HELICAL IMMERSIONS INTO A EUCLIDEAN SPACE

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0. Introduction. Let M be a connected Riemannian manifold and $f: M \to \bar{M}$ an isometric immersion into a Riemannian manifold $\bar{M}$. If the image $f \circ \gamma$ of each geodesic γ in M has constant Frenet curvatures which are independent of the chosen geodesic γ, then f is called a *helical immersion*. Furthermore, if the osculating order of $f \circ \gamma$ in $\bar{M}$ is equal to d, then the helical immersion is said to be of order d. In [8], [9], and [12], helical immersions of order two into real space forms were classified (see also [6]). When the ambient manifold $\bar{M}$ is a sphere, the theory of helical minimal immersions is a submanifold version of compact harmonic manifolds (cf. [1], [13]) and low order cases ($d \leq 5$) were classified in [10] and [15]–[17]. In the present paper, we shall study helical immersions into a Euclidean space.

On the other hand, in [3] and [4], Chen and Verheyen introduced a notion of submanifolds with geodesic normal sections and obtained many results, in particular for the case where the submanifolds are surfaces. Here we recall the definition of a submanifold M with geodesic normal sections in a Euclidean space E^m of dimension m. For each point x in M and vector X tangent to M at x, the intersection of M and the affine subspace through x spanned by X and the normal space at x gives rise to a curve γ in a neighborhood of x. If such curve γ is a geodesic in M, then the submanifold M is called a *submanifold with geodesic normal sections* in E^m. In [3], [4], and [19], surfaces with geodesic normal sections in E^m where $3 \leq m \leq 6$ were classified. In this paper, we shall determine 2- or odd-dimensional complete submanifolds with geodesic normal sections in E^m but without restrictions on m.

Verheyen proved that M is a submanifold with geodesic normal sections in E^m if and only if the inclusion map $\iota: M \to E^m$ is a helical immersion (cf. [19, Theorem 2]). So the concept "helical immersion" coincides with that "submanifold with geodesic normal sections" when the ambient manifold is a Euclidean space. We shall study from the viewpoint of helical immersions, because we can give an explicit expression of a geodesic of M in the ambient space.

In Section 1, we give basic equations used later as well as the accurate definition of helical immersions of order d. In Section 2, making use of an expression of a helical immersion $f: M \to E^m$ in the geodesic polar coordinates around an arbitrarily fixed point, we prove that the extrinsic distance of two points in M is a function of their intrinsic distance. This result is a characterization of helical immersions into E^m. By using this result, we can show that if M is compact then it is a Blaschke manifold, and that if M is complete and noncompact then all points of M are poles. In Section 3, we deal with helical imbeddings of odd order into E^m. We show that if $f: M \to E^m$ is a helical imbedding of odd order and M is

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complete, then $M \approx E^n$ and f is totally geodesic. In Section 4, we are absorbed in the study of the case that the order is even. Our main result is that the order of f is even if and only if M is compact. Finally, we apply Berger's theorem about the Blaschke conjecture and Tsukada's theorem about the rigidity of helical immersions into a sphere to the case $\dim M=2$ or odd.

1. Preliminaries. In the present paper, the differentiability of all geometric objects will be C^∞. Let $f: M \to \bar{M}$ be an isometric immersion of an n-dimensional Riemannian manifold M into an m-dimensional Riemannian manifold $\bar{M}$. We shall identify the tangent space $T_x M$ of M with a subspace $f_* T_x M$ of $T_{f(x)} \bar{M}$. Let ∇ and $\bar{\nabla}$ denote the covariant differential operators of M and $\bar{M}$ respectively. Then the Gauss equation is given by

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y) \tag{1.1}$$

for vector fields X, Y tangent to M, where H denotes the second fundamental form. The Weingarten equation is given by

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi \tag{1.2}$$

for a vector field ξ normal to M, where A_ξ denotes the shape operator corresponding to ξ and $\nabla^\perp$ the normal connection. Clearly A_ξ is related to H as $\langle A_\xi X, Y\rangle = \langle H(X, Y), \xi\rangle$, $\langle\ ,\ \rangle$ being the inner product of vectors.

Let the ambient space $\bar{M}$ be a Euclidean space E^m. Let R be the curvature tensor of M. The structure equation of Gauss and Codazzi are given by

$$R(X, Y)Z = A_{H(Y,Z)} X - A_{H(X,Z)} Y, \tag{1.3}$$

$$(D_X H)(Y, Z) = (D_Y H)(X, Z) \tag{1.4}$$

(respectively), where $(D_X H)(Y, Z)$ is defined as

$$(D_X H)(Y, Z) = \nabla_X^\perp H(Y, Z) - H(\nabla_X Y, Z) - H(Y, \nabla_X Z).$$

We shall denote $(D_X H)(Y, Z)$ by $(DH)(X, Y, Z)$.

Next we explain Frenet curvatures of a curve $\tau: I \to \bar{M}$ parameterized by the arc-length s. Let $\tau_1 = \dot{\tau}$ be the unit tangent vector and put $\lambda_1 = \|\bar{\nabla}_{\dot{\tau}} \tau_1\|$. If λ_1 vanishes on I, then τ is said to be of order 1. If λ_1 is not identically zero, then one defines τ_2 by $\bar{\nabla}_{\dot{\tau}} \tau_1 = \lambda_1 \tau_2$ on $I_1 = \{s \in I: \lambda_1(s) \neq 0\}$. Put $\lambda_2 = \|\bar{\nabla}_{\dot{\tau}} \tau_2 + \lambda_1 \tau_1\|$. If $\lambda_2 \equiv 0$ on I_1, then τ is said to be of order 2. If λ_2 is not identically zero on I_1, then we define τ_3 by $\bar{\nabla}_{\dot{\tau}} \tau_2 = -\lambda_1 \tau_1 + \lambda_2 \tau_3$. Inductively we put $\lambda_d = \|\bar{\nabla}_{\dot{\tau}} \tau_d + \lambda_{d-1} \tau_{d-1}\|$, and if $\lambda_d \equiv 0$ on I_{d-1} then τ is said to be of order d. If τ is of order d, then we have a matrix equation $\bar{\nabla}_{\dot{\tau}}(\tau_1, \tau_2, \ldots, \tau_d) = (\tau_1, \tau_2, \ldots, \tau_d)\Lambda$ on I_{d-1}, where Λ is a $d \times d$ matrix defined by

$$\Lambda = \begin{bmatrix} 0 & -\lambda_1 & & & \mathbf{0} \\ \lambda_1 & 0 & -\lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -\lambda_{d-1} \\ \mathbf{0} & & & \lambda_{d-1} & 0 \end{bmatrix}. \tag{1.5}$$

Equation (1.5), $\{\tau_1, \ldots, \tau_d\}$, and $\lambda_1, \ldots, \lambda_d$ are called (respectively) the Frenet formula, Frenet frame and Frenet curvatures of τ.

Now we give the definition of helical immersion. Let $\gamma: I \to M$ be an arbitrary geodesic in M. If the curve $\tau = f \circ \gamma$ in $\bar{M}$ is of order d and has constant curvatures $\lambda_1, \ldots, \lambda_{d-1}$ $(\neq 0)$, λ_d $(=0)$ which are independent of the chosen geodesic γ, then the isometric immersion $f: M \to \bar{M}$ is called a *helical immersion of order d*. Helical immersions are λ_1-isotropic (cf. [13]). Here we recall the definition of isotropic immersion (cf. [11]). An isometric immersion $f: M \to \bar{M}$ is said to be λ-*isotropic* if $\lambda(x) = \|H(X, X)\|$ is independent of the choice of $X \in U_x M = \{X \in T_x M: \|X\| = 1\}$. In particular, when $\lambda(x)$ is constant on M, f is said to be constant isotropic. It is easily seen that f is λ-isotropic if and only if

$$\mathfrak{S} A_{H(X,Y)} Z = \lambda^2 \mathfrak{S} \langle X, Y \rangle Z \tag{1.6}$$

for every $X, Y, Z \in TM$, where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y, and Z.

2. Helical immersions into E^m. In the sequel, M will be a connected complete Riemannian manifold of dimension n $(n \geq 2)$. Let $f: M \to E^m$ be a helical immersion of order d into an m-dimensional Euclidean space E^m. Let γ be an arbitrary geodesic in M parameterized by the arc-length s. The ith order derivative of the second fundamental form is denoted by $D^i H$, and $(D^i H)(X, \ldots, X)$ is written as $(D^i H)(X^{i+2})$. We have the following.

LEMMA 2.1 (see [13, Theorem 3.1, p. 67; Remark, p. 70]. *The Frenet frame of $\tau = f \circ \gamma$ is given by*

$$\tau_1 = \dot{\gamma},$$
$$\tau_j = (\lambda_1 \cdots \lambda_{j-1})^{-1} \sum a_{ji} (D^{i-2} H)(\dot{\gamma}^i)$$

for $j = 2, \ldots, d$, where i runs over the range $\{2, 4, \ldots, j\}$ if j is even and $\{3, 5, \ldots, j\}$ if j is odd. The coefficients a_{ji}'s are positive constants determined by curvatures $\lambda_1, \ldots, \lambda_{d-1}$ of τ.

Let ${}^t(f_1(s), \ldots, f_d(s))$ be the first column of the matrix $\int_0^s \exp s\Lambda \, ds$. Then the functions f_i $(i = 1, \ldots, d)$ defined on $\mathbf{R}$ satisfy

$$\begin{aligned} f_1' &= 1 - \lambda_1 f_2, \\ f_i' &= \lambda_{i-1} f_{i-1} - \lambda_i f_{i+1} \quad (2 \leq i \leq d-1), \\ f_d' &= \lambda_{d-1} f_{d-1} \end{aligned} \tag{2.1}$$

and $f_i(0) = 0$ for all i. We easily see that f_i is an odd (resp. even) function if i is odd (resp. even). We define normal vectors $\xi(s; X)$ and $\zeta(s; X)$ by

$$\xi(s; X) = \sum_{j:\text{ even}} f_j(s) \tau_j(X),$$
$$\zeta(s, X) = \sum_{j:\text{ odd} \geq 3} f_j(s) \tau_j(X),$$

for all $X \in U_x M$, where $\tau_j(X) = (\lambda_1 \cdots \lambda_{j-1})^{-1} \sum a_{ji} (D^{i-2} H)(X^i)$. We now represent f by the geodesic polar coordinates around a fixed point x.

LEMMA 2.2 (cf. [13, Theorem 5.4]). *For* $s \in \mathbf{R}_+$ *and* $X \in U_x M$, *we have*

$$f(\exp_x sX) = f(x) + f_1(s)X + \xi(s;X) + \zeta(s;X).$$

Proof. Solve the Frenet equation

$$\frac{d}{ds}(\tau_1, \dots, \tau_d) = (\tau_1, \dots, \tau_d)\Lambda$$

with initial conditions $\tau_i(0) = \tau_i(X)$ for $i = 1, \dots, d$, where $\tau(s) = f(\exp_x sX)$. We see that $(\tau_1(s), \dots, \tau_d(s)) = (\tau_1(X), \dots, \tau_d(X)) \exp s\Lambda$. In particular, we obtain $\tau_1(s) = (\tau_1(X), \dots, \tau_d(X)) \exp s\Lambda \cdot e_1$, where $e_1 = {}^t(1, 0, \dots, 0) \in \mathbf{R}^d$. It follows that

$$\tau(s) = f(x) + (\tau_1(X), \dots, \tau_d(X)) \int_0^s \exp s\Lambda \, ds \cdot e_1,$$

which shows the assertion. □

Making use of Lemma 2.2, we have the following.

PROPOSITION 2.3. *Let* δ *denote the distance function on* M. *Then*

$$\|f(x) - f(y)\|^2 = G(\delta(x, y)) \tag{2.2}$$

for every $x, y \in M$, *where* $G = \sum_{i=1}^d f_i^2$. *Thus we can say that the extrinsic distance of two points in* M *is a function of their intrinsic distance. Conversely, if* $f: M \to E^m$ *is an isometric immersion such that* (2.2) *holds for some even function* G, *then* f *is helical.*

Proof. For every $x, y \in M$ there exists a geodesic $s \mapsto \exp_x sX$ such that $\exp_x \delta(x, y)X = y$, since M is connected and complete. By Lemma 2.2., we have $f(y) - f(x) = f_1(s)X + \xi(s;X) + \zeta(s;X)$. Since $\|f_1(s)X + \xi(s;X) + \zeta(s;X)\|^2 = \sum_{i=1}^d f_i^2(s)$, we obtain (2.2). Conversely, if (2.2) holds with some even function G, then for any geodesic γ in M parameterized by the arc-length we have $\|\tau(s) - \tau(t)\|^2 = G(s-t)$, where $|s-t| < \epsilon$: small. It follows that $\langle \dot\tau(s), \dot\tau(t) \rangle = \frac{1}{2}G''(s-t)$ and hence $\langle \tau^{(k)}(s), \tau^{(\ell)}(s) \rangle$ $(k, \ell \geq 1)$ are constants depending only on $G^{(k)}(0)$ (k: even). From the definition of Frenet curvatures, we see that f is helical. □

THEOREM 2.4. *Let* $f: M \to E^m$ *be a helical immersion of a connected complete Riemannian manifold* M *into a Euclidean space* E^m. *If* M *is compact, then* M *is a Blaschke manifold* (*i.e., for each* $x \in M$, *the distance from* x *to its cut points is constant; for details, see* [1]). *If* M *is noncompact, then every point of* M *is a pole.*

Proof. Let $x \in M$ be arbitrarily chosen and γ be a unit speed geodesic such that $\gamma(0) = x$ and $\dot\gamma(0) = X$. It suffices to prove that if $\gamma(s_0)$ is a conjugate point of x, then $f_1(s_0) = 0$ and $G' = 2f_1$. If this assertion has been proved, then the smoothness of the function $\|f(x) - f(\gamma(s))\|^2$ of s for each geodesic γ issuing from x implies that when M is compact it is a Blaschke manifold and, when M is noncompact, $\exp_x: T_x M \to M$ is a diffeomorphism (see the proof of Theorem 6.2 in [13, p. 77]). Let J_V be a Jacobi field along γ such that

$$J_V(0)=0 \quad \text{and} \quad \nabla_X J_V = V \in \{X\}^\perp \cap U_x M.$$

Such Jacobi field is obtained from the variation $(s,\theta)\mapsto \exp_x sX(\theta)$, where $X(\theta)=\cos\theta X+\sin\theta V$. In virtue of Lemma 2.2, we find

$$(2.3) \qquad J_V(s)=f_1(s)V+\frac{d}{d\theta}\{\xi(s;X(\theta))+\zeta(s;X(\theta))\}\,|_{\theta=0}.$$

Thus if $J_V(s_0)=0$, then $f_1(s_0)=0$. We next prove $G'=2f_1$. Equation (2.1) can be rewritten as

$$^t(f_1',\dots,f_d')=\Lambda^t(f_1,\dots,f_d)+{}^t(1,0,\dots,0).$$

Since $G'=2\sum f_i f_i'$ by the definition of G, we have

$$\begin{aligned} G' &= 2(f_1,\dots,f_d)^t(f_1',\dots,f_d') \\ &= 2(f_1,\dots,f_d)\{\Lambda^t(f_1,\dots,f_d)+{}^t(1,0,\dots,0)\} \\ &= 2f_1, \end{aligned}$$

where we have used the fact that Λ is skew-symmetric. □

Next we recall the definition of "geodesic normal sections" (cf. [3], [4]). Let M be a connected n-dimensional ($n\geq 2$) submanifold of an m-dimensional Euclidean space E^m. For $x\in M$ and $X\in U_xM$, the vector X and the normal space N_xM at x determine an $(m-n+1)$-dimensional affine subspace $E(x,X)$ in E^m through x. The intersection of M and $E(x,X)$ gives rise to a curve γ in a neighborhood of x which is called the *normal section* of M at x in the direction X. If every normal section at arbitrary point is a geodesic of M, then M is called a *submanifold with geodesic normal sections.*

In [19], Verheyen proved that if M is a submanifold with geodesic normal sections in E^m, then the inclusion $\iota: M\to E^m$ is helical. The converse is clear from Lemma 2.2. Thus we have the following.

COROLLARY 2.5. *Let M be a connected submanifold with geodesic normal sections in E^m. If M is compact, then M is a Blaschke manifold. If M is noncompact, then every point of M is a pole.*

Concluding this section, we note the following.

LEMMA 2.6 (cf. [13, Corollary 6.3]). *Let $f: M\to E^m$ be a helical immersion of a connected complete Riemannian manifold M into E^m. If f is not injective, then M is isometric to a sphere S^n and $f=\tilde f\circ\pi$, where $\pi: S^n\to \mathbf{R}P^n$ is the covering projection and $\tilde f: \mathbf{R}P^n\to E^m$ is a helical imbedding. Moreover, M is simply connected except for the case that M is diffeomorphic to $\mathbf{R}P^n$.*

Proof. By the same argument as in the proof of Theorem 6.99 ([1, p. 176]) and using Berger's theorem ([1, Appendix D., p. 236]), we have the first assertion. For the second assertion, we have only to consider $f\circ\pi: \hat M\to E^m$ (where $\pi:\hat M\to M$ is the universal Riemannian covering) and note that $f\circ\pi$ is also helical. □

In virtue of Lemma 2.6, we may assume that the helical immersion $f: M\to E^m$ is an imbedding.

3. Helical imbeddings of odd order. At the beginning of this section, we study the functions $f_1, \dots, f_d$. The straightforward computation shows $\det \Lambda = \lambda_1^2 \lambda_3^2 \cdots \lambda_{d-1}^2 \neq 0$ if d is even and $\operatorname{rank} \Lambda = d-1$ if d is odd. Thus the normal form of Λ is given by

$$T^{-1}\Lambda T = \begin{cases} \bigoplus_{i=1}^{d/2} \mathcal{R}(\alpha_i) & \text{if } d \text{ is even,} \\ \bigoplus_{i=1}^{(d-1)/2} \mathcal{R}(\alpha_i) \oplus 0 & \text{if } d \text{ is odd,} \end{cases} \tag{3.1}$$

with some orthogonal matrix T, where

$$\mathcal{R}(\alpha_i) = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix} \quad (0 < \alpha_1 \leq \cdots \leq \alpha_{[d/2]})$$

and, in the case d is odd, $\oplus 0$ means that the (d, d)-element of $T^{-1}\Lambda T$ is zero. We have (cf. [7]) the following.

LEMMA 3.1. *If i is even then*

$$f_i(s) = \sum_{k=1}^{[d/2]} \nu_{ik}(1 - \cos \alpha_k s),$$

and if i is odd then

$$f_i(s) = \begin{cases} \sum_{k=1}^{d/2} \nu_{ik} \sin \alpha_k s & (d\text{: even}), \\ \sum_{k=1}^{(d-1)/2} \nu_{ik} \sin \alpha_k s + \nu_i s & (d\text{: odd}), \end{cases}$$

where ν_{ik} and ν_i are constants determined by $\lambda_1, \dots, \lambda_{d-1}$. Moreover we see that $\alpha_1, \dots, \alpha_{[d/2]}$ are all distinct and $\nu_i \neq 0$ for each odd integer i $(1 \leq i \leq d)$.

Proof. By the definition of f_i and (3.1), we easily have the assertion for f_i. In order to prove that $f_1, \dots, f_d$ are linearly independent, let $\sum a_i f_i \equiv 0$. Since $f_i^{(j)}(0) = 0$ $(j < 1)$ and $f_i^{(i)}(0) = \lambda_1 \cdots \lambda_{i-1}$ for each i because of (2.1), we have inductively $a_1 = \cdots = a_d = 0$. Thus we easily see that $\alpha_1, \dots, \alpha_{[d/2]}$ are all distinct and that $\nu_i \neq 0$ for some i. Using (2.1), we have

$$\lambda_{i-1}\nu_{i-1} - \lambda_i \nu_{i+1} = 0 \quad (i\text{: even} \leq d-1)$$

if d is odd. It follows that $\nu_i \neq 0$ for all odd integers i $(1 \leq i \leq d)$. □

Let $f: M \to E^m$ be a helical imbedding of order d. Assume that d is odd. Let γ be a unit speed geodesic in M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$.

LEMMA 3.2. *If J_V is a Jacobi field along γ such that $J_V(0) = 0$ and $\nabla_X J_V = V \in \{X\}^{\perp} \cap U_x M$, then we have $\lim_{s \to +\infty} \|J_V\| = +\infty$.*

Proof. In the proof of Theorem 2.4, we have shown that $J_V(s) \equiv f_1(s)V \mod N_x M$. It follows that $\|J_V(s)\| \geq |f_1(s)|$. Furthermore, Lemma 3.1 implies that $\lim_{s \to +\infty} |f_1(s)| = +\infty$. □

THEOREM 3.3. *If $f: M \to E^m$ is a helical imbedding of odd order of a connected complete Riemannian manifold M into E^m, then M is isometric to a Euclidean space E^n and f is totally geodesic.*

Proof. At first, we note that M is noncompact because $\delta(x, y) \geq \|f(x)-f(y)\|$ for every $x, y \in M$, and hence $\delta(\gamma(0), \gamma(s))^2 \geq \|\gamma(0)-\gamma(s)\|^2 = G(s) \to +\infty$ as $s \to +\infty$. Therefore, by Theorem 2.4, M is diffeomorphic to E^n and has no conjugate points.

We shall prove

$$\lim_{s \to +\infty} \|H(\dot\gamma, V^*)\| = 0, \tag{3.2}$$

where $V^* = J_V/\|J_V\|$. By Gauss equation (1.1), we have

$$\begin{aligned}\bar\nabla_{\dot\gamma} J_V &= \nabla_{\dot\gamma} J_V + H(\dot\gamma, J_V) \\ &= \nabla_{\dot\gamma} J_V + H(\dot\gamma, V^*)\|J_V\|.\end{aligned}$$

We see from (2.3) that J_V is a linear combination of $f_1, \dots, f_d$ whose coefficients are constant vectors at x. Taking account of Lemma 3.1, the length of $\bar\nabla_{\dot\gamma} J_V$ $(= d/ds\, J_V)$ is bounded. So $\|H(\dot\gamma, V^*)\|\,\|J_V\|$ is bounded. We conclude, from Lemma 3.2, equation (3.2).

We next prove

$$\lim_{s \to +\infty} \langle H(\dot\gamma, \dot\gamma), H(V^*, V^*)\rangle = 0. \tag{3.3}$$

Using Gauss and Weingarten equations (1.1) and (1.2), we find

$$\begin{aligned}\frac{d^2}{ds^2} J_V &= \bar\nabla_{\dot\gamma}^2 J_V \\ &= \nabla_{\dot\gamma}^2 J_V + 2H(\dot\gamma, \nabla_{\dot\gamma} J_V) - A_{H(\dot\gamma, J_V)}\dot\gamma + (DH)(\dot\gamma, \dot\gamma, J_V).\end{aligned}$$

Since J_V is a Jacobi field, we have $\nabla_{\dot\gamma}^2 J_V = R(\dot\gamma, J_V)\dot\gamma$. It follows from Gauss' structure equation (1.3) that

$$\frac{d^2}{ds^2} J_V = -A_{H(\dot\gamma,\dot\gamma)} J_V + 2H(\dot\gamma, \nabla_{\dot\gamma} J_V) + (DH)(\dot\gamma, \dot\gamma, J_V).$$

Thus, by Lemma 3.1,

$$\infty > \left\|\frac{d^2}{ds^2} J_V\right\| \geq \|A_{H(\dot\gamma,\dot\gamma)} J_V\| = \|A_{H(\dot\gamma,\dot\gamma)} V^*\|\,\|J_V\|.$$

Using Lemma 3.2, we obtain

$$\lim_{s \to +\infty} \|A_{H(\dot\gamma,\dot\gamma)} V^*\| = 0.$$

Since

$$\begin{aligned}|\langle H(\dot\gamma, \dot\gamma), H(V^*, V^*)\rangle| &= |\langle A_{H(\dot\gamma,\dot\gamma)} V^*, V^*\rangle| \\ &\leq \|A_{H(\dot\gamma,\dot\gamma)} V^*\|,\end{aligned}$$

we conclude (3.3).

The helical imbedding f is λ_1-constant isotropic, and hence

$$\langle H(\dot\gamma,\dot\gamma), H(V^*,V^*)\rangle + 2\|H(\dot\gamma,V^*)\|^2 = \lambda_1^2$$

because of (1.6). Applying (3.2) and (3.3) to this equation, we obtain $\lambda_1 = 0$ which shows that f is totally geodesic. □

We can say that a connected complete Riemannian manifold does not admit a helical immersion into a Euclidean space such that the order is odd and greater than three.

4. Helical imbeddings of even order. As before, let $f: M \to E^m$ be a helical imbedding of order d. The main purpose of this section is to prove that d is even if and only if M is compact. In the preceding section, we have shown the if part. Thus we assume that d is even and M is a connected complete noncompact Riemannian manifold. So M has no conjugate points and every geodesic in M is a minimizing one in virtue of Theorem 2.4.

LEMMA 4.1. *There exists a divergent sequence* $\{s_k\}_{k=1}^\infty$ *such that*

$$\lim_{k\to\infty} f_i(s_k) = 0 \quad \textit{for } i = 1, 2, \dots, d.$$

Proof. Let γ be a unit speed geodesic in M and $x = \gamma(0)$. Put $x_k = \gamma(k)$ ($k \in \mathbf{Z}_+$). Since G is bounded (Lemma 3.1), we see from (2.2) that $f(x_k)$ is bounded. Therefore a subsequence $\{f(y_k)\}$ of $\{f(x_k)\}$ converges. Put $t_k = \delta(y_k, x)$. Then

$$\lim_{k\to\infty} G(t_k - t_{k-1}) = \lim_{k\to\infty} \|f(y_k) - f(y_{k-1})\|^2 = 0.$$

Define a sequence $\{u_k\}$ by $u_k = t_k - t_{k-1}$ (≥ 1) for every $k \in \mathbf{Z}_+$. If $\{u_k\}$ is bounded, then a subsequence $\{u_k'\}$ of $\{u_k\}$ converges. For this subsequence $\{u_k'\}$ we have

$$G(\lim_{k\to\infty} u_k') = \lim_{k\to\infty} G(u_k') = 0, \qquad \lim_{k\to\infty} u_k' \neq 0,$$

which contradicts the assumption that f is an imbedding. Thus the sequence $\{u_k\}$ has a subsequence $\{s_k\}$ which diverges and satisfies $\lim_{k\to\infty} G(s_k) = 0$. Noting that $G = \sum_{i=1}^d f_i^2$, we obtain the assertion. □

Let γ be a unit speed geodesic in M such that $\gamma(0) = x$ and $\dot\gamma(0) = X$. Since M has no conjugate points, Jacobi fields $\{J_V: J_V(0) = 0, \nabla_X J_V = V \in \{X\}^\perp\}$ along γ span the subspace $\{\dot\gamma(s)\}^\perp$ of $T_{\gamma(s)}M$ at each point $\gamma(s)$ ($s \neq 0$). Let J_V^* be the Jacobi field along γ satisfying $J_V^*(0) = V \in \{X\}^\perp$ and $\nabla_X J_V^* = 0$. There exists a symmetric transformation $S_X(s)$ acting on $\{X\}^\perp$ such that $J_V^*(s) = J_{S_X(s)V}(s)$ for each $s \in \mathbf{R} - \{0\}$. Clearly $S_X(s)$ is smooth with respect to s. The Jacobi field J_V^* is induced from a variation of geodesics $(s,\theta) \to \exp_{\beta(\theta)} sX^*(\theta)$, where $\beta(\theta)$ is a curve in M which satisfies $\beta(0) = x$ and $\dot\beta(0) = V$ and where $X^*(\theta)$ is the parallel vector field along β such that $X^*(0) = X$. Making use of (1.1), (1.2), and Lemma 2.2, we have

$$J_V^*(s) \equiv V - A_{\xi(s;X)}V - A_{\zeta(s;X)}V \tag{4.1}$$

mod N_xM (cf. [15, Theorem 2.1]). It follows from (2.3) and (4.1) that

$$S_X(s)=\frac{1}{f_1(s)}\{I-A_{\xi(s;X)}-A_{\zeta(s;X)}\} \tag{4.2}$$

for each $s\in\mathbf{R}-\{0\}$, where we note that $A_{\xi(s;X)}$ and $A_{\zeta(s;X)}$ leave $\{X\}^\perp$ invariant (cf. [13, Lemma 3.3, p. 68]). Let g_s denote the Riemannian metric induced on the unit tangent sphere U_xM by the map $U_xM\to$ (geodesic sphere with center x and radius s) sending V to $\exp_x sV$. By using the same argument as in the proof of [14, Proposition 2.3, p. 200] or [16, Lemma 3.3], we have the following.

LEMMA 4.2. *The derivative $S_X'(s)$ of $S_X(s)$ satisfies*

$$g_s(S_X'(s)V, W)=-\langle V, W\rangle$$

for every $V, W\in\{X\}^\perp$ and $s\in\mathbf{R}_+$.

The following is a key lemma.

LEMMA 4.3. *There exists a (unique) $u_0\in\mathbf{R}_+$ such that $\langle S_X(u_0)V, V\rangle=0$ for each $X\in U_xM$ and V $(\neq 0)$ $\in\{X\}^\perp$.*

Proof. Consider a function $s\in\mathbf{R}_+\mapsto\langle S_X(s)V, V\rangle$. This function is monotone decreasing because of Lemma 4.2. Furthermore, we have $\xi(0;X)=\zeta(0;X)=0$ and $\lim_{s\to+0}f_1(s)=+0$ since $f_1'(0)=1$. Thus we see that

$$\lim_{s\to+0}\langle S_X(s)V, V\rangle=+\infty.$$

On the other hand, we know from Lemma 4.1 that

$$\lim_{k\to+\infty}f_1(s_k)=0 \quad\text{and}\quad \lim_{k\to+\infty}\xi(s_k;X)=\lim_{k\to+\infty}\zeta(s_k;X)=0$$

for some divergent sequence $\{s_k\}$. Since $\langle S_X(s)V, V\rangle$ is monotone decreasing, (4.2) shows that

$$\lim_{s\to+\infty}\langle S_X(s)V, V\rangle=-\infty.$$

Thus we have proved that there exists a unique $u_0\in\mathbf{R}_+$ such that $\langle S_X(u_0)V, V\rangle=0$. □

REMARK. We can explain Lemma 4.3 geometrically as follows. Let β be a unit speed geodesic such that $\beta(0)=x$ and $\dot\beta(0)=V\in U_xM\cap\{X\}^\perp$. Consider Jacobi field $J=J_V^*-J_{S_X(u_0)V}$ along γ, where u_0 is taken as in Lemma 4.3. This Jacobi field satisfies $J(u_0)=0$, $J(0)=V$, and $\nabla_X J=-S_X(u_0)V$. Since $\langle S_X(u_0)V, V\rangle=0$, we have $\nabla_X J\in\{V\}^\perp$. Therefore $\gamma(u_0)$ is a focal point of β along γ. Conversely if $\gamma(u_0)$ is a focal point of β along γ, then there is a Jacobi field J such that $J(0)=V$, $J(u_0)=0$, and $\nabla_X J\in\{V\}^\perp$. Let $J=J_V^*+J_W$, where $W=\nabla_X J$. Since $J(u_0)=0$, we obtain $f_1(u_0)\{S_X(u_0)V+W\}=0$. Thus if $f_1(u_0)\neq 0$ then $W=-S_X(u_0)V$, and hence $\langle S_X(u_0)V, V\rangle=0$.

By using Lemma 4.3, we show (cf. [5]) the following.

THEOREM 4.4. *Let $f: M\to E^m$ be a helical imbedding of a connected complete Riemannian manifold M into a Euclidean space E^m. If the order of f is even, then M must be compact (and hence a Blaschke manifold).*

Proof. Assume that M is noncompact. Let $\gamma, X \in U_xM$ and $V \in \{X\}^{\perp}$ ($V \neq 0$) as before. By Lemma 4.3, there exist $u_0, u_1 \in \mathbf{R}_+$ such that $\langle S_X(u_0)V, V\rangle = \langle S_{-X}(u_1)V, V\rangle = 0$. Consider a broken Jacobi field

$$\mathcal{J}(s) = \begin{cases} J_V^* - J_{S_X(u_0)V} & \text{if } 0 \leq s \leq u_0, \\ J_V^* - J_{S_X(-u_1)V} & \text{if } -u_1 \leq s \leq 0, \end{cases}$$

along γ. The Jacobi field $\mathcal{J}$ satisfies $\mathcal{J}(0) = V$, $\mathcal{J}(u_0) = \mathcal{J}(-u_1) = 0$, $\mathcal{J}'_+(0) = -S_X(u_0)V$, and $\mathcal{J}'_-(0) = -S_X(-u_1)V$, where $\mathcal{J}'_+$ (resp. $\mathcal{J}'_-$) denotes the right (resp. left) limit of the covariant derivatives of $\mathcal{J}$ with respect to $\dot{\gamma}$. Let I be the index form defined on all piecewise smooth vector fields along γ which vanish at $\gamma(u_0)$ and $\gamma(-u_1)$. Since there is no conjugate point of $\gamma(-u_1)$ along γ, the index form is positive definite (cf. [2]). However, we have

$$\begin{aligned} I(\mathcal{J}, \mathcal{J}) &= \langle -S_X(-u_1)V + S_X(u_0)V, V\rangle \\ &= \langle S_{-X}(u_1)V, V\rangle \\ &= 0, \end{aligned}$$

which is a contradiction. □

In [13], the author showed that if $f: M \to S(1)$ is a helical immersion of a connected complete Riemannian manifold M into a unit sphere $S(1)$, then $\iota \circ f: M \to E$ is a helical immersion of even order, where $\iota: S(1) \to E$ is the inclusion. We therefore have the following.

COROLLARY 4.5. *Every connected complete noncompact Riemannian manifold does not admit a helical immersion into a sphere.*

Moreover, we obtain the following from Theorems 3.3 and 4.4.

COROLLARY 4.6. *Let M be a connected complete submanifold with geodesic normal sections in E^m. If M is noncompact, then it is a totally geodesic submanifold.*

Now we explain helical immersions into a sphere which were given by Tsukada [18]. Let M be a compact rank one symmetric space. Let V_k be the kth eigenspace of the Laplace operator on M and let $\dim V_k = m(k)+1$. We define an inner product $\langle\ ,\ \rangle$ on V_k by $\langle \phi, \psi\rangle = \int_M \phi\psi\, dx$, where dx denotes the canonical measure of M. Taking an orthonormal base $\{\phi_0, \dots, \phi_{m(k)}\}$ in V_k, we define a map $\Phi_k: M \to E^{m(k)+1}$ via $\Phi_k(x) = (\phi_0(x), \dots, \phi_{m(k)}(x))$. Then, under a suitable homothety on M, Φ_k becomes an isometric immersion. Furthermore, it is verified that $\Phi_k(M)$ is contained in a hypersphere $S^{m(k)}$ in $E^{m(k)+1}$ and that $\Phi_k: M \to S^{m(k)}$ is minimal and helical. The isometric immersion Φ_k is called the kth *standard minimal immersion* into $S^{m(k)}$ (cf. [1], [20]). Tsukada defined a helical immersion $\Phi_{k_1,\dots,k_r}$ of M into $S^{m(k_1)+\cdots+m(k_r)+r-1}$ by

$$\begin{aligned} \Phi_{k_1,\dots,k_r}(x) &= (c_1\Phi_{k_1}(x), \dots, c_r\Phi_{k_r}(x)) \\ &\in \mathbf{R}^{m(k_1)+\cdots+m(k_r)+r}, \quad c_1, \dots, c_r > 0. \end{aligned}$$

In Corollary 3.5 ([18, p. 281]) he showed the following.

THEOREM T. *Let $f: M \to S$ be a helical immersion of a compact rank one symmetric space into a sphere. Assume that f is full. Then there exist nonnegative integers $k_1, \ldots, k_r$ such that f is equivalent to $\Phi_{k_1,\ldots,k_r}$, where $k_1, \ldots, k_r$ are distinct and may contain zero (when $k=0$, Φ_k is considered as a nonzero constant map).*

We shall apply Theorem T to a helical imbedding $f: M \to E^m$ of a compact Riemannian manifold M whose dimension is two or odd.

THEOREM 4.7. *Let $f: M \to E^m$ be a helical imbedding. Suppose that M is compact and that* $\dim M = 2$ *or odd integer. Then M is isometric to a sphere or real projective space and f is equivalent to $\iota \circ \Phi_{k_1,\ldots,k_r}$, where $k_1, \ldots, k_r$ are certain nonnegative integers and $\iota: S \to E^m$ is the inclusion map.*

Proof. By Theorem 2.4 and Berger's theorem [1, p. 236], we see that M is isometric to a sphere or real projective space (see also Theorem 7.23 [1, p. 186]). Thus we have only to prove that f is a helical immersion into a hypersphere S of E^m. Let C_f be the centroid of f:

$$C_f = \frac{1}{\mathrm{vol}(M)} \int_M f\,dx,$$

which will become the center of S. Put $r^2(x) = \|f(x) - C_f\|^2$ for $x \in M$. In order to prove that r^2 is constant, we compute $V \cdot r^2$ for any $V \in T_x M$. We have

$$\begin{aligned} \frac{1}{2} V \cdot r^2 &= \langle V, f(x) - C_f \rangle \\ &= \frac{1}{\mathrm{vol}(M)} \left\langle V, \int_M (f(x) - f(y))\,dy \right\rangle \\ &= \frac{1}{\mathrm{vol}(M)} \int_M \langle V, f(x) - f(y) \rangle\,dy. \end{aligned}$$

Let L, dX, and $\theta(s, X)$ be (respectively) the diameter of M, the canonical measure on $U_x M$, and the value of $\sqrt{\det g_s}$ at $X \in U_x M$. Then we obtain

$$\frac{1}{2} \mathrm{vol}(M) V \cdot r^2 = -\int_0^L \int_{U_x M} f_1(s) \langle V, X \rangle \theta(s, X)\,ds\,dX, \tag{4.3}$$

where we have used Lemma 2.2. Since M is isometric to a sphere or real projective space, $\theta(s, X)$ is independent of $X \in U_x M$. Thus the right-hand side of (4.3) vanishes. We have proved that $V \cdot r^2 = 0$ for every $V \in T_x M$, and hence r^2 is constant. ❑

COROLLARY 4.8. *Let M be a compact submanifold with geodesic normal sections in E^m. If* $\dim = 2$ *or odd, then we have the same conclusion for the inclusion map $M \to E^m$ as Theorem* 4.7.

Corollaries 4.6 and 4.8 generalize results obtained in [3], [4], and [19] (see, e.g., [19, Corollary 3]).

REMARK. Chen and Verheyen conjectured that each submanifold with geodesic normal sections in E^m is an open part of an n-plane of E^m or is contained in a hypersphere of E^m. Perhaps C_f will be the center of such sphere. For instance, if M is a D'Atri space (i.e., $\theta(s,X)=\theta(s,-X)$), then the right-hand side of (4.3) vanishes. However, it seems difficult to show that if $f: M \to E^m$ is a helical immersion then M is a D'Atri space.

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