,z'\rangle\notin X$, since $\dim X<k$. Identify a contractible neighborhood $U'\subset U\subset M$ of y such that $X\cap(U'\times z')=\emptyset$, and let γ be a loop in $(U'\times V)\setminus X$. Since

$$2+\dim(K\times\mathbb{R}^k)\le\dim(M\times\mathbb{R}^k),$$

γ is homotopic in $(U'\times V)\setminus X$ to a loop $\gamma'\subset(U'\setminus K)\times V$, which, due to the contractibility of V, is homotopic in $(U'\setminus K)\times V$ to a loop γ'' in $(U'\setminus K)\times\{z'\}$, which, in turn, is homotopic to a constant in $U'\times\{z'\}$. The image of the composite null homotopy lives in $(U'\times V)\setminus X\subset(U\times V)\setminus X$, as required. $\square$

LEMMA 2.3. *Suppose $X = \bigcup_j X_j$ is a compact, $(n-3)$-dimensional subset of an n-manifold Y such that each X_j is 1-LCC in Y. Then X is 1-LCC in Y.*

This is folklore. It is given in different terms as [C1, 2C.4 (see also p. 59)]. One proof involves showing that X is 1-LCC in Y iff, in the space $C(I^2, Y)$ of all maps $I^2 \to Y$ with the (complete) sup-norm metric,

$$A(X) = \{f \in C(I^2, Y) : X \cap f(I^2) = \emptyset\}$$

is dense. Lemma 2.3 quickly follows from the Baire category theorem, since

$$A(X) = \bigcap_j A(X_j).$$

COROLLARY 2.4. *If $X \subset D(\mathbf{C} \times (0,1] \times S^1)$ is a compact, 1-dimensional set and* $\dim(X \cap D(\mathbf{C} \times 1 \times S^1) \le 0$, *then X is tame in Y.*

Proof. Set $X_1 = X \cap D(\mathbf{C} \times 1 \times S^1)$, and for $j > 1$ set

$$X_j = X \cap D(\mathbf{C} \times [2^{-j}, 1 - 2^{-j}].$$

Then Lemma 2.2 gives that X_1 is 1-LCC in Y, since (locally at least) $X_1 \subset Y$ looks like $X_1 \subset C \times \mathbb{R}^1 \subset T \times \mathbb{R}^1$ and X_1 is 0-dimensional by hypothesis. Due to the existence of the embedding $e: S^{n-1} \times (0,1] \to \operatorname{Int} T$, Y looks like $S^{n-1} \times \mathbb{R}^2$ near X_j $(j > 1)$, with

$$X_j \subset \mathbf{C} \times \mathbb{R}^2 \subset S^{n-1} \times \mathbb{R}^2,$$

so Lemma 2.2 also yields that X_j is 1-LCC in Y. Lemma 2.3 does the rest. □

COROLLARY 2.5. *For all $t \in (-1,1) \setminus \{0\}$, the Cantor set of arcs $g(\mathbf{C} \times t \times B^1)$ is tame in Y.*

LEMMA 2.6. *Suppose $g: \mathbf{C} \times (0,1) \times B^1 \to Y$ is an embedding such that* $\operatorname{diam} g(\mathbf{c} \times t \times B^1) \to 0$ *as $t \to 0$ and as $t \to 1$. Suppose also that, for all embeddings $k: \mathbf{C} \times (0,1) \times B^1 \to Y$ with* $\operatorname{image}(k) = \operatorname{image}(g)$, *each Cantor set of arcs $k: \mathbf{C} \times t \times B^1$, $t \in (0,1)$, is tame in Y. Then g can be approximated by a homeomorphism g' of $\mathbf{C} \times (0,1) \times B^1$ onto $g(\mathbf{C} \times (0,1) \times B^1)$ such that, as before,* $\operatorname{diam} g(c \times t \times B^1) \to 0$ *as $t \to 0$ or as $t \to 1$ and such that the decomposition of Y into points and the arcs*

$$\{g'(c \times t \times B^1) : c \in \mathbf{C}, t \in (0,1)\}$$

is shrinkable.

Proof. For any 2-cell F in a manifold Y, pair $\{x, x'\} \subset \partial F$, and parameterization $g((-1,1) \times B^1)$ of $F \setminus \{x, x'\}$, Daverman and Eaton [DE] (see also [D5, Sec. 11] for an elaboration in manifolds of dimension > 3) developed methods depending solely on the presence of many tame arcs in F to adjust g to another homeomorphism g' of $(0,1) \times B^1$ onto $F \setminus \{x, x'\}$, yielding a shrinkable (upper semicontinuous) decomposition of Y into the singletons from $Y \setminus (F \setminus \{x, x'\})$ and the arcs $g'(t \times B^1)$. In light of the hypothesis here promising that every Cantor set's worth of arcs in $g(\mathbf{C}\times(0,1)\times B^1)$ is tame in Y, the same methods accomplish this parameterized version.

An alternative approach involves Edward's characterization [E] of shrinkable decompositions in terms of the disjoint disks property in the image space. In the space $C(I^2, Y)$, defined as above, one extracts a countable dense subset $\mathcal{E}$ consisting of pairwise disjoint embeddings, each intersecting image(g) in a minimal way—due to wildness, the best one can expect is a 0-dimensional intersection, which can be achieved by slipping successively larger 1-complexes in the various disks off image(g). Let F denote the union of the countable family of 0-dimensional intersections. Now g can be adjusted to make the image of any $c\times t\times B^1$ intersect F at most once. The disjoint disks property follows, because given any two maps of I^2 to the decomposition space, one can approximate them by maps descending to the quotient from two of the $I^2\to Y$ in $\mathcal{E}$; no two such images intersect, since no decomposition element meets two disks from $\mathcal{E}$. □

Proof of Proposition 2.1. The crucial part is to produce two items: a homeomorphism $h: Y\setminus(L'\cup D(\mathbf{C}\times(0,1]\times\{\mathbf{i}\}))\to Y\setminus L'$ satisfying conclusion (1), and an associated $\mathbf{C}$-preserving homeomorphism $g: \mathbf{C}\times[(-1,1)\setminus\{0\}]\times B^1\to hD(\mathbf{C}\times(0,1]\times(\mathbf{S}^1_+\setminus\{\pm\mathbf{1},\mathbf{i}\}))$ satisfying conclusion (2). Then the combination of Corollary 2.5 and Lemma 2.6 promises that g can be approximated a $\mathbf{C}$-preserving homeomorphism

$$g': \mathbf{C}\times[(-1,1)\setminus\{0\}]\times B^1\to hD(\mathbf{C}\times(0,1]\times(\mathbf{S}^1_+\setminus\{\pm\mathbf{1},\mathbf{i}\})),$$

so that the decomposition of Y whose nondegenerate elements are the arcs

$$\{g'(c\times t\times B^1): c\in\mathbf{C},\ t\in(-1,1)\setminus\{0\}\}$$

satisfies conclusion (3) as well.

To accomplish the crucial part, observe that $\operatorname{diam} D(c\times t\times \mathbf{S}^1_+)\downarrow 0$ as $t\downarrow 0$, for all $c\in\mathbf{C}$. Choose a sequence

$$1=s_0>s_1>\cdots>s_k>s_{k+1}>\cdots$$

of points in (0,1] such that not only does $s_k\downarrow 0$, but also

$$\operatorname{diam} D(c\times[s_{k+1},s_k]\times\mathbf{S}^1_+)\downarrow 0$$

uniformly as $k\to\infty$, for all $c\in\mathbf{C}$. Since every $D(\mathbf{C}\times[s_{j+1},s_{j-1}]\times\{\mathbf{i}\})$ is tame (Corollary 2.4), we can construct a sequence of controlled homeomorphisms $\psi_j: Y\to Y$, each supported in a neighborhood V_j of $D(\mathbf{C}\times(s_{j+1},s_{j-1}]\times\{\mathbf{i}\})$, such that

$$\psi_j(D(\mathbf{C}\times[s_{j+1},s_{j-1}]\times\{\mathbf{i}\}))=D(\mathbf{C}\times[s_{j+1},s_j]\times\{\mathbf{i}\});$$

this yields the phantom shrinking homeomorphism $h=\lim_{j\to\infty}\psi_j\circ\cdots\circ\psi_1$. Controls on V_j are necessary, in part, to guarantee that h is appropriately defined as a phantom shrinking $Y\setminus(L'\cup D(\mathbf{C}\times(0,1]\times\{\mathbf{i}\}))\to Y\setminus L'$ but, more delicately, to simultaneously fulfill conclusion (2). Details concerning the latter, similar to those of [DE] for shrinking an embedded 2-cell to an arc, are provided below.

We determine successive pairs $\{\theta_j, \theta_j'\}$ of points $\mathbf{S}^1_+$, each bounding a subarc α_j of $\mathbf{S}^1_+$ with $\mathbf{i} \in \operatorname{Int}(\alpha_j) \subset \alpha_j \subset \operatorname{Int}(\alpha_{j-1})$. Our convention is that $\theta_j = \mathbf{a}_j + \mathbf{b}_j\mathrm{i}$ with $a_j > 0$ and $\theta_j' = -\mathbf{a}_j + \mathbf{b}_j\mathbf{i}$; in other words, θ_j lives in the first quadrant, θ_j' in the second quadrant, and the two are symmetric about the imaginary axis. Choose $\{\theta_j, \theta_j'\}$ so that, for $c \in \mathbf{C}$,

$$\operatorname{diam} \psi_j \circ \cdots \circ \psi_1(D(c \times [s_{j+1}, 1] \times \alpha_j) < 2 \cdot \operatorname{diam} D(c \times [s_{j+1}, s_j] \times \{\mathbf{i}\})); \tag{$*$}$$

let $V_j(c)$ denote the component of V_j containing $D(c \times [s_{j+1}, s_{j-1}] \times \{\mathbf{i}\})$. In addition to imposing restrictions on $\{V_j\}$ to make $\{\psi_j \circ \cdots \circ \psi_1(x); j = 1, 2, \ldots\}$ be finite for all $x \in Y \backslash (L' \cup D(\mathbf{C} \times (0, 1] \times \{\mathbf{i}\}))$, we also require that ψ_j be the identity on

$$D(\mathbf{C} \times (0, s_{j+1}] \times \mathbf{S}^1_+) \cup \psi_{j-1} \circ \cdots \circ \psi_1(D(\mathbf{C} \times (0, 1] \times (\mathbf{S}^1_+ \backslash \alpha_{j-1}))),$$

and that $\operatorname{diam} V_j(c) < 2 \cdot \operatorname{diam} D(\mathbf{C} \times [s_{j+1}, s_{j-1}] \times \{\mathbf{i}\})$.

For $j = 1, 2, \ldots$, identify the subarc γ_j of $\mathbf{S}^1_+$ bounded by $\mathbf{1}$ and θ_j, name an arc $A_j = (s_j \times \gamma_j) \cup ([s_j, 1] \times \theta_j) \subset (0, 1] \times \mathbf{S}^1_+$, and let R_j denote the subdisk of $(0, 1] \times \mathbf{S}^1_+$ bounded by A_j and A_{j+1}. Define a homeomorphism λ: $(0, 1) \times B^1 \to (0, 1] \times \beta_+$, where β_+ is the open arc in $\mathbf{S}^1_+$ bounded by $\{\mathbf{1}, \mathbf{i}\}$, such that $\lambda(\{1/j\} \times B^1) = A_j$. Finally, specify the $\mathbf{C}$-preserving homeomorphism

$$g \colon \mathbf{C} \times [(-1, 1) \backslash \{0\}] \times B^1 \to hD(\mathbf{C} \times (0, 1] \times (\mathbf{S}^1_+ \backslash \{\pm \mathbf{1}, \mathbf{i}\}))$$

on $\mathbf{C} \times (0, 1) \times B^1$ by setting $g(c \times t \times b) = hD(c \times \lambda(t, b))$; define it in symmetric fashion on $\mathbf{C} \times (-1, 0) \times B^1$.

Here each R_j splits into two pieces, part of a circular annulus given as $R_j' = R_j \cap ([s_{j+1}, s_j] \times \gamma_j)$ and part of a circular sector (determined by $\theta_j, \theta_{j+1} \in \mathbf{S}^1_+$) given as $R_j'' = R_j \cap ([s_{j+1}, 1] \times \alpha_j)$. See Figure 1. Then, by restrictions on supports of the shrinkings ψ_i,

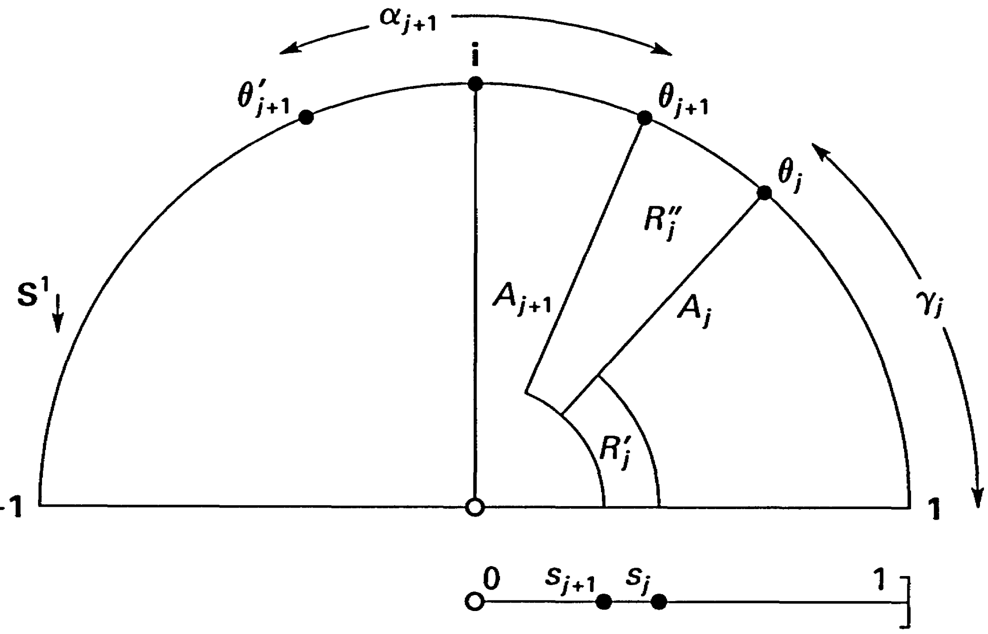

Figure 1

$$\begin{aligned} &hD(c\times R_j'') \\ &\quad = \psi_{j+1}\circ\cdots\circ\psi_1(D(c\times R_j'')) \\ &\quad \subset V_{j+1}(c)\cup D(c\times[s_{j+1},s_j]\times\alpha_j)\cup\psi_j\circ\cdots\circ\psi_1(D(c\times[s_{j+1},1]\times\alpha_j) \\ &\quad \subset V_{j+1}(c)\cup D(c\times[s_{j+1},s_j]\times S_+^1)\cup\psi_j\circ\cdots\circ\psi_1(D(c\times[s_{j+1},1]\times\alpha_j), \end{aligned}$$

whereas

$$\begin{aligned} hD(c\times R_j') = \psi_j\circ\cdots\circ\psi_1(D(c\times R_j')) &= \psi_j(D(c\times R_j')) \\ &\subset D(c\times R_j')\cup V_j(c), \\ &\subset V_j(c)\cup D(c\times[s_{j+1},s_j]\times S_+^1), \end{aligned}$$

from which the diameter constraints of (*) and upon $V_j(c)$ give

$$\operatorname{diam} hD(c\times R_j) < 5\cdot\operatorname{diam} D(c\times[s_{j+1},s_j]\times S_+^1).$$

This indicates why $\operatorname{diam} g'(c\times t\times B^1)\to 0$ as $t\downarrow 0$. □

3. Recognizing the End

This section relies on a grope construction originated by M. A. Stan'ko but developed most extensively by J. W. Cannon and his collaborators [AC; C3; CBL]; see also [D5, Secs. 38–39]. Among other features, a *grope* is an acyclic 2-complex expressed as a locally finite union $\{S_i\}$ of compact, orientable surfaces with connected boundary. The interior of each S_i contains a finite collection of simple closed curves $\{J_{i,j}\}$, called *handle curves,* one for each generator of $H_1(S_i)$. To each $J_{i,j}$ there corresponds a unique surface S_k with

$$S_k\cap S_i = \partial S_k = J_{i,j};$$

one regards the various S_ks as being attached to S_i along the associated handle curves. Except for the identification along handle curves, the various surfaces S_i are pairwise disjoint.

Let V be an n-manifold, $n\geq 5$. Suppose $\gamma_1,\dots,\gamma_k$ are gropes embedded in V as mutually disjoint, closed subsets, and suppose γ_i has a closed neighborhood P_i such that (P_i,γ_i) is pairwise homeomorphic to (Q_i,Γ_i), where Γ_i denotes a copy of γ_i PL-embedded in $\mathbb{R}^n$ as a closed subset and Q_i an infinite regular neighborhood of Γ_i there. Determine another closed neighborhood $P_i'\subset \operatorname{Int} P_i$ of each γ_i such that (P_i',γ_i) is pairwise homeomorphic to (Q_i,Γ_i) and $\operatorname{Cl}(P_i\backslash P_i')$ is a collar on $\operatorname{Fr} P_i$, the frontier of P_i in V. By [CBL, Cor. 3.3], there exist an n-cell B_i and a Cantor set C_i in ∂B_i such that $\partial B_i\backslash C_i$ is homeomorphic to $\operatorname{Fr} P_i'$. The *manifold V' obtained from V by a canonical replacement of the gropes* $\gamma_1,\dots,\gamma_k$ is the space obtained from the disjoint union of $\{\operatorname{Cl}(V\backslash\bigcup P_i'), B_1\backslash C_1,\dots,B_k\backslash C_k\}$ under homeomorphic identification of each $\operatorname{Fr} P_i'$ with $\partial B_i\backslash C_i$.

LEMMA 3.1. *Let V' be obtained from V by a canonical replacement of gropes $\gamma_1,\dots,\gamma_k$. Then each $\pi_1(\operatorname{Fr} P_i')\to\pi_1(P_i')$ and $\pi_1(\operatorname{Cl}(V\backslash\bigcup P_i'))\to$*

$\pi_1(V)$ are isomorphisms. Moreover, $\pi_1(V\setminus\bigcup P_i')\to\pi_1(V')$ is an epimorphism having perfect kernel—namely, the normal closure of all the images $[\pi_1(\mathrm{Fr}\,P_i')\to\pi_1(V')]$.

This is standard. Perfectness is immediate, using $H_1(\mathrm{Fr}\,P_i')\cong H_1(\partial B_i\setminus C_i)\cong 0$.

LEMMA 3.2. *Let D be a compact subset of a grope γ. For any component W_ω of $\gamma\setminus D$, let I_ω denote the union of all conjugates in $\pi_1(\gamma)$ of the (inclusion-induced) image $\pi_1(W_\omega)\to\pi_1(\gamma)$. Then*

$$\mathrm{ncl}(\{I_\omega\colon \textit{all components } W_\omega\};\ \pi_1(\gamma))=\pi_1(\gamma).$$

Proof. For some integer $m\geq 0$, the first m stages of the grope γ contain D. The argument reduces to showing that if every element of $\pi_1(\gamma)$ represented by a loop outside the first j stages of γ belongs to

$$\mathrm{ncl}(\{I_\omega\colon \text{all components } W_\omega\};\ \pi_1(\gamma)),$$

then the same is true of elements represented by loops outside the first $j-1$ stages. Stage $j+1$ is attached to stage j along a complete set of handle curves, so each loop outside stage $j-1$ is freely homotopic to the product of loops outside stage j.

By way of further explanation of this reduction, what happens with the initial stage is typical. Let S denote the first stage of γ. Clearly γ collapses to the union of $\mathrm{Cl}(\gamma\setminus S)$ with finitely many arcs joining the various components of the latter to a base point, where any two such connecting arcs meet only at the base point. □

PROPOSITION 3.3. *Suppose L is a closed n-manifold, $n\geq 5$, and $\gamma_1,\dots,\gamma_k$ are gropes embedded in $L\times(0,1/2]$ as pairwise disjoint, closed subsets. Let V' denote the manifold obtained from $V=L\times(0,1]$ by a canonical grope replacement of $\gamma_1,\dots,\gamma_k$, and suppose $\pi_1(V')$ is finitely presented. Then the end of V' has a neighborhood homeomorphic to $L'\times(0,r)$ for some n-manifold L', and* incl: $L'\times(0,r)\to V$ *is a homotopy equivalence.*

Proof. Quillen's plus construction [Q] provides a compact $(n+1)$-manifold W having boundary components L, L' such that

(1) incl: $L'\to W$ is a homotopy equivalence, and

(2) $\ker(\mathrm{incl}_\#\colon\pi_1(L)\to\pi_1(W))=\ker(\mathrm{incl}_\#\colon\pi_1(L=L\times 1)\to\pi_1(V'))$.

Form a new manifold T by identifying W and V' via the obvious attachment along $L\subset\partial W$ and $L\times 1=\partial V'$. Obviously the end ϵ of T coincides with that of V'. We will show that the inclusion $L'=\partial T\to T$ is a homotopy equivalence, that π_1 is stable at ϵ, and that $\pi_1(\epsilon)\to\pi_1(T)$ is an isomorphism. Then Siebenmann's open collar theorem [S] will yield a homeomorphism between T and $\partial T\times(0,2]=L'\times(0,2]$, giving the desired conclusions.

Let Y denote the manifold obtained by attaching V to W along $L\times 1=\partial V$ and $L\subset\partial W$, just as in the formation of T. Then $\partial Y=L'$ and, clearly, incl: $L'\to Y$ is a homotopy equivalence. Let $\Theta\colon Y'\to Y$ be the universal cover and

$L'' = \Theta^{-1}(L')$. Here $\Theta | L'' : L'' \to L'$ is also the universal cover. By condition (2), for each thickened grope $P_i' \subset L \times (0,1] \subset Y$, $\Theta^{-1}(P_i')$ consists of a pairwise disjoint collection of copies of P_i'. Replace each such copy with a copy of $B_i \setminus C_i$, thereby obtaining a new manifold Y^* with $\partial Y^* = L''$.

First, note that Y^* is simply connected. This follows from the Seifert-van Kampen theorem by viewing Y^* as the union of simply connected pieces $\mathrm{Cl}(Y \setminus \Theta^{-1}(\bigcup P_i'))$ and copies of $B_i \setminus C_i$, where each of the latter meets $\mathrm{Cl}(Y \setminus \Theta^{-1}(\bigcup P_i'))$ in a copy of the connected set $\mathrm{Fr}\, P_i' = \partial B_i \setminus C_i$. Next observe that Y^* is the universal cover of T, with, of course, $L'' = \partial Y^*$ covering $L' = \partial T$. To see why, determine a map $\Theta^* : Y^* \to T$ by restricting to Θ on the complement of all named copies of $B_i \setminus C_i$ and by extending homeomorphically between these corresponding copies in Y^* and T. Checking that Θ^* is a covering map is elementary.

Since $L' \to Y$ is a homotopy equivalence, the same is true of $L'' \to Y'$. Consequently, $L'' \to Y^*$ is a homology equivalence, for the changes made in constructing Y^* from Y' involve replacement of acyclic sets by acyclic sets. In view of simple connectivity, the Whitehead theorem [Sp, p. 399] certifies that $L'' \to Y^*$ is a homotopy equivalence, so the same holds downstairs for the inclusion $L' = \partial T \to T$, as it induces isomorphisms of all homotopy groups. In light of (2) and Lemma 3.1, this means that $\pi_1(T)$ is isomorphic to $\pi_1(V)/\mathrm{ncl}(\{\mathrm{images}[\pi_1(P_i') \to \pi_1(V)]\}; \pi_1(C))$.

Now we take up the stability of π_1 at ϵ. Consider an arbitrary compact subset D of T. For $s \in (0,1]$ let X_s denote the part of $L \times (0,s)$ in T. Produce a neighborhood $N \subset T \setminus D$ of ϵ consisting of some X_s together with sets E_α, where each E_α denotes an open subset of some $B_i \setminus C_i$. Then produce another neighborhood $N' \subset N$ of ϵ consisting of some $X_{s'}$, $0 < s' < s$, together with sets F_β, where $\{F_\beta\}_\beta$ refines $\{E_\alpha\}_\alpha$ and each loop in F_β is null homotopic in some $E_\alpha \supset F_\beta$. Do this so $\pi_1(X_{s'}) \to \pi_1(L \times (0,s))$ is surjective and has kernel equal to that of $\pi_1(X_{s'}) \to \pi_1(X_s)$. As a result,

$$\pi_1(X_{s'}) \to \pi_1(T) \cong \pi_1(V)/\mathrm{ncl}(\{\mathrm{images}[\pi_1(P_i') \to \pi_1(V)]\}; \pi_1(V)) \qquad (\dagger)$$

is surjective. Note that, for each thickened grope $P_i' \subset V'$ involved in the standard replacement, there is a compact subset D_i of γ_i such that the image of $\pi_1(P_i' \cap X_{s'}) \to \pi_1(P_i') \to \pi_1(\gamma_i)$ (ranging over various basepoints, with the second homomorphism induced by a proper collapse $p_i : P_i' \to \gamma_i$) contains the union of all conjugates in $\pi_1(\gamma_i)$ of the (inclusion-induced) image $\pi_1(W_\omega) \to \pi_1(\gamma_i)$, where the union is taken over all components W_ω of $\gamma_i \setminus D_i$. Now, by Lemma 3.2 and (†), inclusion $N' \to T$ restricts to an isomorphism

$$\mathrm{image}[\pi_1(X_{s'}) \to \pi_1(N')] \to \pi_1(T).$$

Finally, choose a third neighborhood $N'' \subset N'$ of ϵ such that every loop in N'' is homotopic in $T \setminus D$ to a loop in $X_{s'} \subset N'$. The point is that

$$\mathrm{image}[(\pi_1(N'') \to \pi_1(T \setminus D)] = \mathrm{image}[(\pi_1(X_{s'}) \to \pi_1(T \setminus D)],$$

implying that inclusion induces an isomorphism

$$\text{image}[(\pi_1(N'') \to \pi_1(T \setminus D)] \to \pi_1(T).$$

Hence, π_1 is stable at ϵ and $\pi_1(\epsilon) \to \pi_1(T)$ is an isomorphism, as required. □

Addendum: The 4-dimensional version of Proposition 3.3 holds when $\pi_1(V')$ is a good group (in the sense of [FQ, p. 99]), because then Quillen's plus construction applies [FQ, Sec. 11.1], exactly as above. All the other steps in the argument routinely carry over when $n = 4$, with no special restrictions on π_1. Consequently, in dimension $n = 5$, Theorem 1.1 is also valid for such good groups G.

4. The 1-Sided Case

A *twisted I-bundle over a base n-manifold* N ($\partial N = \emptyset$) is an $(n+1)$-manifold W which is a locally trivial bundle over N with fiber $I = [-1, 1]$ and for which ∂W is connected. Say that a compact $(n+1)$-manifold with boundary W is a *fake, laminated, twisted I-bundle* if

(a) ∂W is connected,
(b) W contains a closed n-manifold N such that $N \to W$ is a simple homotopy equivalence, and
(c) W admits a crumpled lamination.

THEOREM 4.1. *A closed n-manifold M ($n \geq 6$) bounds a fake, laminated, twisted I-bundle W if and only if there exists a compact $(n+1)$-manifold V with boundary the disjoint union of manifolds M, M' such that*

(1) incl: *$M' \to V$ is a simple homotopy equivalence, and*
(2) *M' admits a fixed-point free involution.*

REMARK. The closed manifold N, which is simple homotopy equivalent to W in the above definition, arises as the orbit space of the manifold M' under the involution named in (2).

Proof of Theorem 4.1. Assume M bounds a fake, laminated, twisted I-bundle W. Let N denote the closed submanifold in W such that $N \to W$ is a simple homotopy equivalence. Here N cannot separate W, for otherwise $H_n(N; Z_2) \to H_n(W; Z_2) \cong H_n(N; Z_2)$ would be trivial. Hence, image[$\text{incl}_\#$: $\pi_1(M) \to \pi_1(W)$] is contained in an index-2 subgroup H of $\pi_1(W)$. To see why, check part of the long exact sequence with Z_2-coefficients for the pair $(W, W \setminus N)$:

$$H_1(W \setminus N) \to H_1(W) \to H_1(W, W \setminus N) \to 0 \cong \tilde{H}_0(W \setminus N)$$

and $H_1(W, W \setminus N) \cong H^n(N) \cong Z_2$, by duality; H is the kernel of the natural composite $\pi_1(W) \to H_1(W; Z_2) \to H_1(W, W \setminus N; Z_2)$. Let $\Theta: W' \to W$ denote the 2–1 cover of W corresponding to H. Then $M' = \Theta^{-1}(N)$ is connected and separates W' into the two lifts of $W \backslash N$. Let V denote the closure of one of the components of $W' \backslash M'$. Since V might fail to be a manifold-with-boundary should N be wildly embedded in W, attach a collar $M' \times [0,1]$ to V^* along $M' = M' \times 0$ to produce a manifold V [D2] having two boundary components, $M' = M' \times 1$ and a copy of M. Here $M' \to V$ is a simple homotopy equivalence and M' admits a fixed-point free involution, since it is a 2–1 cover of N.

Next assume the existence of an $(n+1)$-manifold V with boundary components M, M' satisfying (1) and (2). By the main theorem, V admits a crumpled lamination. Let W be the $(n+1)$-manifold obtained from V by identifying those points of M' in an orbit of the given involution. The crumpled lamination present in V naturally descends to one on W, for the image of distinct manifolds upstairs are distinct manifolds in W, and M' is the only element whose image can be topologically different from its source. In other words, W is a fake, laminated, twisted I-bundle bounded by a copy of M. □

COROLLARY 4.2. *Let M be a closed n-manifold ($n \geq 6$) admitting a fixed-point free involution $\upsilon: M \to M$. Let G denote a finitely presented group and $\mu: \pi_1(M) \to G$ an epimorphism with perfect kernel. Then M bounds a fake, laminated, twisted I-bundle W with*

$$\ker(\mathrm{incl}_{\#}: \pi_1(M) \to \pi_1(W)) = \ker(\mu).$$

Proof. According to [Q], starting with $M \times [0,1]$ we could attach 2-handles and then 3-handles along $M \times 1$ to kill $\ker(\mu)$. This would produce a compact $(n+1)$-manifold V^* with two boundary components $M = M \times 0$ and M^*, where the inclusion $M^* \to V^*$ is a simple homotopy equivalence. The trick simply is to attach handles equivariantly. Use general position to ensure that the attaching 1-sphere S of each 2-handle satisfies $\upsilon(S) \cap S = \emptyset$. Then attach 2-handles along both S and $\upsilon(S)$. This ensures that $M \times 1$ together with all the 2-handles supports a fixed-point free involution extending υ on $M = M \times 1$. Now repeat the same process when attaching 3-handles. In so doing one forms an $(n+1)$-manifold V with boundary components M, M' satisfying conditions (1) and (2) in Theorem 4.1, from which the corollary follows. □

REMARK. Hypothesizing something like the existence of the fixed-point free involution υ is necessary in Corollary 4.2, for obstructions exist to having all M bound a fake laminated twisted I-bundle. For example, any such M must have even Euler characteristic: by Theorem 4.1, M must be homologically equivalent to a manifold M' that 2–1 covers some manifold N, indicating

$$\chi(M) = \chi(M') = 2 \cdot \chi(N).$$

5. Classification

THEOREM 5.1. *An $(n+1)$-manifold U ($n \geq 6$) admits a crumpled lamination if and only if U can be expressed as the union of a collection $\mathfrak{I} = \{W_i\}$*

of compact $(n+1)$-manifolds with boundary, where W_i is either a 1-sided h-cobordism or a fake, laminated, twisted I-bundle, and $W_i \cap W_j \neq \emptyset$ $(i \neq j)$ implies $W_i \cap W_j$ is a boundary component of each.

REMARK. In any such collection $\mathfrak{T}$ for a connected manifold U, at most two elements of $\mathfrak{T}$ can be fake, laminated, twisted I-bundles; all the others must be 1-sided h-cobordisms.

Proof of Theorem 5.1. First, suppose the collection $\mathfrak{T} = \{W_i\}$ exists. Then, by Corollary 1.2 or Theorem 5.1, each $W_i \in \mathfrak{T}$ admits a crumpled lamination $\mathfrak{L}_i$ (with every component of ∂W_i in $\mathfrak{L}_i$), so $\bigcup \mathfrak{L}_i$ provides a crumpled lamination on U.

Next, suppose U admits a crumpled lamination. By [D3, Thm. 6.6], U has what was called there a *quasi-standard formation* $\{V_i\}$, meaning that each V_i is a compact $(n+1)$-manifold with boundary and is endowed with a crumpled lamination, where V_i either is a twisted I-bundle or has two boundary components. In the former case let $W_i = V_i$. In the latter case we will split V_i into two parts, each of which is a 1-sided, laminated h-cobordism. The proof of Corollary 1.3 promises that $\operatorname{Int} V_i$ contains a codimension-1 closed submanifold P_i with incl: $P_i \to V_i$ a homotopy equivalence and with P_i splitting V_i into two compact manifolds with boundary W_i, W_i' such that $W_i \cap W_i' = \partial W_i \cap \partial W_i' = P_i$, where then P_i includes in each of W_i, W_i' as a homotopy equivalence. The collection $\mathfrak{T}$ consisting of all the various W_i and W_i' (wherever defined) fulfills the requirements. □

Here is an alternative to Theorem 5.1 with more stringent conditions on elements of the collection $\mathfrak{T}$. It follows from the proof of the preceding as well as from corresponding improvements concerning quasi-standard formation given in [D3, addendum to Thm. 6.6].

THEOREM 5.2. *An $(n+1)$-manifold U $(n \geq 6)$ admits a crumpled lamination if and only if U can be expressed as the union of a collection $\{W_i\}$ of compact $(n+1)$-manifolds with boundary, where W_i is either a 1-sided h-cobordism or a twisted I-bundle, and $W_i \cap W_j \neq \emptyset$ $(i \neq j)$ implies $W_i \cap W_j$ is a boundary component of each.*

6. Perfect Subgroups of Finitely Presented Groups

If a cobordism (W, M, N) admits a crumpled lamination (W, M, N, p), then $\ker(i_\#)$ is finitely generated as a perfect normal subgroup of $\pi_1(M)$, where $i: M \to W$ is inclusion [DT2, Lemma 2.5]. This paper is the culmination of our efforts to establish a converse to this fact. We remove the extra hypotheses on $\ker(i_\#)$ required for our earlier partial converses.

The main purpose of this section is to catalog examples of crumpled laminations whose constructions require our new methods. In particular, we must show that prior techniques cannot be applied. This determination demands a careful look at the characteristics of $\ker(i_\#)$.

If $\ker(i_\#) \neq 1$, then $p^{-1}(t)$ is wildly embedded in W for at least one $t \in (-1, 1)$. As a by-product of the main result, we obtain a description of the relationship between the algebraic properties of $\ker(i_\#)$ and the geometric properties of the wild embedding. Theorem 1.1 and its corollaries along with simple homotopy theory give us a precise characterization.

This section is very much in the spirit of Howie's classification [H] of countable groups according to properties of their perfect subgroups and his analysis of whether the basic group-theoretic operations preserve these categories. We have changed his labels for the classes to be more suggestive of their properties and have named our new classes accordingly. We warn the reader that the definitions are negations of certain conditions on the groups. Thus, we give each class a label of the form $\mathrm{N}_{\mathrm{xxx}}$ where "N" connotes "no" and "xxx" is a particular property associated with the groups within the class. Though a bit cumbersome, these definitions by negation turn out to be convenient for tracking the behavior of groups under the standard group-theoretic operations. This approach has the additional value of suggesting where one should (or should not) look for particular types of examples. The first two definitions are modified versions of ones given by Howie [H].

DEFINITION 6.1. A group G belongs to $\mathrm{N}_{\mathrm{fgp}}$ if G is finitely presented and contains no nontrivial finitely generated perfect subgroups.

Thus, the label $\mathrm{N}_{\mathrm{fgp}}$ means "no (nontrivial) finitely generated perfect subgroups".

DEFINITION 6.2. A group G belongs to N_{pn} if G is finitely presented and contains no nontrivial perfect, normal subgroups.

Since the normalizer of a perfect subgroup is itself perfect, N_{pn} consists of the finitely presented groups having no nontrivial perfect subgroups whatsoever, which is precisely the class of finitely presented, transfinite metabelian groups.

Recently, Bestvina and Brady resolved an extremely deep question in combinatorial group theory [BB], necessitating the next definition. See [B] for a brief discussion of this question.

DEFINITION 6.3. A group G belongs to $\mathrm{N}_{\mathrm{pfgn}}$ if G is finitely presented and contains no nontrivial perfect normal subgroups that are finitely generated as normal subgroups.

Until Bestvina–Brady, all known examples of perfect normal subgroups of finitely presented groups were, indeed, finitely generated as normal subgroups. Their examples all contain finitely generated, perfect subgroups, and so do not even belong to $\mathrm{N}_{\mathrm{fgp}}$. However, their work is evidence that the classes N_{pn} and $\mathrm{N}_{\mathrm{pfgn}}$ may, indeed, be distinct.

In [DT5] we introduced two intermediate classes of groups.

DEFINITION 6.4. A group G belongs to N_{cl} if whenever (W, M, N, p) is a crumpled lamination, $i: M \to W$ is inclusion, and $\pi_1(M) = G$, then necessarily $\ker(i_\#) = 1$.

Suppose (W, M, N, p) is a crumpled lamination and $j: N \to W$ is a homotopy equivalence; then the hypothesis $\pi_1(M) \in \mathrm{N}_{\mathrm{cl}}$ promises that (W, M, N) is either a product or a nontrivial h-cobordism. In particular, all elements in the standard lamination have the same homotopy type.

Given a finitely presented group G and a nontrivial finitely generated perfect subgroup P of G, we constructed [DT2] a nontrivial crumpled lamination (W, M, N, p) using the mapping cylinder of an acyclic map. The basic strategy was to find a finite acyclic 2-complex K and an embedding $f: K \to M$ with $\mathrm{image}(f_\#: \pi_1(K) \to \pi_1(M)) = P$. Then $f(K)$ became a nontrivial point-preimage of the acyclic map. We call this the *polyhedral acyclic mapping cyclinder construction.*

We then were able to handle a more general situation. A finitely presented group is *almost acyclic* if it is the fundamental group of a finite 2-complex K with $H_1(K; Z)$ free and $H_2(K; Z) = 0$. The *wild group of a group G*, $\mathrm{Wild}(G)$ [C2], is the unique, maximal perfect subgroup of G. In [DT5] we showed how to construct a nontrivial crumpled lamination (W, M, N, p) with $\pi_1(M) \cong G$ whenever given a finitely presented group G, an almost acyclic group H, and a homomorphism $\phi: H \to G$ with $\phi(H) < \mathrm{ncl}(\phi(\mathrm{Wild}(H)); G)$. We label this the *almost acyclic mapping construction.*

DEFINITION 6.5. A group G belongs to N_{aa} if G is finitely presented and, for every almost acyclic group H and homomorphism $\phi: H \to G$, with $\phi(H) < \mathrm{ncl}(\phi(\mathrm{Wild}(H)); G)$, it is necessarily the case that $\phi(H) = 1$.

A group from this class, N_{aa}, admits "no (nontrivial) almost acyclic maps".

We designate the general construction of Theorem 1.1 of this paper as the *perfect normal subgroup construction.*

The ordering of these classes of groups by containment is crucial to understanding which of the three constructions apply to which groups. By definition, $\mathrm{N}_{\mathrm{pn}} \subset \mathrm{N}_{\mathrm{pfgn}}$. Corollary 1.3 and duality require that $\ker(\mathrm{incl}_\#: \pi_1(M) \to \pi_1(W))$ is finitely generated as a perfect, normal subgroup of $\pi_1(M)$, so that $\mathrm{N}_{\mathrm{pfgn}} \subset \mathrm{N}_{\mathrm{cl}}$. Moreover, the almost acyclic mapping construction shows that $\mathrm{N}_{\mathrm{cl}} \subset \mathrm{N}_{\mathrm{aa}}$. Finally, since any acyclic complex is trivially almost acyclic, we have $\mathrm{N}_{\mathrm{aa}} \subset \mathrm{N}_{\mathrm{fgp}}$. Consequently,

$$\mathrm{N}_{\mathrm{pn}} \subset \mathrm{N}_{\mathrm{pfgn}} \subset \mathrm{N}_{\mathrm{cl}} \subset \mathrm{N}_{\mathrm{aa}} \subset \mathrm{N}_{\mathrm{fgp}}.$$

Closer scrutiny of these containments is central to our characterization. In [H] Howie showed that the containment $\mathrm{N}_{\mathrm{pn}} \subset \mathrm{N}_{\mathrm{fgp}}$ is proper. In particular, let $[x, y]$ denote the *commutator of x and y* ($[x, y] = x^{-1}y^{-1}xy$) and let x^y denote *the conjugate of x by y* ($x^y = y^{-1}xy$). For

$$G_1 = \langle y, t \colon y = [y, y^t] \rangle \quad \text{(Adams's group)},$$

Howie proved that $G_1 \in (\mathrm{N_{fgp}} - \mathrm{N_{pn}})$. Later on, we will show that $G_1 \in (\mathrm{N_{aa}} - \mathrm{N_{pn}})$. In [DT5] we demonstrated properness of the containment $\mathrm{N_{aa}} \subset \mathrm{N_{fgp}}$ by verifying that

$$G_2 = \langle y, t \colon y = [y, y^{(y^t)}] \rangle \in \mathrm{N_{fgp}} - \mathrm{N_{aa}}.$$

The main consequence of Theorem 1.1 for this section is that $\mathrm{N_{pfgn}} = \mathrm{N_{cl}}$.

PROPOSITION 6.6. *The classes of finitely presented groups* $\mathrm{N_{pfgn}}$ *and* $\mathrm{N_{cl}}$ *coincide.*

Proof. We need only show that $\mathrm{N_{cl}} \subset \mathrm{N_{pfgn}}$ or (equivalently) that $\mathrm{C(N_{pfgn})} \subset \mathrm{C(N_{cl})}$, where $\mathrm{C}(*)$ denotes the complement of $*$. Let $G \in \mathrm{C(N_{pfgn})}$. Then G is finitely presented with nontrivial perfect normal subgroup K, which is finitely generated as a normal subgroup. Theorem 1.1 provides a nontrivial crumpled lamination (W, M, N, p) with $\pi_1(M) = G$ and $\ker(i_\#) = K \neq 1$ $(i \colon M \to W)$. Thus, $G \in \mathrm{C(N_{cl})}$. □

To summarize:

$$\mathrm{N_{pn}} \subset \mathrm{N_{pfgn}} = \mathrm{N_{cl}} \subsetneq \mathrm{N_{aa}} \subsetneq \mathrm{N_{fgp}}.$$

We give a new more details. Our techniques for demonstrating proper containment include an elaboration, within each of these five classes, about closure with respect to three basic group-theoretic operations: free products (Free), split amalgamated free products (SplitAmalg), and split HNN extensions (SplitHNN). Recall than an amalgamated free product $G = H *_D K$ *splits* if D is a retract of either H or K. Similarly, an HNN-extension $G = \langle H, t \colon D^t = \psi(D) \rangle$ splits if either D or $\psi(D)$ is a retract of H. In [DT5, Sec. 4] we made the determinations shown in Table 1 for all three operations

Table 1

	$\mathrm{N_{pn}}$	$\mathrm{N_{cl}}$	$\mathrm{N_{aa}}$	$\mathrm{N_{fgp}}$
Free	Yes	Yes	Yes	Yes
SplitAmalg	Yes	?	Yes	Yes
SplitHNN	No	No	No	Yes

within all classes, except for $\mathrm{N_{cl}}$. Proposition 6.6 and our next result allow completion of Table 1.

PROPOSITION 6.7. *The class* $\mathrm{N_{pfgn}}$ *is closed under split amalgamated products.*

Proof. Without loss of generality, let $H, K \in \mathrm{N_{pfgn}}$, $G = H *_D K$, and $r \colon H \to D$ be a retraction. Then r extends to a retraction $r \colon G \to K$. Let Q be a perfect, normal subgroup of G which is finitely generated as a normal subgroup. Since r is surjective, $r(Q)$ is a perfect, normal subgroup of K that is finitely generated as a normal subgroup. But $K \in \mathrm{N}_{\mathrm{fgn}}$, so $r(Q) = 1$; that is, $Q \subset \ker(r)$.

In particular, the intersection of Q with each conjugate of D must be trivial. By any standard subgroup theorem for amalgamated free products (our specific reference is the subgroup theorem of Lyndon and Schupp [LS, Chap. I, Sec. 11]), Q, as a subgroup of G, naturally has the structure of a free product:

$$Q = F * \prod_{\alpha} \{H^{g_\alpha} \cap Q\},$$

where F is a free group and $\{H^{g_\alpha}\}$ is a special collection of conjugates of H. Since Q is perfect, F must be trivial. Since Q is normal in G, there is a retraction

$$\eta : \ker(r) \cong \prod_{\alpha} \{H^{g_\alpha}\} \to H$$

that takes Q to $Q \cap H$ and trivializes each factor of $\ker(r)$ other than H. It follows that $Q \cap H$ is a perfect normal subgroup of H.

If we show that $Q \cap H$ is finitely generated as a normal subgroup of H, then $Q \cap H = 1$ since $H \in \mathrm{N}_{\mathrm{pfgn}}$ and necessarily $Q = 1$, completing the proof. To that end, recall that Q is finitely generated as a normal subgroup of G, and assume (without loss of generality) that the generating set $\{h_1, \ldots, h_m\}$ of Q lies in H. Then any $h \in H$ can be written as

$$h = \prod_{1}^{k} h_{i_j}^{g_j},$$

where $g_j \in G$. Moreover, each $g_j = g_{\alpha_j} q_j$, where $g_{\alpha_j}^{-1} H g_{\alpha_j} \cap Q$ is a factor in the free-product representation of Q and where $q_j \in Q$ [LS, p. 78]. Applying η to $h \in H \cap Q$, we have

$$h = \eta(h) = \eta\left(\prod_{1}^{k} h_{i_j}^{g_{\alpha_j} q_j}\right) = \prod_{1}^{k} \eta(h_{i_j}^{g_{\alpha_j}})^{\eta(q_j)}.$$

But $\eta(h_{i_j}^{g_{\alpha_j}}\}$ equals 1 if $g_{\alpha_j} \neq 1$ and equals h_{i_j} if $g_{\alpha_j} = 1$. Also, q_j is in the domain of η, so $\eta(q_j) \in H$ and $h = \eta(h)$ belongs to the normal closure of $\{h_1, \ldots, h_m\}$ in H. □

The completed Table 2 summarizes closure under group operations of these classes in light of Propositions 6.6 and 6.7. We use Table 2 to provide details

Table 2

	N_{pn}	$\mathrm{N}_{\mathrm{pfgn}}$	N_{cl}	N_{aa}	$\mathrm{N}_{\mathrm{fgp}}$
Free	Yes	Yes	Yes	Yes	Yes
SplitAmalg	Yes	Yes	Yes	Yes	Yes
SplitHNN	No	No	No	No	Yes

regarding which inclusions $\mathrm{N}_{\mathrm{pn}} \subset \mathrm{N}_{\mathrm{pfgn}} = \mathrm{N}_{\mathrm{cl}} \subsetneq \mathrm{N}_{\mathrm{aa}} \subsetneq \mathrm{N}_{\mathrm{fgp}}$ are proper. The crux involves scrutiny of the groups G_1 and G_2 previously given.

Tietze transformations show that both G_1 and G_2 belong to $\mathrm{N}_{\mathrm{fgp}}$, since

$$\begin{aligned} G &\cong \langle y, x, t \colon y = [x, y], x = y^t \rangle \cong \langle y, x, t \colon y^2 = y^x, x = y^t \rangle \\ &\cong \langle \langle y, x \colon y^2 = y^x \rangle, t \colon x = y^t \rangle. \end{aligned}$$

Thus, G_1 is obtained by a sequence of two split HNN extensions beginning with $\mathbf{Z}$. But $\mathbf{Z} \in \mathrm{N}_{\mathrm{pn}} \subset \mathrm{N}_{\mathrm{fgp}}$. Since $\mathrm{N}_{\mathrm{fgp}}$ is closed under split HNN extensions, $G_1 \in \mathrm{N}_{\mathrm{fgp}}$ and contains no nontrivial finitely generated perfect subgroups. Similarly,

$$\begin{aligned} G_2 &\cong \langle x, y, z, t \colon y = [y, x], x = y^z, z = y^t \rangle \\ &\cong \langle \langle x, y, z \colon y = [y, x], x = y^z \rangle, t \colon z = y^t \rangle \end{aligned}$$

and G_2 is a split HNN extension of a group isomorphic to G_1. Thus, $G_2 \in \mathrm{N}_{\mathrm{fgp}}$ as well. But $G_2 \notin \mathrm{N}_{\mathrm{aa}}$, so $\mathrm{N}_{\mathrm{aa}} \subsetneq \mathrm{N}_{\mathrm{fgp}}$.

Now, $G_1 \notin \mathrm{N}_{\mathrm{pfgn}} = \mathrm{N}_{\mathrm{cl}}$ since $[G_1, G_1] = \mathrm{ncl}(\{y\}; G_1)$ is a nontrivial perfect subgroup of G_1. However, we can show $G_1 \in \mathrm{N}_{\mathrm{aa}}$. Name a homomorphism $\phi \colon H \to G_1$ of an almost acyclic group H into G_1 with

$$\phi(H) < \mathrm{ncl}(\phi(\mathrm{Wild}(H)); G_1);$$

then $\phi(H) \subset \mathrm{Wild}(G_1) \subset [G_1, G_1]$. Any basic subgroup theorem for HNN extensions gives a presentation for $[G_1, G_1]$ as the infinite amalgamated tree product $(-\infty < m < \infty)$:

$$\cdots * K_{m-1} *_{D_{m-1}} K_m *_{D_m} K_{m+1} * \cdots,$$

where

$$K_m \cong \langle y_m, x_m \colon y_m = [y_m, x_m] \rangle$$

and the amalgamation of K_m and K_{m+1} is along $x_m \equiv y_{m+1}$. Since H is finitely generated,

$$\phi(H) \subset K_r *_{D_r} K_{r+1} *_{D_{r+1}} K_{r+2} * \cdots *_{D_{r+s-1}} K_{r+s}$$

for some r and s, $-\infty < r < \infty$, $0 \le s < \infty$. But each $K_m \in \mathrm{N}_{\mathrm{pn}}$, so

$$K_r *_{D_r} K_{r+1} *_{D_{r+1}} K_{r+2} * \cdots *_{D_{r+s-1}} K_{r+s} \in \mathrm{N}_{\mathrm{pn}}$$

as it results from a sequence of s split amalgamated products of K_m. Since the homomorphic image of a perfect group is perfect, $\phi(\mathrm{Wild}(H)) = 1$ and hence $\phi(H) = 1$. Thus, $G_1 \in \mathrm{N}_{\mathrm{aa}} - \mathrm{N}_{\mathrm{pfgn}}$, giving $\mathrm{N}_{\mathrm{pfgn}} \subsetneq \mathrm{N}_{\mathrm{aa}}$ or (equivalently) $\mathrm{N}_{\mathrm{cl}} \subsetneq \mathrm{N}_{\mathrm{aa}}$. Thus, G_1 is an example that cannot be handled by techniques from prior papers.

We now examine the topology of wildness arising from each class of groups. More precisely, suppose that (W, M, N) is an $(n+1)$-dimensional cobordism $(n \ge 6)$ with $\mathrm{incl} \colon N \to W$ a homotopy equivalence. Recall that $\mathrm{incl}_{\#} \colon \pi_1(M) \to \pi_1(W)$ has perfect kernel, denoted by P. Using any of the three constructions alluded to above, we build a cobordism W' that admits a crumpled lamination (W', M, N', p), with the kernel of the inclusion-induced homomorphism $M \to W'$ equal to P and with the precise properties of the chosen construction. The basic result states that these two cobordisms are topologically the same up to h-cobordism class.

LEMMA 6.8 [DT2, Thm. 5.1]. *Suppose (W, M, N) and (W', M, N') are $(n+1)$-dimensional cobordisms ($n \geq 5$, $\operatorname{bdy}(M) = \emptyset$) such that the inclusions $N \to W$ and $N' \to W'$ are homotopy equivalences and the inclusion-induced homomorphisms $\pi_1(M) \to \pi_1(W)$ and $\pi_1(M) \to \pi_1(W')$ have equal kernels. Then W is homeomorphic to $W' \cup_{N'} W''$, where (W'', N', N'') is an h-cobordism (possibly nontrivial).*

Lemma 6.8 ensures that, up to h-cobordism, (W, M, N) depends only on which of the three constructions we select to build (W', M, N'), and this in turn depends only on $\pi_1(M)$ and $\ker(\mathrm{incl}_{\#}\colon \pi_1(M) \to \pi_1(W'))$. The attached h-cobordism admits a trivial crumpled lamination with all the lamination elements tamely embedded and of the same homotopy type. We summarize this as follows.

THEOREM 6.9. *Let (W, M, N) be an $(n+1)$-dimensional cobordism ($n \geq 6$) with $N \to W$ a homotopy equivalence and*

$$Q = \ker(\mathrm{incl}_{\#}\colon \pi_1(M) \to \pi_1(W)) \neq 1.$$

(a) *(W, M, N) is homotopy equivalent to the crumpled lamination resulting from a perfect normal subgroup construction on M with π_1-kernel $= Q$.*
(b) *If $\pi_1(M) \in \mathrm{N}_{\mathrm{aa}}$ then (W, M, N) is not homotopy equivalent to the crumpled lamination resulting from any almost acyclic mapping construction on M with kernel Q.*
(c) *If $\pi_1(M) \in \mathrm{N}_{\mathrm{fgp}} - \mathrm{N}_{\mathrm{aa}}$ and $Q = \mathrm{ncl}(\phi(\mathrm{Wild}(H)); \pi_1(M))$ for an almost acyclic group H, then (W, M, N) is homotopy equivalent to the crumpled lamination resulting from an almost acyclic mapping construction. However, (W, M, N) is not homotopy equivalent to the crumpled lamination resulting from any polyhedral acyclic mapping construction on M with kernel Q.*
(d) *If $\pi_1(M) \notin \mathrm{N}_{\mathrm{fgp}}$ and $Q = \mathrm{ncl}(P; \pi_1(M))$, where P is a finitely generated perfect subgroup of $\pi_1(M)$, then (W, M, N) is homotopy equivalent to the crumpled lamination resulting from a polyhedral acyclic mapping construction.*

COROLLARY 6.10. *If the appropriate h-cobordism (W', N, N') (possibly nontrivial) is attached to W along N, then "homotopy equivalent" may be replaced by "homeomorphic" in Theorem 6.9.*

The polyhedral acyclic mapping construction arising from the presence of a finitely generated perfect subgroup is quite special. For all practical purposes, we may view the point-preimages of the acyclic maps as manifold, homology cells [DT2, p. 348] or, more generally, as sets with finitely generated perfect fundamental groups. A more complicated crumpled lamination might conceivably arise from an acyclic map of manifolds but not from the polyhedral acyclic mapping construction.

A natural rephrasing asks for a partial converse to Theorem 6.9(d).

QUESTION 6.11. Suppose $f: M \to N$ is an acyclic map of closed manifolds. Must $\ker(f_{\#}: \pi_1(M) \to \pi_1(N))$ equal the normal closure in $\pi_1(M)$ of a finitely generated perfect group?

A negative answer to Question 6.11 would imply the existence of a nontrivial crumpled lamination with an acyclic mapping cylinder structure not arising from the polyhedral acyclic mapping construction. Thus, the point-preimages of f would be acyclic compacta that are not polyhedral and not even ANR-like. Though we do not know the answer to Question 6.11, we can show that large classes of fundamental groups cannot exhibit any type of acyclic mapping structure. In particular, the group G_2 defined in this section, which admits a crumpled lamination with an almost acyclic mapping construction, cannot admit a nontrivial acyclic mapping construction. We shall explain why not.

Suppose $f: M \to N$ is an acyclic map of closed n-manifolds ($n \geq 6$) with $\ker(f_{\#}: \pi_1(M) \to \pi_1(N)) \neq 1$. In [DT4, Lemma 12], we show how to always find $y \in N$ so that for every neighborhood U of $f^{-1}(y)$ in M, the inclusion-induced map $\mathrm{incl}_{\#}: \pi_1(U) \to \pi_1(M)$ is nontrivial but the composition $f_{\#} \circ \mathrm{incl}_{\#}: \pi_1(U) \to \pi_1(M) \to \pi_1(N)$ is trivial. We say f has *local π_1-kernel over y*.

Let L be a closed neighborhood of y in N. Then $f \mid f^{-1}(\mathrm{Int}\, L)$ is an acyclic map of open manifolds. However, if S is the upper semicontinuous decomposition of M into points and acyclic sets, $\{f^{-1}(t) \mid t \in L\}$, then the decomposition space M/S need not be an ANR. In particular, if $g: M \to M/S$ is the decomposition map then M/S may not be locally contractible at points $g \circ f^{-1}(t)$ for $t \in \mathrm{bdy}(L)$. The question of whether M/S is an ANR is closely related to the question identified by Daverman and Walsh [DW] of whether, in this setting, $M/f^{-1}(t)$ is necessarily an ANR for each $t \in N$.

In any case, $g \circ f^{-1}(\mathrm{int}(L))$ will be an ANR. To study whether particular groups will admit nontrivial acyclic maps, we define a new class of finitely presented groups. The description of this class is quite technical because of the unresolved question alluded to in the previous paragraph. Generally, though, groups in this class admit only acyclic maps with trivial local π_1-kernel.

DEFINITION 6.12. A finitely presented group G belongs to $\mathrm{N_a}$ if, whenever $f: M \to X$ is an acyclic map from a compact manifold with $\pi_1(M) \cong G$ onto a finite-dimensional metric space X and V is an open and locally contractible subset of X with $\mathrm{incl}: f^{-1}(V) \to M$, then necessarily

$$\mathrm{incl}_{\#}(\ker\{(f \mid f^{-1}(V))_{\#}: \pi_1(f^{-1}(V)) \to \pi_1(V)\}) = 1.$$

THEOREM 6.13. *The class N_a is closed under both the operations of split amalgamated products and split HNN-extensions.*

Thus, a group belongs to N_a if it is obtained by a finite sequence of split amalgamated products or split HNN-extensions beginning with a group (or groups) belonging to N_a. In particular, the group G_2 above belongs to $N_a - N_{aa}$. Consequently, a nontrivial crumpled lamination (W, M, N) with $\pi_1(M) \cong G_2$ cannot have the structure of an acyclic mapping cyclinder of any kind. The question of whether $N_{aa} \subset N_a$ remains unresolved, although some progress has been made on this front (see [DLT]). Ultimately, we definitively know only that

$$N_{pn} \subset N_{pfgn} = N_{cl} \subsetneq N_a \subset N_{fgp}.$$

We close with the proof of Theorem 6.13. The following lemma is essential.

LEMMA 6.14. *Suppose $f: M \to X$ is an acyclic map of a compact manifold M onto a finite-dimensional metric space X, V is an open, locally contractible subset of X with* incl: $f^{-1}(V) \to M$, *and*

$$\mathrm{incl}_\#(\ker\{(f \mid f^{-1}(V))_\#: \pi_1(f^{-1}(V)) \to \pi_1(V)\}) \neq 1.$$

Suppose $\pi_1(M) \notin N_a$ and $\psi: \pi_1(M) \to G_1$ is a homomorphism to a finitely presented group G_1 such that

$$\psi(\mathrm{incl}_\#(\ker\{(f \mid f^{-1}(V))_\#: \pi_1(f^{-1}(V)) \to \pi_1(V)\})) \neq 1.$$

Then $G_1 \notin N_a$.

Proof. By adding a finite number of 1- and 2-handles to $M \times [0,1]$ along $M \times \{1\}$, we obtain a cobordism (W, M, N) where $\pi_1(W) \cong G_1$, $\mathrm{incl}_\#: \pi_1(N) \to \pi_1(W)$ is an isomorphism, and $\mathrm{incl}_\#: \pi_1(M) \to \pi_1(W)$ induces the commutative diagram

$$\begin{array}{ccc}
\pi_1(M) & \cong & G \\
\downarrow & & \downarrow{\scriptstyle\psi} \\
\pi_1(W) & \cong & G_1 \\
\uparrow{\scriptstyle\cong} & & \| \\
\pi_1(N) & \cong & G_1.
\end{array}$$

Consider the acyclic decomposition G of W whose nondegenerate elements are

$$\{(f^{-1}(x), s) \in M \times [0,1] \mid x \in \mathrm{Cl}(V) \text{ and } s \in [\tfrac{1}{4}, \tfrac{1}{2}]\}.$$

Let $f': W \to X' = W/G$ be the decomposition map, $V' = f'(f^{-1}(V) \times (\frac{1}{4}, \frac{1}{2}))$, and incl: $f'^{-1}(V') \to W$. It is easy to check that

$$\mathrm{incl}_\#(\ker\{(f' \mid f'^{-1}(V'))_\#: \pi_1(f'^{-1}(V')) \to \pi_1(V')\}) \neq 1 \in \pi_1(W).$$

Thus, $G_1 \notin N_a$. □

Proof of Theorem 6.13. We first prove that N_a is closed under free products. To that end, suppose $G = H * K$ is a free product where $H, K \in N_a$, and that M, X, $f: M \to X$, and V are as in the definition of N_a. Let

$$E = \mathrm{incl}_\#(\ker\{(f \mid f^{-1}(V))_\#: \pi_1(f^{-1}(V)) \to \pi_1(V)\}).$$

Let $r: G \to H$ and $s: G \to K$ be the obvious retractions. Since $H, K \in N_a$, necessarily $E \subset (\ker(r) \cap \ker(s))$. Thus, E is contained in a free subgroup of G, implying $E = 1$ since free groups contain no nontrivial perfect subgroups.

Second, suppose $G = H *_D K$ is a free product with amalgamated subgroup D, $H, K \in N_a$, and $r: K \to D$ is a retraction. Then r extends to a retraction $r: G \to H$. But $\ker(r)$ trivially intersects each conjugate of D, so $\ker(r)$ is the free product of a free group and a collection of conjugates of K intersected with $\ker(r)$:

$$\ker(r) = F * \prod(K^{g_\alpha} \cap \ker(r))$$

[LS, Chap. 2].

If M, X, f, V, and E are as above, then $E \subset \ker(r)$. Let $q: M' \to M$ be the cover of M corresponding to $\ker(r)$. Then $\pi_1(M') \cong \ker(r)$ and $q_\#$ is injective. Let $x \in V$. Since V is locally contractible and $f^{-1}(x)$ is acyclic, it follows that $f^{-1}(x)$ has a compact manifold neighborhood L_x and a lifting $j: L_x \to M'$. Hence image$(j_\#)$ is contained in a subgroup of $\pi_1(M')$ that is the free product of a finitely generated free group and finitely many conjugates of K. By the free product case, this subgroup belongs to N_a. To apply Lemma 6.14, let P be any compact neighborhood of x in V with $f^{-1}(P) \subset \mathrm{int}(L_x)$ and, with the image of $(f \mid f^{-1}(\mathrm{int}(P))_\#$ trivial in $\pi_1(V)$, let G be the decomposition of L_x consisting of points and nondegenerate elements $\{f^{-1}(y) \mid y \in P\}$; let $f': L_x \to L_x/G$ be the decomposition map. Now, $f'(f^{-1}(\mathrm{Int}\, P)) \approx \mathrm{Int}\, P$, and Lemma 6.14 requires that image$((j \mid f^{-1}(\mathrm{Int}\, P))_\#)$ be trivial in $\pi_1(M')$. Thus, $\mathrm{incl}_\#(\pi_1(f^{-1}(\mathrm{Int}\, P)))$ is trivial in $\pi_1(M)$. It follows easily that E is trivial in $\pi_1(M)$ and G belongs to N_a.

Finally, suppose K belongs to N_a. Let $G = \langle K, t \mid D^t = \phi(D)\rangle$, $r: K \to D$ be a retraction, and $s: G \to Z$ be the homomorphism sending K to 0 and t to 1. Then $\ker(s)$ is the split, amalgamated tree product of conjugates of K:

$$\ker(s) \cong \cdots *_{D_{-1}} K_{-1} *_{D_0} K_0 *_{D_1} K_1 * \cdots,$$

where $K_j = K^{t^j}$, $D_j = D^{t^j}$, and $(r \mid K_j): K_j \to D_j$ is a retraction. Since $\mathbf{Z}$ contains no nontrivial perfect subgroups, $E \subset \ker(s)$. By again localizing to a compact manifold neighborhood L_x of any $f^{-1}(x)$, we see that $\pi_1(L_x)$ includes into a finite, split, amalgamated tree product that must—by the second part of this theorem—belong to N_a. Thus, E is trivial in $\pi_1(M) \cong G$, and G belongs to N_a. □

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R. J. Daverman
Department of Mathematics
University of Tennessee—Knoxville
Knoxville, TN 37996-1300

F. C. Tinsley
Department of Mathematics
Colorado College
Colorado Springs, CO 80903