Weakly Outer Polynomials

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1. Introduction

The notion of a weakly outer function has its origins in the prediction theory of 2-parameter stationary random fields. Evidently, those stationary fields with the so-called commutation properties provide a natural medium in which to seek out multiparameter extensions of many classical 1-parameter results [5; 8; 11]. In [1; 2] it is proved that such a field possesses the "weak commutation property" if and only if its spectral density is the squared modulus of a weakly outer function in $H^2(\mathbf{T}^2)$ (a slightly weaker result is obtained in [8]). The related function theory and further applications to prediction are treated in [3].

In the present work, another prediction-theoretic result is obtained. It states that for weakly commutative stationary fields, the past, conditioned on the future (in some sense), is finite-dimensional if and only if the associated weakly outer function is a certain type of rational function. This, in turn, points to the need to characterize the weakly outer polynomials. A complete characterization in terms of zero sets is found.

2. Notation and Preliminaries

Let $\mathbf{D}$ be the open unit disc, and $\mathbf{T}$ the unit circle, in the complex plane $\mathbf{C}$. Normalized Lebesgue measure on $\mathbf{T}$ is written $d\sigma$, and $d\sigma_r$ is the associated product measure on the torus $\mathbf{T}^r$. By the symbols $N_*(\mathbf{D}^r)$ and $H^p(\mathbf{D}^r)$ we mean the usual Nevanlinna and Hardy classes of analytic functions on the polydisc $\mathbf{D}^r$ (see [6; 9]). We shall identify a function on the polydisc with its radial limit function on the torus, whenever the latter exists; likewise, an integrable function on the torus will be identified with its harmonic extension into the polydisc.

The symbol $\hat{}$ indicates a Fourier coefficient. Thus, if $f \in L^1(\mathbf{T}^2)$ then

$$\hat{f}_{m,n} = \int f(e^{is}, e^{it}) e^{-ims-int}\, d\sigma_2(e^{is}, e^{it}).$$

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For $f \in L^p(\mathbf{T}^r)$, we write $f \mathrel{\hat{\in}} S$ if the Fourier coefficients of f vanish outside the subset S of $\mathbf{Z}^r$.

The Cauchy kernel in one variable will be denoted by C_z. Thus $C_z(e^{is}) = (1-ze^{-is})^{-1}$. A function $f \in N_*(\mathbf{D}^r)$ is *outer* if

$$\log|f(0)| = \int \log|f|\, d\sigma_r.$$

In this case,

$$f(z_1, \dots, z_r) = c \cdot \exp \int (2C_{z_1} \cdots C_{z_r} - 1) \log|f|\, d\sigma_r$$

with some $|c| = 1$. A polynomial in one variable is outer precisely when its zeros lie outside the disc. If a polynomial in r variables is outer then it cannot vanish in the polydisc $\mathbf{D}^r$; by examining the "slice functions" [9, p. 44] we see that this condition is also sufficient.

A function $f \in N_*(\mathbf{D}^2)$ is *weakly outer* if $f(e^{is}, \cdot)$ is outer in the disc for almost every fixed e^{is} and if $f(\cdot, e^{it})$ is outer in the disc for almost every fixed e^{it}. An outer function in $N_*(\mathbf{D}^2)$ is weakly outer, but the converse is false. Some elementary properties of weakly outer functions are explored in [3]. In particular, the following structural information is obtained [3, Thm. 2.9].

THEOREM 2.1. *Let $f \in N_*(\mathbf{D}^2)$. Then f is weakly outer if and only if there are outer functions $g(e^{is}, e^{it})$ and $h(e^{is}, e^{it})$, and 1-variable unimodular functions $a(e^{is})$ and $b(e^{it})$, such that*

$$f(e^{is}, e^{it}) = g(e^{is}, e^{it})\bar{h}(e^{is}, e^{-it})a(e^{is}) \tag{1}$$

$$= g(e^{is}, e^{it})h(e^{is}, e^{-it})\bar{b}(e^{-it}). \tag{2}$$

We may take $h(0,0)$ to be positive, and

$$g(z_1, z_2) = \exp \int (2C_{z_1}C_{z_2} - 1) \log|f|\, d\sigma_2; \tag{3}$$

in this case, the representation above is unique.

Thus, a weakly outer function f factors into an outer part g and a "purely" weak outer part h.

The information provided by Theorem 2.1 is not fully satisfactory, however. The outer function h must be quite unusual in order to satisfy conditions (1) and (2). For a polynomial f this issue is addressed in the next section.

3. Principal Results

Let us consider a problem in the prediction theory of stationary random processes. Suppose that $w(e^{i\theta})$ is the spectral density for a regular stationary process on $\mathbf{Z}$ (regularity in this case is equivalent to the integrability of $\log w$). We may identify the "past" and "future" of the process with the subspaces

$$\mathcal{P} = \bigvee\{e^{in\theta} : n \le 0\} \quad \text{and} \quad \mathcal{F} = \bigvee\{e^{in\theta} : n \ge 0\}$$

of $L^2(w\,d\sigma)$. A classical result (see [7, §4.7]) states that the projection of $\mathcal{P}$ onto $\mathcal{F}$ is finite-dimensional if and only if $w(e^{i\theta})$ is a rational function. Equivalently, w factors into the squared modulus of an outer rational function.

For a 2-parameter version of this problem, we start with a weight function $w(e^{is}, e^{it})$ on the torus. We define the "right" and "top" subspaces of $L^2(w\,d\sigma_2)$ by

$$\mathcal{R} = \bigvee\{e^{ims+int} : m \ge 0\} \quad \text{and} \quad \mathfrak{T} = \bigvee\{e^{ims+int} : n \ge 0\}.$$

The needed regularity condition here is

$$\bigcap_{m=0}^{\infty} e^{ims}\mathcal{R} = (0) \quad \text{and} \quad \bigcap_{n=0}^{\infty} e^{int}\mathfrak{T} = (0),$$

which occurs exactly when

$$\int \log w(e^{i\theta}, e^{it})\, d\sigma(e^{i\theta}) > -\infty,$$

$$\int \log w(e^{is}, e^{i\theta})\, d\sigma(e^{i\theta}) > -\infty$$

for almost every fixed e^{is} and e^{it} (see [4]). If $\mathcal{M}$ is a subspace of $L^2(w\,d\sigma_2)$, we write $P_{\mathcal{M}}$ for the projection operator of $L^2(w\,d\sigma_2)$ onto $\mathcal{M}$. For the "past" and "future" of $L^2(w\,d\sigma_2)$, we take

$$\mathcal{P} = \bar{\mathcal{R}} \cap \bar{\mathfrak{T}} \quad \text{and} \quad \mathcal{F} = \mathcal{R} \cap \mathfrak{T},$$

where the overline denotes pointwise complex conjugation.

The space $L^2(w\,d\sigma_2)$ has the *weak commutation property* if the projection operators $P_{\mathcal{R}}$ and $P_{\mathfrak{T}}$ commute; in this case, their product is the projection onto $\mathcal{F}$ as defined above. By [2, Thm. 1.1], $L^2(w\,d\sigma_2)$ is weakly commutative exactly when $w = |f|^2$ for some weakly outer function f in $H^2(\mathbf{T}^2)$.

Here, then, is an extension of the classical result to the 2-parameter scenario.

THEOREM 3.1. *Suppose that f is weakly outer in $H^2(\mathbf{T}^2)$. Then the subspace $P_{\mathcal{F}}\mathcal{P}$ of $L^2(|f|^2\,d\sigma_2)$ is finite-dimensional if and only if f has the structure*

$$f(z_1, z_2) = \frac{p(z_1, z_2)}{q_1(z_1)\,q_2(z_2)}, \tag{4}$$

where $p(z_1, z_2)$ is a weakly outer polynomial and $q_1(z_1)$ and $q_2(z_2)$ are 1-variable outer polynomials.

This theorem suggests the problem of describing all the weakly outer polynomials. An immediate necessary and sufficient condition for a polynomial to be weakly outer is that it have no zeros on $\mathbf{T} \times \mathbf{D}$ or $\mathbf{D} \times \mathbf{T}$. In fact, much more can be said. Let us write $\mathbf{A}$ for the set $\{z \in \mathbf{D} : |z| > 1\}$.

THEOREM 3.2. *A polynomial $f(z_1, z_2)$ is weakly outer if and only if there are polynomials $p(z_1, z_2)$ and $q(z_1, z_2)$ such that* (i) *p has no zeros in* $\mathbf{D}^2$; (ii) *q has no zeros outside of* $(\mathbf{D}^2 \cup \mathbf{T}^2 \cup \mathbf{A}^2)$; *and* (iii) $f = pq$.

Hence the outer and purely weak outer parts of a weakly outer polynomial are themselves polynomials; this is not at all trivial from (1) and (2). Note that the zero set of the purely weak factor is quite restricted.

It is straightforward to check that the following polynomials have property (ii), and are in fact weakly outer: $q(z_1, z_2) = z_1^m + z_2^n$, where m and n are positive integers; $q(z_1, z_2) = \xi + z_1 - z_2 - \bar{\xi} z_1 z_2$, where $|\xi| < 1$.

4. Proofs

Proof of Theorem 3.1. First, let us recall some concrete representations of the spaces $\mathfrak{F}$ and $\mathcal{P}$ and the projection $P_{\mathfrak{F}}$. Take Q to be the projection operator of $L^2(\mathbf{T}^2)$ onto $H^2(\mathbf{T}^2)$. Again, by $\bar{f}$ we mean the pointwise complex conjugate of f; for a set of functions $\mathfrak{M}$, we write $\bar{\mathfrak{M}}$ for the set $\{\bar{f} : f \in \mathfrak{M}\}$.

LEMMA 4.1. *If f is weakly outer in $H^2(\mathbf{T}^2)$, and the supspaces $\mathcal{P}$ and $\mathfrak{F}$ of $L^2(|f|^2\, d\sigma)$ are defined as above, then* (i) *we can make the identifications*

$$f\mathfrak{F} = H^2(\mathbf{T}^2) \quad \textit{and} \quad \bar{f}\mathcal{P} = \bar{H}^2(\mathbf{T}^2);$$

(ii) *the projection $P_{\mathfrak{F}}$ has the realization*

$$P_{\mathfrak{F}} h = (1/f) Q(fh).$$

Part (i) of the Lemma is from [11], while part (ii) comes from [2].

Now suppose that $P_{\mathfrak{F}}\mathcal{P}$ has finite dimension N. Consider the $N+1$ functions $1, e^{-is}, e^{-2is}, \ldots, e^{-iNs}$ in $\mathcal{P}$. It may be that $P_{\mathfrak{F}} e^{-ijs} = 0$ for some j. Otherwise, the set $\{P_{\mathfrak{F}} e^{-ijs}\}_{j=0}^{N}$ is linearly dependent, and there are coefficients $\{a_j\}_{j=0}^{N}$, not all zero, such that

$$P_{\mathfrak{F}}\left(\sum_{j=0}^{N} a_j e^{ijs}\right) = 0.$$

Either way, $P_{\mathfrak{F}}$ annihilates a nonzero polynomial g in e^{-is}. By Lemma 4.1,

$$(1/f(e^{is}, e^{it}))\, Q(f(e^{is}, e^{it})\, g(e^{-is})) = 0;$$
$$Q(f(e^{is}, e^{it})\, g(e^{-is})) = 0.$$

Thus, $(f(e^{is}, e^{it})\, g(e^{is})) \hat{\in} \{(m, n) : -N \leq m < 0,\, n \geq 0\}$.

Let J be the smallest nonnegative integer for which $e^{iJs} g(e^{-is})$ is a polynomimal in e^{is}, and call this polynomial $q_1(e^{is})$. We have shown that

$$fq_1 \hat{\in} \{(m, n) : 0 \leq m < N,\, n \geq 0\}.$$

A similar argument in terms of the second variable produces a polynomial $q_2(e^{it})$ of degree K, $0 \leq K < N$, such that

$$fq_2 \hat{\in} \{(m, n) : m \geq 0,\, 0 \leq n < N\}.$$

Now observe that $fq_1q_2 \hat{\in} \{(m,n): 0 \le m < N, 0 \le n < N\}$, and hence is a polynomial $p(e^{is}, e^{it})$. This verifies the necessity of the representation of f.

For sufficiency, suppose that f has the form (4). We assume that all of the nontrivial common factors have been canceled. Since f is weakly outer in $H^2(\mathbf{T}^2)$, the polynomials q_1 and q_2 must have no zeros in the closed unit disc. It further follows that p is weakly outer. Let M be the smallest positive integer for which $e^{-iMs}q_1(e^{is})$, $e^{-iMt}q_2(e^{it})$, and $e^{-iMs-iMt}p(e^{is}, e^{it})$ all lie in $\bar{H}^2(\mathbf{T}^2)$. Then, as subspaces of $L^2(d\sigma_2)$,

$$fP_{\mathfrak{F}}\mathcal{P} = Q(f\mathcal{P}) \tag{5}$$

$$= Q\left(\frac{f(e^{is}, e^{it})}{\bar{f}(e^{is}, e^{it})}\bar{H}^2\right) \tag{6}$$

$$= \bigvee\left\{Q\left(\frac{f(e^{is}, e^{it})}{\bar{f}(e^{is}, e^{it})}e^{-ijs-ikt}\right): j \ge 0, k \ge 0\right\}$$

$$= \bigvee\left\{Q\left(\frac{p(e^{is}, e^{it})\bar{q}_1(e^{is})\bar{q}_2(e^{it})}{\bar{p}(e^{is}, e^{it})q_1(e^{is})q_2(e^{it})}e^{-ijs-ikt}\right): j \ge 0, k \ge 0\right\}$$

$$\subseteq \bigvee\left\{Q\left(\frac{p(e^{is}, e^{it})}{\bar{p}(e^{is}, e^{it})q_1(e^{is})q_2(e^{it})}e^{-ijs-ikt}\right): j \ge 0, k \ge 0\right\}.$$

This last line follows because Q is linear, and for each $m \ge 0$ and $n \ge 0$ the function

$$\frac{p}{\bar{p}q_1q_2}\bar{q}_1\bar{q}_2e^{-ims-int}$$

belongs to the linear span of $\{(p/\bar{p}q_1q_2)e^{-ijs+ikt}: j \ge 0, k \ge 0\}$. The chain of inclusions continues with

$$\subseteq \bigvee\left\{Q\left(\frac{p(e^{is}, e^{it})}{\bar{p}(e^{is}, e^{it})q_1(e^{is})q_2(e^{it})}e^{-ijs-ikt}\right): 0 \le j \le M, 0 \le k \le M\right\}$$

$$+ \bigvee\left\{Q\left(\frac{p(e^{is}, e^{it})}{\bar{p}(e^{is}, e^{it})q_2(e^{it})}e^{-ijs-ikt}\right): j > M, k \ge 0\right\}$$

$$+ \bigvee\left\{Q\left(\frac{p(e^{is}, e^{it})}{\bar{p}(e^{is}, e^{it})q_1(e^{is})}e^{-ijs-ikt}\right): j \ge 0, k > M\right\} \tag{7}$$

To see this, first observe that for each $j \ge M$, the function e^{-ijs} is linearly dependent on $q_1(e^{is})e^{-ijs}, e^{-i(j-1)s}, \ldots, e^{-is}, 1$. Thus

$$\begin{aligned}\bar{H}^2 &= \bigvee\{e^{-ijs-ikt}: 0 \le j \le M, 0 \le k \le M\} \\ &\quad + \bigvee\{e^{-ijs-ikt}: j > M, k \ge 0\} \\ &\quad + \bigvee\{e^{-ijs-ikt}: j \ge 0, k > M\} \\ &= \bigvee\{e^{-ijs-ikt}: 0 \le j \le M, 0 \le k \le M\} \\ &\quad + \bigvee\{q_1(e^{is})e^{-ijs-ikt}: j > M, k \ge 0\} \\ &\quad + \bigvee\{e^{-ijs-ikt}: 0 \le j \le M, k \ge 0\} +\end{aligned}$$

$$+\bigvee\{q_2(e^{it})e^{-ijs-ikt}: j\geq 0,\, k>M\}$$
$$+\bigvee\{e^{-ijs-ikt}: j\geq 0,\, 0\leq k\leq M\}.$$

But $\bigvee\{e^{-ijs-ikt}: 0\leq j\leq M,\, k\geq 0\}$ is already a subspace of $\bigvee\{e^{-ijs-ikt}: 0\leq j\leq M,\, 0\leq k\leq M\}+\bigvee\{q_2(e^{it})e^{-ijs-ikt}: j\geq 0,\, k>M\}$. Thus

$$\begin{aligned}\bar{H}^2 = {} & \bigvee\{e^{-ijs-ikt}: 0\leq j\leq M,\, 0\leq k\leq M\}\\ & +\bigvee\{q_1(e^{is})e^{-ijs-ikt}: j>M,\, k\geq 0\}\\ & +\bigvee\{q_2(e^{it})e^{-ijs-ikt}: j\geq 0,\, k>M\}.\end{aligned}$$

The chain of inclusions continues from (7) with

$$\begin{aligned}(fP_{\mathfrak{F}}\mathcal{P}) \subseteq {} & \bigvee\left\{Q\left(\frac{p(e^{is},e^{it})}{\bar{p}(e^{is},e^{it})q_1(e^{is})q_2(e^{it})}e^{-ijs-ikt}\right): 0\leq j\leq M,\, 0\leq k\leq M\right\}\\ & +\bigvee\left\{Q\left(\frac{e^{-iMs}p(e^{is},e^{it})}{\bar{p}(e^{is},e^{it})q_2(e^{it})}e^{-is}e^{-ijs-ikt}\right): j\geq 0,\, k\geq 0\right\}\\ & +\bigvee\left\{Q\left(\frac{e^{-iMt}p(e^{is},e^{it})}{\bar{p}(e^{is},e^{it})q_1(e^{is})}e^{-it}e^{-ijs-ikt}\right): j\geq 0,\, k\geq 0\right\} \qquad (8)\\ \subseteq {} & \bigvee\left\{Q\left(\frac{p(e^{is},e^{it})}{\bar{p}(e^{is},e^{it})q_1(e^{is})q_2(e^{it})}e^{-ijs-ikt}\right): 0\leq j\leq M,\, 0\leq k\leq M\right\}\\ & +\bigvee\left\{Q\left(\frac{e^{-iMs}p(e^{is},e^{it})}{\bar{p}(e^{is},e^{it})q_2(e^{it})}e^{-is}e^{-ijs-ikt}\right): j\geq 0,\, k\in\mathbf{Z}\right\}\\ & +\bigvee\left\{Q\left(\frac{e^{-iMt}p(e^{is},e^{it})}{\bar{p}(e^{is},e^{it})q_1(e^{is})}e^{-it}e^{-ijs-ikt}\right): j\in\mathbf{Z},\, k\geq 0\right\} \qquad (9)\\ \subseteq {} & \bigvee\left\{Q\left(\frac{p(e^{is},e^{it})}{\bar{p}(e^{is},e^{it})q_1(e^{is})q_2(e^{it})}e^{-ijs-ikt}\right): 0\leq j\leq M,\, 0\leq k\leq M\right\}\\ & +\bigvee\left\{Q\left(\frac{e^{-iMs}p(e^{is},e^{it})}{q_2(e^{it})}e^{-is}e^{-ijs-ikt}\right): j\geq 0,\, k\in\mathbf{Z}\right\}\\ & +\bigvee\left\{Q\left(\frac{e^{-iMt}p(e^{is},e^{it})}{q_1(e^{is})}e^{-it}e^{-ijs-ikt}\right): j\in\mathbf{Z},\, k\geq 0\right\}. \qquad (10)\end{aligned}$$

In this last step, the weak outer property of p provides that $\bigvee\{e^{ijs+ikt}: j\geq 0,\, k\in\mathbf{Z}\}=\bigvee\{p(e^{is},e^{it})e^{ijs+ikt}: j\geq 0,\, k\in\mathbf{Z}\}$ (see [3, Thm. 3.6]). The last two spaces in (10) are trivial, since Q is operating there on functions with zero Fourier coefficients throughout the first quadrant; the first space is of dimension at most $(M+1)^2$. This establishes sufficiency. □

Proof of Theorem 3.2. Let $f(z_1,z_2)$ be a weakly outer polynomial. Let g, h, a, and b be the functions provided by Theorem 2.1, with g given by (3). Consider the auxiliary function

$$
\begin{aligned}
F(z_1,z_2) &= g(z_1,z_2)^2 \cdot \exp \int (-2C_{z_1}+1)\log|f|\,d\sigma_2 \\
&\quad \cdot \exp \int (-2C_{z_2}+1)\log|f|\,d\sigma_2 \cdot \exp \int \log|f|\,d\sigma_2 \\
&= \exp \int (2C_{z_1}-1)(2C_{z_2}-1)\log|f|\,d\sigma_2.
\end{aligned}
$$

Fix z_1 and z_2.

Since f is weakly outer, $f(z,e^{it})$ is outer in z for e^{it} fixed. Hence

$$\int (2C_{z_1}-1)\log|f(e^{is},e^{it})|\,d\sigma(e^{is}) = \operatorname{Log} f(z_1,e^{it}) + i\gamma(e^{it}),$$

where Log is a branch of the logarithm, and γ is chosen so that $e^{i\gamma(e^{it})}f(0,e^{it})$ is positive. We now need to interpret the logarithm as a function of e^{it}. To do this, note that for z_1 fixed, $f(z_1,e^{it})$ is a polynomial in e^{it}. Thus we may write

$$f(z_1,e^{it}) = b(z_1)(e^{it}-\alpha_1)\cdot\cdots\cdot(e^{it}-\alpha_J)\cdot(e^{it}-\beta_1)\cdot\cdots\cdot(e^{it}-\beta_K), \tag{11}$$

where each α_j and β_k are dependent on z_1, and $|\alpha_j|<1$ and $|\beta_k|>1$. (There are no unimodular roots, as the weak outer property of f essentially precludes zeros on $\mathbf{D}\times\mathbf{T}$ or $\mathbf{T}\times\mathbf{D}$.)

Let us examine the behavior of the factors arising in (11). If $|\beta|>1$, then

$$\operatorname{Log}(e^{it}-\beta) = \log|e^{it}-\beta| + i\arg(e^{it}-\beta).$$

The right side is a well-defined function of e^{it}. If $|\alpha|<1$, then

$$
\begin{aligned}
\operatorname{Log}(e^{it}-\alpha) &= \operatorname{Log}[e^{it}(1-\alpha e^{-it})] \\
&= \log|1-\alpha e^{-it}| + i\arg(1-\alpha e^{-it}) + i\arg(e^{it}) \\
&= \log|1-\bar{\alpha}e^{it}| - i\arg(1-\bar{\alpha}e^{it}) + i\arg(e^{it}).
\end{aligned}
$$

Again, the last expression is a well-defined function of e^{it} modulo $2\pi i$.

We will need to integrate such expressions against the kernel $(2C_{z_2}-1)$. For this step recall that $(2C_z-1)=P_z+iQ_z$, where P_z is the Poisson kernel and Q_z is the conjugate kernel. Thus, if u and v are real integrable functions on the circle then

$$\int (P_z+iQ_z)(u+iv)\,d\sigma = u(z)+i\tilde{u}(z)+iv(z)-\tilde{v}(z),$$

where $\tilde{}$ represents Fourier conjugation. Applying this to the two types of factors in (11) yields

$$
\begin{aligned}
\int (2C_{z_2}-1)\operatorname{Log}(e^{it}-\beta)\,d\sigma &= \log|z_2-\beta| + i\arg(z_2-\beta) + i\arg(z_2-\beta) \\
&\quad + \log|z_2-\beta| - \eta_1(z_1) \\
&= \operatorname{Log}(z_2-\beta)^2 - \eta_1(z_1)
\end{aligned} \tag{12}
$$

and

$$\int (2C_{z_2}-1)\operatorname{Log}(e^{it}-\alpha)\,d\sigma = \log|1-\bar{\alpha}z_2| + i\arg(1-\bar{\alpha}z_2) - i\arg(1-\bar{\alpha}z_2) \\ -\log|1-\bar{\alpha}z_2| - \eta_2(z_1) - \delta(z_2). \tag{13}$$

Each η_j term arises from the e^{it}-constant difference between $-v$ and $(\tilde{v})^{\sim}$, and δ takes the contribution from $i\arg(e^{it})$.

Let us grant for the moment that J and K are independent of z_1. Then there are exactly J terms of the type $i\arg(e^{it})$ when (13) is applied to the appropriate factors of (11). Thus, from (12) and (13) we have

$$F(z_1,z_2) = \exp\int (2C_{z_2}-1)[\operatorname{Log} f(z_1,e^{it}) + i\gamma(e^{it})]\,d\sigma(e^{it}) \\ = \Phi(z_2)\Psi(z_1)(z_2-\beta_1)^2\cdots(z_2-\beta_K)^2,$$

where Φ and Ψ are genuinely functions of one variable.

Repeat this argument with the $d\sigma(e^{is})$ integral taken first, to get

$$F(z_1,z_2) = \Phi'(z_1)\Psi'(z_2)(z_1-\beta_1')^2\cdots(z_1-\beta_{K'}')^2.$$

It follows that

$$Q(z_1,z_2) = \Psi(z_1)\Phi'(z_1)^{-1}(z_2-\beta_1)^2\cdots(z_2-\beta_K)^2 \\ = \Psi'(z_2)\Phi(z_2)^{-1}(z_1-\beta_1')^2\cdots(z_1-\beta_{K'}')^2$$

is the square of an outer polynomial in each separate variable z_1 and z_2. Now

$$g(z_1,z_2)^2 = Q(z_1,z_2)\Phi'(z_1)\Phi(z_2)\cdot\exp\int (2C_{z_1}-1)\log|f|\,d\sigma_2 \\ \cdot\exp\int (2C_{z_2}-1)\log|f|\,d\sigma_2\cdot\exp\int(-\log|f|)\,d\sigma_2.$$

Fix z_2, and isolate $\Phi'(z_1)$ to see that it is outer. Likewise, $\Phi(z_2)$ is outer. It follows that $Q(z_1,z_2)^{1/2}$ is an outer polynomial on the bidisc $\mathbf{D}^2$. We have proved the following fact about the outer part g.

LEMMA 4.2. *The outer part $g(z_1,z_2)$ of a weakly outer polynomial $f(z_1,z_2)$ is an outer polynomial, multiplied by 1-variable outer functions.*

In establishing this, we needed J and K to be constants independent of z_1. Let us now verify that this is indeed true. First, we write

$$f(z_1,z_2) = f_0(z_1) + f_1(z_1)z_2 + \cdots + f_n(z_1)z_2^n,$$

where f_n is not identically zero. For all but finitely many z_1 (namely, the zeros of the polynomial f_n), $f(z_1,\cdot)$ has exactly n roots. Suppose, for now, that f is irreducible.

Apply the inverse function theorem [10, Thm. 9.28] to f, viewed as a function from $\mathbf{R}^4$ into $\mathbf{R}^2$. The hypothesis "A_x is invertible at the zero (a,b) of f" here takes the form

$$\det\begin{bmatrix} \dfrac{\partial \Re f}{\partial x_1} & \dfrac{\partial \Re f}{\partial y_1} \\ \dfrac{\partial \Im f}{\partial x_1} & \dfrac{\partial \Im f}{\partial y_1} \end{bmatrix}(a,b) \neq 0.$$

By the Cauchy–Riemann equations, this is equivalent to

$$\frac{\partial f}{\partial z_1}(a,b) \neq 0. \tag{14}$$

By Bezout's theorem [12, p. 29], f and $\partial f/\partial z_1$ can have only finitely many common zeros (since f was presumed to be irreducible); that is, (14) holds for all but finitely many points (a,b). Away from these exceptional points, [10, Thm. 9.28] provides open sets $U \subseteq \mathbf{C}\times\mathbf{C}$ $(=E$ in the notation of [10]) and $W \subseteq \mathbf{C}$, with $a \in W$ and $(a,b) \in U$, such that for every $z_1 \in W$ there exists $z_2 \in \mathbf{C}$ with $(z_1, z_2) \in U$ and $f(z_1, z_2) = 0$. If the given root (a,b) lies in $\mathbf{D}\times\mathbf{T}$, then the conclusion is that for all e^{it} in some unit arc, $f(z, e^{it}) = 0$ for some $z \in \mathbf{D}$—this contradicts f being weakly outer. Thus f can only have a root in $\mathbf{D}\times\mathbf{T}$ if (14) fails there.

Repeat the above argument, taking the set E to be first $\mathbf{D}\times\mathbf{D}$ and then $\mathbf{D}\times\mathbf{A}$ in [10, Thm. 9.28]. The conclusion is that the functions

$$\mathfrak{N}_1(z_1) = \#\{z_2 \in \mathbf{D} : f(z_1, z_2) = 0\},$$

$$\mathfrak{N}_2(z_1) = \#\{z_2 \in \mathbf{A} : f(z_1, z_2) = 0\}$$

are continuous in the disc, minus the finite collection of exceptional z_1. But these functions are integer-valued and hence are constant. In fact, this shows that $\mathfrak{N}_1(z_1) = J$ and $\mathfrak{N}_2(z_1) = K$.

If f is not irreducible then the previous observation can be applied to each of the irreducible factors of f, each of which is easily seen to be weakly outer. This verifies that J and K are constants independent of z_1.

From Lemma 4.2 we have that $g(z_1, z_2) = g_0(z_1, z_2)\phi(z_1)\psi(z_2)$, where g_0 is an outer polynomial and ϕ and ψ are 1-variable outer. We can "absorb" the factors ϕ and ψ by redefining g, h, a, and b as follows:

$$g(e^{is}, e^{it}) \leftarrow g_0(e^{is}, e^{it});$$

$$h(e^{is}, e^{it}) \leftarrow h(e^{is}, e^{it})/\phi(e^{is})\bar{\psi}(e^{-it});$$

$$a(e^{is}) \leftarrow a(e^{is})\bar{\phi}(e^{is})/\phi(e^{is});$$

$$b(e^{it}) \leftarrow b(e^{it})\psi(e^{-it})/\bar{\psi}(e^{-it}).$$

These exchanges preserve the structural assertion of Theorem 2.1, but g is now a polynomial.

From (1) and (2) we get

$$\frac{f(e^{is}, e^{it})\bar{g}(e^{is}, e^{it})}{\bar{f}(e^{is}, e^{it})g(e^{is}, e^{it})} = a(e^{is})\bar{b}(e^{-it}),$$

which implies that a and b are rational functions. Another look at (1) reveals that h must then be a rational function as well.

From (1) and (2) again, we have

$$\bar{h}(e^{is}, e^{it})a(e^{is}) = h(e^{is}, e^{it})\bar{b}(e^{it}).$$

Without loss of generality, we can assume that $h(e^{is}, 1)$ and $h(1, e^{it})$ are outer functions, and that

$$a(e^{is}) = h(e^{is}, 1)/\bar{h}(e^{is}, 1)\, b(1);$$

$$b(e^{it}) = h(1, e^{it})/\bar{h}(1, e^{it})\, a(1).$$

This can be rearranged to show that the outer function

$$R(e^{is}, e^{it}) = \frac{h(e^{is}, e^{it})\, a(1)^{1/2} b(1)^{1/2}}{h(e^{is}, 1)\, h(1, e^{it})}$$

is real-valued on the torus. Apply the reflection principle to the 1-variable outer rational functions $R(e^{is}, \cdot)$ and $R(\cdot, e^{it})$ to deduce that R has no zeros or poles on $\mathbf{T}\times\mathbf{A}$, $\mathbf{T}\times\mathbf{D}$, $\mathbf{A}\times\mathbf{T}$, or $\mathbf{D}\times\mathbf{T}$.

Now $h(z_1, 1)$ is an outer rational function in z_1. By the previous observations about R, we see that any factors of $h(z_1, 1)$ of the form $(z_1 - \xi)$, $|\xi| > 1$, are necessarily also factors of $h(z_1, z_2)$. Let us group such factors of $h(z_1, 1)$ into the rational function $j_1(z_1)$. Thus, $h(z_1, 1) = j_1(z_1)k_1(z_1)$, where $k_1(z_1)$ has zeros and poles only on $\mathbf{T}$. Similarly decompose $h(1, z_2) = j_2(z_2)k_2(z_2)$. We once again redefine g, h, a, and b while preserving their structural roles:

$$g(e^{is}, e^{it}) \leftarrow g(e^{is}, e^{it})\, j_1(e^{is})\, \bar{j}_2(e^{-it})$$

$$h(e^{is}, e^{it}) \leftarrow R(e^{is}, e^{it})\, k_1(e^{is})\, k_2(e^{it})\, a(1)^{1/2} b(1)^{1/2}$$

$$a(e^{is}) \leftarrow a(e^{is})\, \bar{j}_1(e^{is})/j_1(e^{is})$$

$$b(e^{it}) \leftarrow b(e^{it})\, k_2(e^{it})/\bar{k}_2(e^{it}).$$

Thus redefined, $a(e^{is}) = a(1)e^{iMs}$, where M is the number of roots of k_1 minus the number of poles; similarly, $b(e^{it}) = b(1)e^{iNt}$.

With that we examine the rational function $h(z_1, z_2)$. By extending meromorphically off $\mathbf{T}^2$, we have

$$h(z_1, z_2) = a(1)\, b(1)\, \bar{h}(1/\bar{z}_1, 1/\bar{z}_2) z_1^M z_2^N; \tag{15}$$

$$h(z_1, 1/\bar{z}_2) z_2^N = a(1)\, b(1)\, \bar{h}(1/\bar{z}_1, \bar{z}_2) z_1^M. \tag{16}$$

Since h is outer, it cannot vanish in $\mathbf{D}^2$. By (15), it cannot vanish in $\mathbf{A}^2$, either. Zeros in $\mathbf{T}\times\mathbf{A}$, $\mathbf{T}\times\mathbf{D}$, $\mathbf{A}\times\mathbf{T}$, or $\mathbf{D}\times\mathbf{T}$ were ruled out before. Only $\mathbf{D}\times\mathbf{A}$, $\mathbf{T}^2$, and $\mathbf{A}\times\mathbf{D}$ remain.

The denominator of h cannot vanish in $\mathbf{D}^2$ or $\mathbf{A}^2$. Furthermore, the function in (16) is weakly outer, so the denominator of h also cannot vanish in $\mathbf{D}\times\mathbf{A}$ or $\mathbf{A}\times\mathbf{D}$. Zeros of the denominator of h in $\mathbf{T}\times\mathbf{A}$, $\mathbf{T}\times\mathbf{D}$, $\mathbf{A}\times\mathbf{T}$, or $\mathbf{D}\times\mathbf{T}$ were ruled out before. This says that such zeros could only arise in

$\mathbf{T}^2$, which in turn cannot happen at all. Thus the polynomial in the denominator must be constant.

We conclude that h is a polynomial with no zeros outside of $(\mathbf{D}\times\mathbf{A})\cup(\mathbf{T}\times\mathbf{T})\cup(\mathbf{A}\times\mathbf{D})$. Furthermore, with g redefined as in (15), g is also a polynomial. Thus, with $p(z_1,z_2)=g(z_1,z_2)$ and $q(z_1,z_2)=\bar{h}(1/\bar{z}_1,\bar{z}_2)a(1)z_1^M$, the conditions (i), (ii), and (iii) hold in Theorem 2.1.

The converse is immediate. □

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