EQUIVARIANT IMBEDDINGS OF G-COMPLEXES INTO REPRESENTATION SPACES

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Let G be a finite group. We are concerned with the problem of finding a representation space V of G of *minimal dimension* with respect to the property that every k-dimensional G-complex X can be equivariantly imbedded in V. In case $G=\{e\}$ we have the well-known classical result that every k-dimensional complex can be imbedded in $\mathbf{R}^{2k+1}$ and that this is best possible; that is, the minimal imbedding dimension in the non-equivariant case is $2k+1$. The general fact that every G-complex has an equivariant imbedding into some representation space follows from Mostow's theorem [9], or one can also prove directly that every G-complex has a proper p.l. G-imbedding into some representation space of G (see for example Proposition 1.1 in [6]). But these results do not determine the minimal imbedding dimension in the equivariant case; in fact, they do not even give any reasonable information about the required dimensions of the representation spaces in question. In this paper we solve, among other things, the problem of the minimal imbedding dimension and the minimal representation space in the case of finite nilpotent groups.

In the case when G is a finite cyclic group $\mathbf{Z}_p$ of prime order, Copeland and de Groot [4] proved that every k-dimensional metrizable $\mathbf{Z}_p$-space can be equivariantly imbedded in a representation space of dimension $3k+2$ or $3k+3$, depending on whether p and k are odd or even, and they also showed that their result is best possible, that is, gives the minimal imbedding dimension in this case. In [8] Kister and Mann obtained results on the required dimensions of the representation spaces in the case of equivariant imbeddings of actions of compact abelian Lie groups, with a finite number of orbit types, on finite-dimensional separable metrizable spaces. Their results on the required dimensions of the representation spaces are not in general best possible, at least not in the case of equivariant imbeddings of G-complexes with G a finite abelian group.

In [7, Theorem 4.2] we proved a general equivariant imbedding result which, for an arbitrary finite group G, gives conditions that are sufficient for the existence of equivariant imbeddings of G-complexes into a given representation space V. Furthermore, we showed in [7] that in the case of a finite cyclic group $\mathbf{Z}_m$, $m\geq 2$, this general equivariant imbedding result gives the best possible result, and we also explicitly determined the minimal imbedding dimension in this case. In the present paper we prove that Theorem 4.2 of [7] is in fact best possible when G is a finite nilpotent group. A somewhat simplified version of our main result is as follows. (See Theorem 3.1 for a more general formulation.)

THEOREM I. *Suppose G is a finite nilpotent group and that $(H_1),\dots,(H_q)$ are G-isotropy types. Let V be a representation space of G. In order for every*

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k-dimensional G-complex with isotropy types among (H_i), $1 \leq i \leq q$, to have an equivariant imbedding into V, the following are necessary and sufficient:
(i) $\dim V^{H_i} - \dim V^{>H_i} \geq k+1$ *and*
(ii) $\dim V^{H_i} \geq 2k+1$
for $i = 1, \ldots, q$.

Here we have denoted $V^{>H} = \{y \in V \mid H \subsetneqq G_y\}$. In the more general formulation given in Theorem 3.1, the role of the fixed integer k is replaced by an arbitrary dimension function $n\colon \{(H_i)\}_{i=1}^{q} \to \mathbf{N}$. If each H_i $(1 \leq i \leq q)$ is normal in G, we may drop the assumption that G is nilpotent. In particular this applies to the case of equivariant imbeddings of G-complexes with free or semi-free G-actions, and the following two results hold for G an arbitrary finite group.

THEOREM II. *Let G be a finite group and V a linear representation space of G. In order that every k-dimensional free G-complex can be equivariantly imbedded in V it is necessary and sufficient that*
(1) $\dim V - \dim V^{>\{e\}} \geq k+1$ *and*
(2) $\dim V \geq 2k+1$.

THEOREM II$'$. *Otherwise as in Theorem* II *but change "free" into "semi-free" and condition* (2) *into*
(2$'$) $\dim V^G \geq 2k+1$.

In the special case of equivariant imbeddings of semi-free compact finite-dimensional metric $\mathbf{Z}_m$-spaces X, with a given imbedding $i\colon X^{\mathbf{Z}_m} \to \mathbf{R}^d$ of the fixed point set, an imbedding dimension is obtained in Allen [1, Theorems 1 and 2] and the corresponding result for semi-free $\mathbf{Z}_m$-complexes (not necessarily compact) appears as Corollary 4.5 of [7]. Although not shown in [1] or [7], it is easy to see that this imbedding dimension is the minimal imbedding dimension in this case.

Theorem I solves the problem of determining the minimal imbedding dimension in the case of finite nilpotent groups, in the following sense. By $w_{[G;(H_1),\ldots,(H_q)]}(k)$, or simply by $w_G(k)$, we denote the minimal imbedding dimension in the case of equivariant imbeddings into representation spaces of k-dimensional G-complexes with isotropy types among (H_i), $1 \leq i \leq q$. That is, $w_G(k)$ is the least integer for which it is true that each k-dimensional G-complex as above has an equivariant imbedding into some representation space of dimension $w_G(k)$. From Theorem I we obtain the following result.

COROLLARY III. *Let G be a finite nilpotent group and let $(H_1), \ldots, (H_q)$ be G-isotropy types. Then the minimal imbedding dimension $w_G(k) = w_{[G;(H_1),\ldots,(H_q)]}(k)$ is given by*

$$w_G(k) = \min\{\dim V \mid V \text{ is a representation space of } G \text{ which satisfies (i) and (ii) in Theorem I}\}.$$

Thus the determination of the minimal imbedding dimension $w_G(k)$ is completely reduced to the linear representation theory of G. In particular it follows from Corollary III that we can always find *one* representation space W, of the minimal dimension $w_G(k)$ such that every k-dimensional G-complex with the specified isotropy types has an equivariant imbedding into this same representation space W. In Section 3 we also construct, for each integer $k \geq 0$, a specific k-dimensional G-complex with specified isotropy types which cannot be equivariantly imbedded in any representation space of dimension less than $w_G(k)$. In most cases these G-complexes can be chosen to be connected. These constructions are in fact also carried out in the greater generality where, instead of dealing with a fixed integer k as dimension estimate, we are dealing with an arbitrary dimension function $n: \{(H_i)\}_{i=1}^q \to \mathbf{N}$.

In the case of equivariant imbeddings of free and semi-free G-complexes into linear representation spaces we obtain (respectively) from Theorem II and II′, in complete analogy with Corollary III, the determination of the minimal imbedding dimensions in these cases. These results hold for G an arbitrary finite group.

In Section 4 we apply our results to the case of elementary abelian p-groups $(\mathbf{Z}_p)^t$, $t \geq 2$. We consider equivariant imbeddings of free $(\mathbf{Z}_p)^t$-complexes into representation spaces and establish a rough lower bound for the minimal imbedding dimension in this case. Our results here give *counterexamples* to results in Allen ([2]; see also [3]). A detailed discussion of this matter is given in Section 4.

In Section 5 we study the case $G = \mathbf{Z}_p \oplus \mathbf{Z}_p$, where p is a prime, and explicitly determine the minimal imbedding dimension in the three different cases of equivariant imbeddings of free, semi-free, and arbitrary $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes into representation spaces. For example, for p an odd prime the minimal imbedding dimension in the case of equivariant imbeddings of k-dimensional semi-free $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes is given by

$$w_1(k) = \begin{cases} 3k+5+2[k/2p] & \text{if } k \text{ is even,} \\ 3k+4+2[k/2p] & \text{if } k \text{ is odd,} \end{cases}$$

and in the case of k-dimensional arbitrary $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes by

$$w(k) = \begin{cases} 3k+3+p(k+2) & \text{if } k \text{ is even,} \\ 3k+2+p(k+1) & \text{if } k \text{ is odd.} \end{cases}$$

Here $[x]$ denotes the integer part of x. For the minimal imbedding dimensions in the other cases we refer to Theorems 5.1.a and b. The results by Kister and Mann [8] give imbedding dimensions that are much larger than the minimal imbedding dimensions, and for example in the case of arbitrary $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes, p an odd prime, the Kister–Mann imbedding dimension is greater than twice the minimal imbedding dimension $w(k)$ given above. For more details on this matter see Section 5.

The main equivariant imbedding results are proved in Section 3; see Theorems 3.1, 3.2, and 3.2′. These theorems are established in the general form where, instead of considering k-dimensional G-complexes, we consider G-complexes satisfying $\dim X^{H_i} \leq n(H_i)$ for $1 \leq i \leq q$, where $n: \{(H_i)\}_{i=1}^q \to \mathbf{N}$ is any given dimension function, that is, a function satisfying $(H_i) \geq (H_j) \Rightarrow n(H_i) \leq n(H_j)$. The fact that condition (ii) of Theorem I is a necessary condition is very easy to see. Condition (i) of Theorem I is roughly speaking a necessary condition already for the general existence of isovariant maps from k-dimensional G-complexes into V. To be precise we have that the condition

$$(*) \qquad \dim V^{H_i} - \dim V^{>H_i} \geq k+1 \quad \text{for all } 1 \leq i \leq q \text{ with } H_i \neq G,$$

is necessary and sufficient in order for every k-dimensional finite G-complex to have an isovariant map into V. See Proposition 2.1, where this result is proved in the more general setting described above. The sufficiency part of $(*)$ follows from results in [7], and the necessity part of the condition $(*)$ is proved in Section 2.

1. Notation, terminology and preliminaries. Let G be a finite group and X a G-space. By G_x we denote the isotropy subgroup of G at the point $x \in X$. Let H be any subgroup of G. Then X^H denotes the fixed point set of H, and by $X^{>H}$ we denote the set of points in X with isotropy subgroup strictly greater than H, that is,

$$X^{>H} = \{x \in X \mid G_x \supsetneqq H\}$$

which we may also write in the form

$$X^{>H} = \bigcup_{K \supsetneqq H} X^K.$$

Observe that in particular we have

$$X^{>\{e\}} = \bigcup_{g \in G - \{e\}} X^g,$$

where X^g denotes the fixed point set of g. The set $X_H = X^H - X^{>H}$ equals the set of all points in X with isotropy subgroup exactly equal to H.

An equivariant map or a G-map $f: X \to Y$, where X and Y are G-spaces, is a map such that $f(gx) = gf(x)$ for every $x \in X$ and all $g \in G$. It follows that $G_x \subset G_{f(x)}$ for every $x \in X$. An *isovariant* map $f: X \to Y$ is by definition a G-map for which $G_x = G_{f(x)}$ for every $x \in X$. That is, a G-map $f: X \to Y$ is isovariant if and only if $f(X_H) \subset Y_H$ for every subgroup H of G.

For any subgroup H of G we let (H) denote the conjugacy class of H in G. Such a conjugacy class (H) is also called a G-isotropy type, or simply an isotropy type in cases where no misunderstanding can arise.

By a G-complex we mean a countable, locally finite, and finite-dimensional simplicial complex on which G acts simplicially. One often needs to work with G-complexes that satisfy some additional technical conditions. In [7] we reserved the term "equivariant simplicial complex" for such a notion of a G-complex which satisfies some additional conditions; see Section 1 of [7]. Although it was essential to work with equivariant simplicial complexes in [7], we shall not really need to work with them in this paper (except for one very minor point) and hence we will be content with referring to Section 1 of [7] for more information on this matter. We shall only note here that in an equivariant simplicial complex we have that all points in any open simplex $\mathring{s}$ have the same isotropy subgroup; that is, $\mathring{s} \subset X_H$ for some subgroup H of G. This fact is used in the proof of Lemma 1.2.

In the proof of Proposition 2.1 we need to find proper G-maps from G-complexes into representation spaces. (A map is proper if the inverse image of any compact set is compact.) For this purpose let us first record the following obvious fact.

LEMMA 1.1. *Let X be a G-complex and V a representation space of G with* $\dim V^G \geq 1$. *Then there exists a proper G-map $\bar{f}: X \to V$.*

Proof. Since V^G contains a 1-dimensional linear subspace it is immediate that there exists a proper G-map $\bar{f}: X \to V^G \subset V$. □

In the case when $V^G = \{0\}$ we use Lemma 1.2 below, which is a reformulation of a special case of Lemma 4.1 of [7]. In fact the actual proof of Lemma 4.1 in [7] establishes Lemma 1.2, but we can also deduce Lemma 1.2 from the statement of Lemma 4.1 of [7] as we do below.

LEMMA 1.2. *Let G be a finite group and let $(H_1), \dots, (H_q)$ be G-isotropy types. Let V be a representation space of G and X a G-complex such that the orbit types occurring in X are among (H_i), $1 \leq i \leq q$, and X^G is a finite complex or empty. Assume that*

$$\dim(X - X^G)^H + 1 \leq \dim V^H \quad \textit{for every } H \in \{H_1, \dots, H_q\}, \ H \neq G.$$

Then there exists a proper G-map $\bar{f}: X \to V$.

Proof. We may assume that X is an equivariant simplicial complex, since this can be achieved by taking the second barycentric subdivision of X. In case $X^G \neq \varnothing$ we define $\bar{f}_0: X^G \to V$ by $\bar{f}_0(x) = 0$ for every $x \in X^G$. Since X^G is compact $\bar{f}_0$ is a proper G-map. By Lemma 4.1 of [7] we have that there exists a proper G-map $\bar{f}: X \to V$, extending $\bar{f}_0$ for the case where $X^G \neq \varnothing$, if

$$(\#) \qquad \dim(X - X^G)^K + 1 \leq \dim V^K \quad \text{for every subgroup } K \text{ of } G.$$

Since $\dim \varnothing = -1$ this always holds when $(X - X^G)^K = \varnothing$. (In particular (1) holds for $K = G$ even when $V^G = \{0\}$.) Now assume that $(X - X^G)^K \neq \varnothing$ and let s be a simplex of $(X - X^G)^K$ of maximal dimension, and denote $\dim s = \dim(X - X^G)^K = m$. Let H be the principal isotropy subgroup of s, that is, $\mathring{s} \subset X_H$. Then (H) equals one of the G-orbit types $(H_1), \dots, (H_q)$ and $H \neq G$, and furthermore $K \subset H$. Thus we have

$$\dim(X - X^G)^H + 1 \leq \dim V^H \leq \dim V^K,$$

that is, $m + 1 \leq \dim V^K$, which shows that (#) holds for K. □

Lemma 1.3 below is of crucial importance in this paper. It corresponds to the arguments used in Copeland–de Groot [4, Section 4]. The $\mathbf{Z}_p$-spaces $Y^k(p)$ defined below are the same ones as Copeland and de Groot used in [4] in showing that their result gives the minimal imbedding dimension in the case of $\mathbf{Z}_p$-spaces, p a prime.

Let p be a prime. We write the group $\mathbf{Z}_p$ multiplicatively and denote $\mathbf{Z}_p = \{e, t, \dots, t^{p-1}\}$. For any non-negative integer k we define a k-dimensional free $\mathbf{Z}_p$-complex $Y^k(p)$ as follows.

(1) Assume k is odd. Let U denote $\mathbf{C}^a$, where $a = (k+1)/2$, with $\mathbf{Z}_p$-action given by $t(z_1, \dots, z_a) = (\xi z_1, \dots, \xi z_a)$. Here $\xi = \exp(2\pi i/p)$. We define

$$(1) \qquad Y^k(p) = S(U),$$

where $S(U)$ denotes the unit sphere in $\mathbf{C}^a \cong \mathbf{R}^{k+1}$. Then $\mathbf{Z}_p$ acts freely on $Y^k(p)$, and $Y^k(p)$ can easily be triangulated so that $\mathbf{Z}_p$ acts simplicially and thus $Y^k(p)$ becomes a finite free $\mathbf{Z}_p$-complex.

(2) Assume k is even. Then let U be $\mathbf{C}^b$, where $b = k/2$, with $\mathbf{Z}_p$-action as above. Let $D(U)$ denote the unit ball in $\mathbf{C}^b \cong \mathbf{R}^k$ and consider the product $D(U) \times \mathbf{Z}_p$ with diagonal $\mathbf{Z}_p$-action. Here $\mathbf{Z}_p$ acts on $\mathbf{Z}_p$ by multiplication. Now define

$$(2) \qquad Y^k(p) = (D(U) \times \mathbf{Z}_p)/\sim$$

where $(z, g) \sim (z', g')$ if and only if $z = z' \in S(U)$. The diagonal $\mathbf{Z}_p$-action on $D(U) \times \mathbf{Z}_p$ induces a $\mathbf{Z}_p$-action on $Y^k(p)$ and this action is free. Moreover $Y^k(p)$ can be triangulated so that $\mathbf{Z}_p$ acts simplicially and thus $Y^k(p)$ becomes a finite free $\mathbf{Z}_p$-complex.

Let W denote a linear representation space for $\mathbf{Z}_p$. Then we have the following.

LEMMA 1.3. *The existence of an isovariant map $f: Y^k(p) \to W$ implies that* $\dim W - \dim W^{\mathbf{Z}_p} \geq k+1$.

Proof. Since the $\mathbf{Z}_p$-action on $Y^k(p)$ is free and f is isovariant we have

$$\operatorname{im} f \cap W^{\mathbf{Z}_p} = \varnothing.$$

The composite

$$\bar{f}: Y^k(p) \xrightarrow{f} W \xrightarrow{\pi} W/W^{\mathbf{Z}_p} = W_1,$$

where π denotes the natural projection, is a $\mathbf{Z}_p$-map with $0 \notin \operatorname{im} \bar{f}$. Hence we obtain a $\mathbf{Z}_p$-map

$$\hat{f}: Y^k(p) \longrightarrow S(W_1)$$

by defining $\hat{f}(y) = \bar{f}(y)/\|\bar{f}(y)\|$ for every $y \in Y^k(p)$. We claim that the existence of such a $\mathbf{Z}_p$-map $\hat{f}$ implies that $\dim S(W_1) \geq k$, that is, that

$$(*) \qquad \dim W - \dim W^{\mathbf{Z}_p} - 1 \geq k.$$

(1) Assume that k is odd, and hence $Y^k(p) = S(U)$. If $\dim S(W_1) < k = \dim S(U)$ there exists a $\mathbf{Z}_p$-map $h: S(W_1) \to S(U)$, and the composite map $h \circ \hat{f}: S(U) \to S(U)$ has degree 0 and hence Lefschetz number equal to 1. But since the $\mathbf{Z}_p$-action on $S(U)$ is free and $h \circ \hat{f}$ is a $\mathbf{Z}_p$-map, the Lefschetz number of $h \circ \hat{f}$ must be divisible by p, a contradiction. Thus $(*)$ holds in this case.

(2) Assume that k is even, and thus $Y^k(p)$ is given by (2). Since $S(W_1) \neq \varnothing$ it follows that $(*)$ holds for $k = 0$, and hence we may assume that $k \geq 2$. If $\dim S(W_1) \leq k - 1 = \dim S(U)$ there exists a $\mathbf{Z}_p$-map $h: S(W_1) \to S(U)$. Then the restriction $(h \circ \hat{f}) \mid : S(U) \to S(U)$ is a $\mathbf{Z}_p$-map which has degree 0, since it extends to a map from $D(U)$ to $S(U)$. But as we saw above in case (1) this leads to a contradiction, and hence $(*)$ also holds in this case. □

2. Existence of isovariant maps into representation spaces. Let $(H_1), \ldots, (H_q)$ be G-isotropy types. We say that a function $n: \{(H_i)\}_{i=1}^q \to \mathbf{N}$ is a *dimension function* if it satisfies

$$(H_i) \geq (H_j) \Rightarrow n(H_i) \leq n(H_j) \quad \text{for all } 1 \leq i, j \leq q.$$

For G a finite nilpotent group we give in the result below a condition that is both necessary and sufficient for the general existence of isovariant maps into a representation space V.

PROPOSITION 2.1. *Let G be a finite nilpotent group and V a representation space of G. Let $(H_1),\dots,(H_q)$ be G-isotropy types and $n:\{(H_i)\}_{i=1}^q\to\mathbf{N}$ a dimension function. In order that there exist an isovariant map $f:X\to V$ for each finite G-complex X with* $\dim X^{H_i}\le n(H_i)$ *and isotropy types among (H_i), $1\le i\le q$, it is necessary and sufficient that*

$$(*)\qquad \dim V^{H_i}-\dim V^{>H_i}\ge n(H_i)+1 \quad \text{for all } 1\le i\le q \text{ with } H_i\ne G.$$

ADDENDUM 2.1. *Let X be a G-complex as above except that X need not be a finite complex. In case $G\notin\{H_1,\dots,H_q\}$ and hence $X^G=\varnothing$, or more generally in case X^G is a finite complex, condition $(*)$ is sufficient for the existence of an isovariant proper* p.l. *map $f:X\to V$. If X^G is not a finite complex we have that $(*)$ together with $\dim V^G\ge 1$ are sufficient conditions for the existence of an isovariant proper* p.l. *map $f:X\to V$. Moreover these facts hold for G an arbitrary finite group.*

Proof. We first prove Addendum 2.1. Let G be a finite group and assume that condition $(*)$ holds. Let X be a not necessarily finite G-complex with $\dim X^{H_i}\le n(H_i)$ and isotropy types among (H_i), $1\le i\le q$. For any $H\in\{H_1,\dots,H_q\}$, $H\ne G$, we then have

$$\begin{aligned}\dim(X-X^G)^H+1&\le\dim X^H+1\le n(H)+1\\&\le\dim V^H-\dim V^{>H}\le\dim V^H.\end{aligned}$$

In case X^G is a finite complex or empty we therefore have by Lemma 1.2 that there exists a proper G-map $\bar{f}:X\to V$. In case X^G is not finite but we are assuming that $\dim V^G\ge 1$, there also exists a proper G-map $\bar{f}:X\to V$; see Lemma 1.1. By Theorems 3.1 and 3.5 in [7], (cf. the Remark in [7, p. 139]), condition $(*)$ and the existence of a proper G-map $\bar{f}:X\to V$ imply that there exists an isovariant proper p.l. map $f:X\to V$. This proves Addendum 2.1 and hence in particular the sufficiency part of Proposition 2.1.

Now assume that G is a finite nilpotent group. We shall prove that in this case condition $(*)$ is also necessary for the existence of isovariant maps. Let V be a representation space of G such that there exists an isovariant map $f:X\to V$ for every finite G-complex X with $\dim X^{H_i}\le n(H_i)$ and orbit types among (H_i), $1\le i\le q$. In particular there then exist isovariant maps $f_i:G/H_i\to V$ and hence $V^{H_i}-V^{>H_i}=V_{H_i}\ne\varnothing$ for $1\le i\le q$. Thus $\dim V^{>H_i}<\dim V^{H_i}$, $1\le i\le q$.

For each subgroup $H\in\{H_1,\dots,H_q\}$ with $H\ne G$, we shall construct a finite G-complex X with the following properties. We have $\dim X^{H_i}\le n(H_i)$ for $1\le i\le q$ and each point in X has isotropy type equal to (H), and the existence of an isovariant map $f:X\to V$ implies that $\dim V^H-\dim V^{>H}\ge n(H)+1$.

So let $H\in\{H_1,\dots,H_q\}$, where $H\ne G$. As we observed above, we then have

$$(1)\qquad \dim V^{>H}<\dim V^H.$$

Since $H\ne G$ there exists a subgroup $K\supsetneq H$ such that

(2) $$\dim V^K = \dim V^{>H}.$$

Since G is nilpotent also K is nilpotent and hence $H \subsetneqq K$ implies (see, e.g., [10, §5.2.4]) that

(3) $$H \subsetneqq N_K(H) = N(H) \cap K.$$

Now choose $g \in (N(H) \cap K) - H$ such that $g^p \in H$, for some prime p, and let

(4) $$P = \langle g, H \rangle.$$

Since $H \subsetneqq P \subset K$ we have $\dim V^K \leq \dim V^P \leq \dim V^{>H}$, and hence (2) implies that

(5) $$\dim V^P = \dim V^{>H}.$$

Furthermore H is normal in P and $P/H \cong \mathbf{Z}_p$ with gH as a generator. The quotient group P/H acts on V^H and we have

(6) $$V^P = (V^H)^{P/H}.$$

Let $Y = Y^{n(H)}(p)$ be the $n(H)$-dimensional finite free P/H-complex defined in Section 1. (In order to be specific let us say that we identify P/H with $\mathbf{Z}_p$ by letting the generator $gH \in P/H$ correspond to the generator $t \in \mathbf{Z}_p$.) We make Y into a P-space through the natural projection $\pi: P \to P/H$, and form the twisted product

(7) $$X = X^{n(H)}(P; H) = G \times_P Y.$$

Then X is an $n(H)$-dimensional G-complex, and since every point in the P-space Y has isotropy subgroup equal to H it follows that every point in X has G-isotropy type equal to (H). For $H' \in \{H_1, \ldots, H_q\}$ with $(H') \leq (H)$ we have $\dim X^{H'} = n(H) \leq n(H')$, and in case $(H') \not\leq (H)$ we have $X^{H'} = \phi$. Thus $\dim X^{H_i} \leq n(H_i)$ for $1 \leq i \leq q$.

Now assume that $f: X \to V$ is an isovariant G-map. Since $Y \subset X^H$ we have an induced map $f|: Y \to V^H$, which is an isovariant P/H-map. Thus we have by Lemma 1.3 and (5) and (6) that

$$\begin{aligned} n(H) = \dim Y &\leq \dim V^H - \dim (V^H)^{P/H} - 1 \\ &= \dim V^H - \dim V^{>H} - 1. \end{aligned}$$ □

REMARK. In the proof of Proposition 2.1 the assumption that G is nilpotent was used only to establish (3), that is, to show that $H \subsetneqq K \subset G$ implies that $H \subsetneqq N_K(H) = N(H) \cap K$. Therefore we may replace the assumption that G is nilpotent by the assumption that H is normal in G. Hence Proposition 2.1 is valid for G an arbitrary finite group if we assume that the subgroups $H_1, \ldots, H_q$ are normal in G. In particular this remark applies to the case of G-complexes with free or semi-free actions, and hence we obtain the following.

PROPOSITION 2.2. *Let G be a finite group and V a representation space of G. In order that there exist an isovariant map $f: X \to V$ for every k-dimensional finite free* [*semi-free*] *G-complex X it is necessary and sufficient that*

(**) $$\dim V - \dim V^{>\{e\}} \geq k+1.$$ □

If we wish to construct *one* finite G-complex $\bar{X}$ such that the existence of an isovariant map $f:\bar{X}\to V$ forces condition $(*)$ in Proposition 2.1 to hold, we proceed as follows. As before, G denotes a finite nilpotent group, $(H_1),\dots,(H_q)$ are G-isotropy types, and $n\colon \{(H_i)\}_{i=1}^{q}\to \mathbf{N}$ is a given dimension function. We then define

$$\bar{X}=\dot{\bigcup}\,\dot{\bigcup}\,X^{n(H)}(P;H)$$

where the first disjoint union is over all $H\in\{H_1,\dots,H_q\}$ with $H\neq G$, and for a fixed subgroup H the second disjoint union is over all subgroups P of G such that H is normal in P and P/H is cyclic of prime order. Here $X^{n(H)}(P;H)$ is defined by (7) in the proof of Proposition 2.1. Clearly $\bar{X}$ is a finite G-complex and the isotropy types occurring in $\bar{X}$ are exactly all $(H_1),\dots,(H_q)$ with $H_i\neq G$. Using the fact that n is a dimension function it follows immediately that $\dim \bar{X}^{H_i}\leq n(H_i)$ for $1\leq i\leq q$. The proof of Proposition 2.1 shows that if there exists an isovariant map $f:\bar{X}\to V$ then condition $(*)$ in Proposition 2.1 must hold. In case $H_i\lhd G$, $1\leq i\leq q$, we need not assume that G is nilpotent; see the Remark after Proposition 2.1.

For any finite group G we set

$$\bar{X}^k_{\text{free}}=\dot{\bigcup}\,X^k(P;\{e\})=\dot{\bigcup}\,G\times_P Y^k$$

where the disjoint union is over all cyclic subgroups P of prime order of G. (Here the P-space Y^k equals $Y^k(p)$, where $p=|P|$ and $Y^k(p)$ is as in Section 1.) Then $\bar{X}^k_{\text{free}}$ is a k-dimensional finite free G-complex, and the existence of an isovariant map $f:\bar{X}^k_{\text{free}}\to V$ into a representation space V of G implies that

$$\dim V-\dim V^{>\{e\}}\geq k+1.$$

The G-complex $\bar{X}$ defined above is not connected. In case either G or $\{e\}$ are among the given G-isotropy types $(H_1),\dots,(H_q)$, that is, in case either fixed points or free orbits are allowed, we may easily modify $\bar{X}$ such that we obtain a *connected* G-complex which serves the same purpose as $\bar{X}$.

In case $G\in\{H_1,\dots,H_q\}$ we simply add a disjoint fixed point $\{*\}$ to $\bar{X}$ and then join each component of $\bar{X}$ by suitable 1-simplexes to $\{*\}$ in such a way that we obtain a G-complex $\bar{X}_*$. Then $\bar{X}_*$ is a finite connected G-complex containing $\bar{X}$ as a G-subcomplex. We have $(\bar{X}_*)^G=\{*\}$ and $\dim(\bar{X}_*)^H=\max(1,n(H))$ for all $H\in\{H_1,\dots,H_q\}$ with $H\neq G$, and the orbit types occurring in $\bar{X}_*$ are exactly all $(H_1),\dots,(H_q)$. Thus under the very mild additional assumption that $n(H_i)\geq 1$ for every (H_i), $1\leq i\leq q$, we have that $\dim(\bar{X}_*)^{H_i}\leq n(H_i)$ for $1\leq i\leq q$, and hence in this case $\bar{X}_*$ is a finite *connected* G-complex of the required type such that the existence of an isovariant map $f:\bar{X}_*\to V$ into a representation space V implies that condition $(*)$ of Proposition 2.1 holds.

In case $\{e\}\in\{H_1,\dots,H_q\}$ we can proceed as follows in order to change $\bar{X}$ into a finite connected G-complex. Let F be a connected 1-dimensional finite simplicial complex on which G acts freely and simplicially. We then join each component of $\bar{X}$ to F by suitable 1-simplexes in such a way that we obtain a G-complex $\bar{X}^*$. Then $\bar{X}^*$ is a finite connected G-complex containing $\bar{X}$ as a G-complex. Furthermore we have $\dim\bar{X}^*=\max(1,n(\{e\}))$ and $\dim(\bar{X}^*)^H=n(H)$ for $H\neq\{e\}$, and the orbit types occurring in $\bar{X}^*$ are exactly the same ones as the ones occurring in $\bar{X}$. Thus in case $\{e\} \in \{H_1, \ldots, H_q\}$ and $n(\{e\}) \geq 1$ we may as well replace the G-complex $\bar{X}$ by the *connected* G-complex $\bar{X}^*$.

3. Existence of equivariant imbeddings into representation spaces. In this section we establish the main result which, for G a finite nilpotent group, gives conditions that are both necessary and sufficient for the general existence of equivariant imbeddings of G-complexes into a representation space. In the case of G-complexes with free or semi-free actions we need not assume that G is nilpotent. The main part of the proof that the conditions are necessary conditions was given already in the proof of Proposition 2.1.

THEOREM 3.1. *Let G be a finite nilpotent group and V a representation space of G. Let $(H_1), \ldots, (H_q)$ be G-isotropy types and $n: \{(H_i)\}_{i=1}^q \to \mathbf{N}$ a dimension function. In order for every G-complex X with isotropy types among (H_i) and* $\dim X^{H_i} \leq n(H_i)$ $(1 \leq i \leq q)$ *to have an equivariant imbedding into V, it is necessary and sufficient that*

(i) $\dim V^{H_i} - \dim V^{>H_i} \geq n(H_i) + 1$ *and*

(ii) $\dim V^{H_i} \geq 2n(H_i) + 1$

for $i = 1, \ldots, q$.

ADDENDUM 3.1. *Conditions* (i) *and* (ii) *are in fact sufficient conditions for the existence of a proper* p.l. *G-imbedding $f: X \to V$, and moreover this holds for G an arbitrary finite group. On the other hand, for G a finite nilpotent group, conditions* (i) *and* (ii) *are necessary conditions even in the case of topological G-imbeddings of finite G-complexes.*

Proof. By Theorem 4.2 in [7] the above conditions are sufficient for the existence of an equivariant proper p.l. imbedding $f: X \to V$, and in fact this holds for G an arbitrary finite group.

Let $H \in \{H_1, \ldots, H_n\}$. By $A = A^{n(H)}$ we denote the $n(H)$-skeleton of the standard $(2n(H)+2)$-simplex. Then $G/H \times A$ is an $n(H)$-dimensional finite G-complex. Every point in $G/H \times A$ has G-isotropy type equal to (H) and $\dim(G/H \times A)^{H'} = n(H) \leq n(H')$ for every $H' \in \{H_1, \ldots, H_q\}$, with $(H') \leq (H)$ and $(G/H \times A)^{H'} = \varnothing$ in case $(H') \not\leq (H)$. If $f: G/H \times A \to V$ is a G-imbedding we obtain an induced imbedding $f|: A \to V^H$, and hence by [5] we have $\dim V^H \geq 2n(H)+1$. This fact and Proposition 2.1 show that (i) and (ii) are necessary conditions for the existence of an equivariant imbedding. □

REMARK. As was the case with Proposition 2.1, Theorem 3.1 also remains valid for G an arbitrary finite group if we assume that $H_i \triangleleft G$ for $1 \leq i \leq q$. In particular we have, in the case of G-complexes with free or semi-free actions, the following results.

THEOREM 3.2. *Let G be a finite group and V a representation space of G. In order for every k-dimensional free G-complex to have an equivariant imbedding into V, it is necessary and sufficient that*

(1) $\dim V - \dim V^{>\{e\}} \geq k+1$ *and*
(2) $\dim V \geq 2k+1$. $\square$

THEOREM 3.2′. *Let G and V be as in Theorem* 3.2 *and let* $n\colon \{\{e\}, G\} \to \mathbf{N}$ *be a dimension function; that is,* $n(\{e\}) \geq n(G)$. *In order for every semi-free G-complex X with* $\dim X \leq n(\{e\})$ *and* $\dim X^G \leq n(G)$ *to have an equivariant imbedding into V, it is necessary and sufficient that*
(1) $\dim V - \dim V^{>\{e\}} \geq n(\{e\})+1$,
(2′) $\dim V^G \geq 2n(G)+1$, *and* $\dim V \geq 2n(\{e\})+1$. $\square$

As in the case of isovariant maps considered in Section 2, we may also here analogously construct *one* finite G-complex such that the existence of a G-imbedding $f\colon \hat{X} \to V$ into a representation space V forces conditions (i) and (ii) in Theorem 3.1 to hold. Let G, $(H_1), \ldots, (H_q)$, and $n\colon \{(H_i)\}_{i=1}^q \to \mathbf{N}$ be as in Theorem 3.1. We define

$$\hat{X} = \dot{\bigcup_{i=1}^{q}} (G/H_i \times A^{n(H_i)}) \cup \bar{X},$$

where $\bar{X}$ is the G-complex defined in Section 2 and A^k denotes the k-skeleton of the standard $(2k+2)$-simplex. Then $\hat{X}$ is a finite G-complex whose isotropy types are exactly $(H_1), \ldots, (H_q)$ and $\dim \hat{X}^{H_i} \leq n(H_i)$ for $1 \leq i \leq q$. The proof of Theorem 3.1 shows that the existence of a G-imbedding $f\colon \hat{X} \to V$, where V is a linear representation space of G, implies that conditions (i) and (ii) in Theorem 3.1 must hold. In case $H_i \lhd G$, $1 \leq i \leq q$, we may drop the assumption that G is nilpotent.

In particular we have that, for any finite group G, the k-dimensional finite free G-complex

$$\hat{X}^k_{\text{free}} = G \times A^k \,\dot{\cup}\, \bar{X}^k_{\text{free}}$$

is such that the existence of a G-imbedding $f\colon \hat{X}^k_{\text{free}} \to V$ implies that

$$\begin{cases} \dim V - \dim V^{>\{e\}} \geq k+1, \\ \dim V \geq 2k+1. \end{cases}$$

The question of replacing $\hat{X}$ by a *connected* G-complex is completely analogous to the corresponding problem for the G-complex $\bar{X}$ in Section 2. In case $G \in \{H_1, \ldots, H_q\}$ we can easily construct a connected G-complex $\hat{X}_*$ with $\dim(\hat{X}_*) = \max(1, n(H))$ for all $H \in \{H_1, \ldots, H_q\}$, and which otherwise has the same properties as $\hat{X}$ and serves the same purpose as $\hat{X}$. In case $\{e\} \in \{H_1, \ldots, H_q\}$ the same construction as in Section 2 gives us a connected G-complex $\hat{X}^*$, with $\dim \hat{X}^* = \max(1, n(\{e\}))$, that otherwise has the same properties as $\hat{X}$ and serves the same purpose as $\hat{X}$.

4. Free $(\mathbf{Z}_p)^t$-complexes. In this section we apply our results to the case where G is an elementary abelian p-group $(\mathbf{Z}_p)^t$, $t \geq 2$. We consider isovariant maps and equivariant imbeddings of finite *free* $(\mathbf{Z}_p)^t$-complexes $(t \geq 2)$ into representation spaces, and give counterexamples to results in Allen [2].

First we need the following simple fact.

LEMMA 4.1. *Let V be a linear representation space of the group $(\mathbf{Z}_p)^t$, where p is a prime and $t \geq 2$. Then either $V^{>\{e\}} = V$ or we have*

$$\dim V^{>\{e\}} \geq \begin{cases} 2t-2 & \text{if } p \text{ is odd and } \dim V \text{ is even,} \\ 2t-1 & \text{if } p \text{ is odd and } \dim V \text{ is odd,} \\ t-1 & \text{if } p=2. \end{cases}$$

Proof. Assume that p is odd and that $\dim V = 2n$ is even. Then V is a direct sum of 2-dimensional representation spaces (i.e., of 1-dimensional complex representation spaces) and the corresponding representation is a homomorphism

$$\rho: (\mathbf{Z}_p)^t \to (\mathbf{Z}_p)^n \subset (U(1))^n.$$

If ρ is not injective we have $V^{>\{e\}} = V$. If ρ is injective we have $t \leq n$ and denote $q = n-t+1$. Since $t+q > n$ it follows by general position that

$$\operatorname{im} \rho \cap (\{e\}^{n-q} \oplus (\mathbf{Z}_p)^q) \neq \{e\}.$$

Thus there exists $g \in (\mathbf{Z}_p)^t$, $g \neq e$, such that $\rho(g) \in \{e\}^{n-q} \oplus (\mathbf{Z}_p)^q$, and hence $\dim V^{>\{e\}} \geq \dim V^g \geq 2(n-q) = 2t-2$.

In case p is odd and $\dim V$ is odd we have $V = V^{2n} \oplus \mathbf{R}$, with trivial $(\mathbf{Z}_p)^t$-action on $\mathbf{R}$. This completes the proof when p is odd. The proof for the case $p=2$ is completely analogous. □

Let $\bar{X}^k_{\text{free}}$ denote the k-dimensional finite free G-complex defined at the end of Section 2. Recall that $\bar{X}^k_{\text{free}}$ is such that if $f: \bar{X}^k_{\text{free}} \to V$ is an isovariant map into a representation space V, then we must have $\dim V - \dim V^{>\{e\}} \geq k+1$.

COROLLARY 4.2. (a) *Let $G = (\mathbf{Z}_p)^r$, where p is an odd prime and $r \geq 2$, and let $k \geq 0$ be a non-negative integer. Then the k-dimensional finite free G-complex $\bar{X}^k_{\text{free}}$ is such that if $f: \bar{X}^k_{\text{free}} \to V$ is an isovariant map we must have*

$$\dim V \geq \begin{cases} 2r+k & \text{if } k \text{ is even,} \\ 2r+k-1 & \text{if } k \text{ is odd.} \end{cases}$$

(b) *In case $G = (\mathbf{Z}_2)^s$, where $s \geq 2$, the corresponding conclusion is that*

$$\dim V \geq s+k.$$

Proof. This follows directly from Lemma 4.1 and the fact that we must have $\dim V - \dim V^{>\{e\}} \geq k+1$. □

Corollary 4.2 shows that Theorem 1.1 in Allen [2] is incorrect. Let p denote an odd prime. According to Theorem 1.1 in [2], every compact k-dimensional metric space X with a free $(\mathbf{Z}_p)^r$-action [$(\mathbf{Z}_2)^s$-action] should have an equivariant imbedding into a representation space of dimension equal to $\max(2k+1, 2r)$ [$\max(2k+1, s)$]. Thus for p an odd prime and any $r \geq 2$, the 2-dimensional finite free $(\mathbf{Z}_p)^r$-complex $\bar{X}^2_{\text{free}}$ is a counterexample to Theorem 1.1 in [2], since the existence of an isovariant map $f: \bar{X}^2_{\text{free}} \to V$ already implies that we must have

$\dim V \geq 2r+2 > \max(5, 2r)$. In case $p=2$ and $s \geq 3$ the 1-dimensional $(\mathbf{Z}_2)^s$-complex $\bar{X}^1_{\text{free}}$ provides a counterexample to Theorem 1.1 in [2].

A mistake in Allen [2] can be found in the proof of Theorem 5.7, where it is incorrectly stated that a certain representation space is free, that is, that the action in question is free outside the origin. (See [2, p. 30, lines 15 and 24–25].) Of course groups like $(\mathbf{Z}_p)^t$, $t \geq 2$, do not have any free representations at all. In [2], Theorem 1.1 is obtained as a special case of Theorem 5.7, and Theorems 6.2 and 6.3 are based upon Theorem 5.7 and also contain it as a special case. Hence the above counterexamples show that Theorems 1.1, 5.7, 6.2, and 6.3 in [2] are incorrect.

So far we did not give a counterexample to the results of [2] for the case when $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. In order to do so we consider the 1-dimensional finite semi-free $(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$-complex $\hat{X}^1_{\text{s.f.}} = A^1 \dot{\cup} \bar{X}^1_{\text{free}}$. (See the discussion at the end of Section 3.) Here A^1 denotes the 1-skeleton of the standard 4-simplex, with trivial $(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$-action. Now assume that $f: \hat{X}^1_{\text{s.f.}} \to V$ is a G-imbedding into a linear representation space V of $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Then we have the induced imbedding $f^G: A^1 \to V^G$ and hence $\dim V^G \geq 3$. Moreover, the composite map

$$\bar{X}^1_{\text{free}} \xrightarrow{f|} V \xrightarrow{\pi} V/V^G = V_1,$$

where π denotes the natural projection, is an isovariant map; hence we have by Corollary 4.2.b that $\dim V_1 \geq 3$. Thus $\dim V \geq 6$. This is a counterexample to Theorem 5.7 in [2] when $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, because according to [2, Theorem 5.7] $\hat{X}^1_{\text{s.f.}}$ should have a G-imbedding into a 5-dimensional linear representation space. (In the next section we will see that in fact 6 is the minimal imbedding dimension in this case.)

5. The minimal imbedding dimensions for free, semi-free and arbitrary $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes. Let p be a prime. In this section we determine the minimal imbedding dimensions in the three different cases of equivariant imbeddings into representation spaces of free, semi-free and arbitrary $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes. These minimal imbedding dimensions depend on whether p is odd or even, and they are given in Theorem 5.1.a and 5.1.b, respectively. Let us here recall the general fact that the minimal imbedding dimensions do not depend on whether we consider only finite G-complexes or arbitrary G-complexes or if we consider equivariant proper p.l. imbeddings of arbitrary G-complexes. As we already pointed out in the introduction, it is also a general fact that we can always choose *one* representation space W with dimension equal to the minimal imbedding dimension $w(k)$ such that every k-dimensional G-complex, of the appropriate kind, has an equivariant (proper p.l.) imbedding into W. In the proof of Theorem 5.1.a we also explicitly give such a minimal representation space in each of the three cases being considered.

A. *p is odd prime.* Write any representation space of $\mathbf{Z}_p \oplus \mathbf{Z}_p$ as a direct sum of irreducible representation spaces. We have the trivial 1-dimensional representation, and every non-trivial irreducible representation of $\mathbf{Z}_p \oplus \mathbf{Z}_p$ (p an odd prime) is 2-dimensional. Each non-trivial 2-dimensional representation of $\mathbf{Z}_p \oplus \mathbf{Z}_p$ can be considered as a 1-dimensional complex representation and has as kernel a non-trivial cyclic subgroup of $\mathbf{Z}_p \oplus \mathbf{Z}_p$. Moreover, each non-trivial cyclic subgroup of $\mathbf{Z}_p \oplus \mathbf{Z}_p$ occurs as the kernel of some non-trivial 1-dimensional complex representation of $\mathbf{Z}_p \oplus \mathbf{Z}_p$, and there are exactly $p+1$ non-trivial cyclic subgroups of $\mathbf{Z}_p \oplus \mathbf{Z}_p$.

In the following we let U^u denote a complex representation space of $\mathbf{Z}_p \oplus \mathbf{Z}_p$ of complex dimension u. We define $\psi: \mathbf{N} \to \mathbf{N}$ by

$$\psi(u) = \min\{\dim_{\mathbf{C}} U^{>\{e\}} \mid \dim_{\mathbf{C}} U = u\}.$$

It follows from the above remarks that we have

$$\psi(u) = \begin{cases} d & \text{if } u = (p+1)d, \\ d+1 & \text{if } u = (p+1)d + r,\ 1 \le r \le p, \end{cases}$$

that is,

$$\psi(u) = \left[\frac{u+p}{p+1}\right].$$

Furthermore, we have the following. Let $H_1, \ldots, H_{p+1}$ denote the nontrivial cyclic subgroups of $\mathbf{Z}_p \oplus \mathbf{Z}_p$, and let U_i be a 1-dimensional complex representation space of $\mathbf{Z}_p \oplus \mathbf{Z}_p$ with kernel H_i, $1 \le i \le p+1$. Given any integer $u \ge 0$ we set $u = (p+1)d + r$, where $0 \le r \le p$, and define

$$\bar{U} = (U_1 \oplus \cdots \oplus U_{p+1})^d \oplus U_1 \oplus \cdots \oplus U_r. \tag{1}$$

Then $\bar{U}$ is a u-dimensional complex representation space for which $\dim_{\mathbf{C}} \bar{U}^{>\{e\}}$ attains the minimal value; that is, we have $\psi(u) = \dim_{\mathbf{C}} \bar{U}^{>\{e\}}$.

For any integer $k \ge 0$ we now define $u(k)$ as the least integer u for which there exists a u-dimensional complex representation space U such that

$$\dim_{\mathbf{C}} U - \dim_{\mathbf{C}} U^{>\{e\}} \ge \tfrac{1}{2}(k+1). \tag{2}$$

Assume $k \ge 0$ given and let $U_0 = U_0^{u(k)}$ be a complex representation space which satisfies (2) and has the minimal dimension $u_0 = u(k)$. We claim that we then have

$$\dim_{\mathbf{C}} U_0^{>\{e\}} = \psi(u_0). \tag{3}$$

If (3) does not hold we would have $\dim_{\mathbf{C}}^{>\{e\}} \ge \psi(u_o) + 1$. Let U be a complex representation space of dimension $u_0 - 1$ with $\dim_{\mathbf{C}} U^{>\{e\}} = \psi(u_0 - 1)$. Since $\psi(u_0 - 1) \le \psi(u_0)$ we then have

$$\begin{aligned} \dim_{\mathbf{C}} U - \dim_{\mathbf{C}} U^{>\{e\}} &= u_0 - 1 - \psi(u_0 - 1) \ge u_0 - 1 - \psi(u_0) \\ &\ge u_0 - \dim_{\mathbf{C}} U_0^{>\{e\}} \ge \tfrac{1}{2}(k+1), \end{aligned}$$

which contradicts the minimality of $u_0 = u(k)$. Thus (3) holds.

It follows from the above that the function $u: \mathbf{N} \to \mathbf{N}$ is determined by the fact that $u(k)$ is the least integer for which

$$u(k) - \psi(u(k)) \ge \tfrac{1}{2}(k+1);$$

that is, $u(k)$ equals the least integer for which

$$u(k)-\left[\frac{u(k)+p}{p+1}\right]\geq\frac{1}{2}(k+1).$$

Hence it follows by elementary considerations that we have

$$u(k)=\begin{cases}\dfrac{k+2}{2}+1+\left[\dfrac{k}{2p}\right] & \text{for } k \text{ even},\\ \dfrac{k+1}{2}+1+\left[\dfrac{k}{2p}\right] & \text{for } k \text{ odd}.\end{cases}$$

Moreover, the discussion above shows that a solution of (2) of the minimal dimension $u(k)$ is given by the complex representation space $\bar{U}$ defined in (1), where $u(k)=(p+1)d+r$ and $0\leq r\leq p$.

Now observe that if V is a real representation space of $\mathbf{Z}_p\oplus\mathbf{Z}_p$ of minimal dimension with respect to the inequality

(4) $$\dim V-\dim V^{>\{e\}}\geq k+1$$

then we must have $V^{(\mathbf{Z}_p\oplus\mathbf{Z}_p)}=\{0\}$. Hence we may consider V as a complex representation space, and therefore $\dim V=\dim_{\mathbf{R}} V=2u(k)$. By considering $\bar{U}$ defined by (1) as a real representation space, where $u(k)=(p+1)d+r$ and $0\leq r\leq p$, we obtain a specific $2u(k)$-dimensional representation space which satisfies (4).

We are now ready to prove the following.

THEOREM 5.1.a. *Let p be an odd prime. The minimal imbedding dimension for equivariant imbeddings of k-dimensional free $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complexes equals*

$$w_0(k)=\begin{cases}4 & \textit{for } k=0,1,\\ 6 & \textit{for } k=2,\\ 2k+1 & \textit{for } k\geq 3.\end{cases}$$

In the case of semi-free $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complexes, the minimal imbedding dimension is given by

$$w_1(k)=\begin{cases}3k+5+2[k/2p] & \textit{if } k \textit{ is even},\\ 3k+4+2[k/2p] & \textit{if } k \textit{ is odd},\end{cases}$$

and in the case of arbitrary $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complexes by

$$w(k)=\begin{cases}3k+3+p(k+2) & \textit{if } k \textit{ is even},\\ 3k+2+p(k+1) & \textit{if } k \textit{ is odd}.\end{cases}$$

Proof. By Theorem 3.2 and the above discussion the minimal imbedding dimension for equivariant imbeddings of k-dimensional free $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complexes equals

$$w_0(k)=\max\{2u(k),2k+1\}.$$

Observe that $2u(k)>2k+1$ for $0\leq k\leq 2$ and $2k+1>2u(k)$ for $k\geq 3$. For each $k\geq 0$, the real representation space

$$W_0=\bar{U}\oplus\mathbf{R}^{a(k)}$$

(where $\bar{U}$ is as above, $a(k)=0$ if $0\leq k\leq 2$, and $a(k)=2k+1-2u(k)$ if $k\geq 3$) is a representation space of the minimal imbedding dimension $w_0(k)$ such that every k-dimensional free $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complex has an equivariant proper p.l. imbedding in W_0.

It follows by Theorem 3.2′ and the discussion proceeding Theorem 5.1.a that the minimal imbedding dimension in the case of semi-free $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complexes equals $w_1(k)=2k+1+2u(k)$. As a specific $w_1(k)$-dimensional representation space into which every k-dimensional semi-free $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complex has an equivariant proper p.l. imbedding, we may take $W_1=\mathbf{R}^{2k+1}\oplus\bar{U}$.

In the case of arbitrary $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complexes we have, by Theorem 3.1, that in order for a representation V to be such that every k-dimensional $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complex has an equivariant imbedding into V, it is necessary and sufficient that the following conditions hold:

(i) $\dim V_i-\dim V^{>\{e\}}\geq k+1$,

(ii) $\dim V^{H_i}-\dim V^G\geq k+1$, and

(iii) $\dim V^G\geq 2k+1$,

where $H_1,\dots,H_{p+1}$ denote the non-trivial cyclic subgroups of $G=\mathbf{Z}_p\oplus\mathbf{Z}_p$. Assume that W is a representation space of minimal dimension which satisfies (ii). Then $W^G=\{0\}$, and we may consider W as a complex representation space and write W as a direct sum of 1-dimensional irreducible complex representations. Since $\dim W^{H_i}\geq k+1$ the number of irreducible summands in W with kernel H_i must be greater than $\frac{1}{2}(k+1)$ for $i=1,\dots,p+1$. Thus

$$\dim W\geq\begin{cases}(p+1)(k+2) & \text{if } k \text{ is even,}\\ (p+1)(k+1) & \text{if } k \text{ is odd.}\end{cases}$$

A representation space of minimal dimension which satisfies (ii) is given by

$$\bar{W}=(U_1\oplus\cdots\oplus U_{p+1})^{b(k)},$$

where $b(k)=[(k+2)/2]$ and U_i is a 1-dimensional complex representation space with kernel H_i, $1\leq i\leq p+1$. Observe that

$$\dim\bar{W}-\dim\bar{W}^{>\{e\}}=2pb(k)>k+1$$

and hence $\bar{W}$ automatically satisfies (i). It follows that the minimal dimension for equivariant imbeddings of arbitrary $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complexes equals

$$w(k)=\begin{cases}2k+1+(p+1)(k+2) & \text{if } k \text{ is even,}\\ 2k+1+(p+1)(k+1) & \text{if } k \text{ is odd.}\end{cases}$$

As a specific $w(k)$-dimensional representation space into which every k-dimensional $(\mathbf{Z}_p\oplus\mathbf{Z}_p)$-complex has an equivariant proper p.l. imbedding, we may take

$$W=\mathbf{R}^{2k+1}\oplus\bar{W}. \qquad \square$$

Let us now compare the various minimal imbedding dimensions given by Theorem 5.1.a with earlier known equivariant imbedding results. The result by Kister and Mann (see [8, Theorem 1, §4]) shows that every separable metrizable k-dimensional free or semi-free $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-space can be equivariantly imbedded in a representation space of dimension

$$N(k) = \begin{cases} 4k+5 & \text{if } k \text{ is even}, \\ 4k+3 & \text{if } k \text{ is odd}. \end{cases}$$

(The reason for the fact that the Kister–Mann result gives the same dimension result in the case of both free and semi-free actions is that in their proof a factor $\mathbf{R}^{2k+1}$ comes from an imbedding of the orbit space X/G, and no account is taken of the possibility that the fixed point set X^G may be very low-dimensional or even empty.) In the case of separable metrizable spaces with arbitrary $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-actions, the result by Kister and Mann [8, Theorem 2, §4] shows that every such k-dimensional space has an equivariant imbedding in a representation space of dimension

$$M(k) = \begin{cases} 6k+9+2p(k+2) & \text{if } k \text{ is even}, \\ 6k+5+2p(k+1) & \text{if } k \text{ is odd}. \end{cases}$$

Thus we see that for $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes the results by Kister and Mann are far from best possible. In the case of semi-free actions the Kister–Mann result comes closest to the minimal imbedding dimension, and for $k=0$ or 1 it gives the minimal imbedding dimension. But for 2-dimensional semi-free $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes the Kister–Mann result gives the imbedding dimension 13, whereas the minimal imbedding dimension in this case equals 11. In the two other cases (i.e., for free or arbitrary $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes) the imbedding dimension given by the Kister–Mann result is strictly greater than the minimal imbedding dimension for all $k \geq 0$, and except for free $(\mathbf{Z}_p \oplus \mathbf{Z}_p)$-complexes of dimension 0 or 1 it is in fact greater than twice the minimal imbedding dimension.

B. *The case* $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. The procedure here is completely analogous to the one in case A. This time we define $\psi\colon \mathbf{N} \to \mathbf{N}$ by

$$\psi(v) = \min\{\dim V^{>\{e\}} \mid \dim V = v\},$$

where V denotes a real representation space of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $\dim = \dim_{\mathbf{R}}$. Since the irreducible representations of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ are 1-dimensional and there are three non-trivial different cyclic subgroups of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, it follows that

$$\psi(v) = \begin{cases} d & \text{if } v = 3d, \\ d+1 & \text{if } v = 3d+r,\ r = 1 \text{ or } 2, \end{cases}$$

that is,

$$\psi(v) = \left[\frac{v+2}{3}\right].$$

Let H_1, H_2, and H_3 denote the non-trivial cyclic subgroups of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and let V_i be a 1-dimensional real representation space with kernel H_i, $1 \leq i \leq 3$. For each integer $v \geq 0$ we set $v = 3d+r$, $0 \leq r \leq 2$, and define

$$(1) \qquad \bar{V}=(V_1\oplus V_2\oplus V_3)^d\oplus V_1\oplus\cdots\oplus V_r.$$

Then $\bar{V}$ is a v-dimensional representation space for which $\dim \bar{V}^{>\{e\}}$ attains the minimal value; that is, we have

$$\psi(v)=\dim \bar{V}^{>\{e\}}.$$

For any integer $k\geq 0$ we define $v(k)$ as the least integer v for which there exists a v-dimensional representation space V satisfying

$$(2) \qquad \dim V-\dim V^{>\{e\}}\geq k+1.$$

In the same way as in case A we see that $v: \mathbf{N}\to\mathbf{N}$ is determined by the fact that $v(k)$ is the least integer for which

$$v(k)-\psi(v(k))\geq k+1;$$

that is, $v(k)$ equals the least integer for which

$$v(k)-\left[\frac{v(k)+2}{3}\right]\geq k+1.$$

It follows that

$$v(k)=k+2+\left[\frac{k}{2}\right].$$

Moreover, $\bar{V}$ defined by (1), where $v(k)=3d+r$, $0\leq r\leq 2$, is a representation space of the minimal dimension $v(k)$ which satisfies (2).

THEOREM 5.1.b. *The minimal embedding dimension for equivariant imbeddings of k-dimensional free* $(\mathbf{Z}_2\oplus\mathbf{Z}_2)$*-complexes equals*

$$w_0(k)=\begin{cases}2 & \text{if } k=0,\\ 2k+1 & \text{if } k\geq 1.\end{cases}$$

In the case of semi-free $(\mathbf{Z}_2\oplus\mathbf{Z}_2)$*-complexes the minimal imbedding dimension is given by*

$$w_1(k)=3k+3+\left[\frac{k}{2}\right],$$

and in the case of arbitrary $(\mathbf{Z}_2\oplus\mathbf{Z}_2)$*-complexes by*

$$w(k)=5k+4.$$

Proof. The proof is completely analogous to the proof of Theorem 5.1.a. We leave the details to the reader. □

The results of Kister and Mann [8, Theorems 1 and 2] show that every k-dimensional free or semi-free $(\mathbf{Z}_2\oplus\mathbf{Z}_2)$-complex has an equivariant imbedding in a representation space of dimension $4k+3$; in the case of arbitrary $(\mathbf{Z}_2\oplus\mathbf{Z}_2)$-complexes the corresponding dimension is $10k+9$.

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