STABILIZING SURFACE SYMMETRIES

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This paper concerns free, orientation preserving actions of finite groups on closed, connected, oriented surfaces. Given such a group G and surface M, such an action consists by definition of an injective homomorphism $\phi: G \to \mathrm{Diff}_+(M)$, where $\mathrm{Diff}_+(M)$ is the group of orientation preserving diffeomorphisms of M, and $\phi(g)$ is fixed point free for all $g \in G$, other than the identity. Two such actions, ϕ_1 and ϕ_2, of G on surfaces M_1 and M_2 are called *equivalent* if there is an orientation preserving diffeomorphism $h: M_1 \to M_2$ such that $h \circ \phi_1(g) \circ h^{-1} = \phi_2(g)$ for all $g \in G$.

A general motivating question concerning such actions is, for fixed group, to determine all possible equivalence classes of actions of that group on a given surface. This has been done for cyclic groups by Nielsen [3] and for abelian and metacyclic groups by Edmonds [1; 2]. The results in those cases essentially state that two actions are equivalent if they are freely bordant. We note here that the free bordism group, $\Omega_2^{\mathrm{free}}(G)$, is isomorphic to $H_2(G, \mathbf{Z})$. For further results including the nonorientable case see [4; 5].

As an example, the $\mathbf{Z}_5$ actions on $S^1 \times S^1$ generated by the maps $(x, y) \to (x, \omega^i y)$, where ω is a primitive fifth root of unity and i is 1 or 2, are equivalent. Finding the equivalence h is an enlightening exercise.

Given an action of G on a surface M there is a natural way to stabilize the action to an action on $M \#_k T^2$, where $k = \mathrm{order}(G)$, and $M \#_k T^2$ denotes the connected sum of M with k copies of $T^2 = S^1 \times S^1$. Essentially we let G freely permute the added tori. This will be defined precisely in Section 1. Two actions are called *stably equivalent* if upon repeated stabilization they become equivalent. The main result of this paper is the following:

THEOREM. Let ϕ_1 *and* ϕ_2 *be free, orientation preserving actions on connected closed oriented surfaces* M_1 *and* M_2. *(*M_1 *and* M_2 *need not be homeomorphic.) Then* ϕ_1 *and* ϕ_2 *are stably equivalent if they are freely bordant.*

The converse is trivially true, as any action is bordant to its stabilization, and equivalent actions are bordant.

Whether or not stable equivalence implies equivalence for group actions is an open question. As noted earlier, such an implication holds in the case of cyclic, abelian, and metacyclic groups. In addition, it has been verified for a variety of other groups.

An outline of the paper is as follows. In Section 1 preliminary material is presented. Section 2 contains a proof of the main theorem. In the final section extensions of the main result to the unoriented or nonfree setting are summarized.

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I would like to thank Allan Edmonds for pointing out the problem of stable equivalence of group actions to me. In addition, the observations concerning the extension of the main theorem to the nonfree setting which are described in Section 3 are essentially his.

1. Preliminaries. Throughout this paper G represents a fixed finite group. Manifolds are always smooth, compact, and oriented. Surfaces are closed. Group actions are always free.

This section begins with a precise definition of stabilization. Following that, it is shown how covering space theory yields a reformulation of statements concerning group actions into statements concerning representations of surface groups into G. The section concludes with the proof of a key lemma.

Stabilization. Let ϕ be an action of G on M. It is possible to find an embedding of $D^2 \times G$ (where G has the discrete topology) into M such that $\phi(g)(x, g_i) = (x, gg_i)$, $(x, g_i) \in D^2 \times G \subseteq M$. To find such an embedding consider the orbit of a regular neighborhood of a point on M.

Let F be a punctured torus; that is, $T^2 - \text{int}(D^2)$, where D^2 is an embedded disk on T^2. There is an action ψ of G on $F \times G$ as above. As the actions of ϕ and ψ are equivalent when restricted to $\partial(M - \text{int}(D^2 \times G))$ and $-\partial(F \times G)$, they combine to give an action ϕ_s of G on $(M - \text{int}(D^2 \times G)) \cup_h (F \times G)$, where $h: \partial(F \times G) \to -\partial(D^2 \times G)$ is that equivalence. Note that the resulting space is diffeomorphic to $M \#_k T^2$.

The action ϕ_s is well defined up to equivalence.

Covering spaces. Dropping the restriction that M be 2-dimensional temporarily, note that the free action ϕ of G on M determines a quotient space $\bar{M}$ of which M is a covering space. Fixing a base point in $\bar{M}, x_0$, that covering space is determined by a representation $\bar{\phi}: \pi_1(\bar{M}, x_0) \to G$. As M is connected, $\bar{\phi}$ is surjective.

Conversely, a surjective homomorphism $\bar{\phi}: \pi_1(\bar{M}, x_0) \to G$ determines a G action, ϕ, on the associated connected covering space. The choice of base point does not affect the equivalence class of the action on the covering space. More generally, any diffeomorphism h of $\bar{M}$ determines a new representation $\bar{\phi} \circ h_*$. The associated group actions are equivalent. Except where explicitly needed, we will drop references to base points.

We now proceed to interpret stabilization in this setting. Given a representation $\bar{\phi}: \pi_1(\bar{M}) \to G$, we wish to describe an associated representation $\bar{\phi}_s: \pi_1(M \# T^2) \to G$, such that $\bar{\phi}_s$ and ϕ_s correspond under the correspondence described in the previous paragraph.

First we describe the connected sum. Let D be an embedded disk on $\bar{M}$, and D_1 and D_2 be disjoint embedded disks in the interior of D. $\bar{M} \# T^2 \simeq (\bar{M} - \text{int}(D_1 \cup D_2)) \cup S^1 \times I$.

Generators of $\pi_1(\bar{M} \# T^2)$ can be described as follows. Pick a basepoint x_0 for $\bar{M}$ on ∂D. Pick a second point x_1 on ∂D_1 and an arc α_1 on $D - \text{int}(D_1 \cup D_2)$ from x_0 to x_1. Similarly, pick a point x_2 and arc α_2 using D_2. Pick an embedded arc α on $S^1 \times I$ running from x_2 to x_1. Finally, let β be the path running around ∂D_1, based at x_1. Define $m = \alpha_1 * \beta * \alpha_1^{-1}$ and $l = \alpha_2 * \alpha * \alpha_1^{-1}$. A set of paths $\{m_i, l_i\}_{i=1,\dots,n}$ in $\bar{M} - \operatorname{int}(D)$, with $n = \operatorname{genus}(\bar{M})$ can be chosen to represent the standard generators of $\pi_1(\bar{M})$. The collection $\{m_i, l_i\}_{i=1,\dots,n} \cup \{m, l\}$ represents a standard generating set for $\pi_1(\bar{M} \# T^2)$.

To define $\bar{\phi}_s$, proceed as follows. Let $g \in G$ be some fixed element of G. Define $\bar{\phi}_{s,g}$ by $\bar{\phi}_{s,g}(m_i) = \bar{\phi}(m_i)$, $\bar{\phi}_{s,g}(l_i) = \bar{\phi}(l_i)$, $\bar{\phi}_{s,g}(m) = 1$, and $\bar{\phi}_{s,g}(l) = g$. $\bar{\phi}_{s,g}$ is well defined, since $\bar{\phi}_{s,g}(\prod_{i=1}^n [m_i, l_i]) = 1$ and $\bar{\phi}_{s,g}([m, l]) = 1$. Hence $\bar{\phi}_{s,g}$ determines a well-defined homomorphism from $\pi_1(\bar{M} \# T^2)$ since it vanishes on $(\prod_{i=1}^n [m_i, l_i])([m, l])$.

Set $\bar{\phi}_s = \bar{\phi}_{s,1}$. $\bar{\phi}_s$ corresponds to ϕ_s via the representation, group action relationship described above.

MAIN LEMMA. *If $\bar{\phi}: \pi_1(\bar{M}) \to G$ is surjective and $g \in G$, then there is a diffeomorphism $h: \bar{M} \# T^2 \to \bar{M} \# T^2$ such that $\bar{\phi}_{s,g} \circ h_* = \bar{\phi}_s$.*

Proof. Pick a smooth closed path γ based at x_1, contained in $\bar{M} - \operatorname{int}(D_1 \cup D_2)$, such that $\bar{\phi}(\alpha_1 * \gamma * \alpha_1^{-1}) = g^{-1}$. Note that this implies $\bar{\phi}_{s,g}(\alpha_1 * \gamma * \alpha_1^{-1}) = g^{-1}$ as well. Viewing γ as is an isotopy of maps of a point into $\bar{M}$, we can use the isotopy extension theorem to construct a family of diffeomorphisms h_t of $\bar{M}$ to itself such that $h_t(x_1) = \gamma(t)$. With a little care it can be arranged that $h_t \mid_{D_2} = \mathrm{id}$ for all t, $h_0 = \mathrm{id}$, $h_1 \mid_{D_1} = \mathrm{id}$, and $h_t(x_0) = x_0$ for all t. (For that last point it is necessary to arrange that γ misses x_0.)

Define $F_t = (\bar{M} - \operatorname{int}(h_t(D_1 \cup D_2))) \cup S^1 \times I$. Since $h_1(D_1 \cup D_2) = D_1 \cup D_2$, F_0 and F_1 are identical surfaces. More precisely, there is a diffeomorphism from F_0 to F_1 which is the identity on $\bar{M} - (D_1 \cup D_2)$. We use that diffeomorphism to identify each with $\bar{M} \# T^2$.

On the other hand, for all t, h_t can be used to define a diffeomorphism of F_0 to F_t. Hence $h_1: F_0 \to F_1$ is a diffeomorphism, which by the above identification can be viewed as a self diffeomorphism of $\bar{M} \# T^2$. h_1 is the desired diffeomorphism. It remains to analyze $\bar{\phi}_{s,g} \circ h_{1*}$.

Define a 2-complex $C_t = F_t \cup h_t(D_1 \cup D_2)$. The identification of F_0 with F_1 extends to one of C_0 with C_1. h_t extends to a homeomorphism $j_t: C_0 \to C_t$. As $\bar{\phi}_{s,g}$ and $\bar{\phi}_s$ vanish on m, they factor through representations $\bar{\psi}_{s,g}$ and $\bar{\psi}_s$ of $\pi_1(C_0) \to G$. We have reduced the problem to one of showing that $\bar{\psi}_{s,g} \circ j_{1*} = \bar{\psi}_s$.

The relationship $\bar{\phi}_{s,g} \circ j_{1*} = \bar{\psi}_s$ follows from the following observations: $j_1(m_i)$ is homotopic (fixing x_0) to m_i on C_0. Similarly, $j_1(l_i)$ is homotopic to l_i on C_0, again fixing x_0. $j_1(m)$ is null homotopic on C_0, as is m. Finally, $j_1(l) = l * \alpha_1 * \gamma * \alpha_1^{-1}$. From the last observation, $\bar{\psi}_{s,g} \circ j_{1*}(l) = g \circ g^{-1} = 1 = \bar{\psi}_s(l)$.

2. Proof of Theorem. Let ϕ_1 and ϕ_2 be freely bordant actions of G on surfaces M_1 and M_2. By definition, this means that there is a 3-manifold W with a free action ϕ of G and a diffeomorphism $h: M_1 \amalg -M_2 \to \partial W$ such that $\phi(g) \circ h \mid_{M_i} = h \circ \phi_i(g)$, for $i = 1$ and 2. We denote $h(M_1) = \partial_+(W)$ and $h(-M_2) = \partial_-(W)$. It is sufficient to prove that $\phi \mid_{\partial_+ W}$ and $\phi \mid_{-\partial_- W}$ are stably equivalent.

As in the previous section, denote by $\bar{W}$ the quotient of W under ϕ. Denote the two components of $\partial(\bar{W})$ by $\partial_+\bar{W}$ and $\partial_-\bar{W}$. $\bar{\phi}$ denotes the associated representation of $\pi_1(\bar{W})$ to G. It induces representations $\bar{\phi}_+$ and $\bar{\phi}_-$ of $\pi_1(\partial_+\bar{W})$ and $\pi_1(\partial_-\bar{W})$ to G. It remains to show that $\bar{\phi}_+$ and $\bar{\phi}_-$ are stably equivalent.

Using a standard handlebody or Morse theory argument, we can find disjoint embedded surfaces in $\bar{W}$, $\partial_+\bar{W}=F_1, F_2, \dots, F_k$ and $G_m, G_{m-1}, \dots, G_1=\partial_-\bar{W}$ with the following properties: (1) $F_k=-G_m$; (2) F_i and $-F_{i+1}$ cobound a region in $\bar{W}$ which contains no other F_i or G_i, and which is diffeomorphic to $F_i\times I\cup$(1-handle); and (3) G_i and $-G_{i+1}$ similarly cobound a region in $\bar{W}$ containing no other F_i or G_i, and which is diffeomorphic to $G_i\times I\cup$(1-handle).

If we can show that $\bar{\phi}\,|_{F_{i+1}}$ is stably equivalent to $\bar{\phi}\,|_{F_i}$, and similarly for $\bar{\phi}\,|_{G_{i+1}}$ and $\bar{\phi}\,|_{G_i}$ we will be finished. This would show that $\bar{\phi}\,|_{\partial_+\bar{W}}$ is stably equivalent to $\bar{\phi}\,|_{F_k}$, and that $\bar{\phi}\,|_{\partial_-\bar{W}}$ is stably equivalent to $\bar{\phi}\,|_{G_m}$. As $F_k=-G_m$, it follows that $\bar{\phi}_{\partial_+\bar{W}}$ is stably equivalent $\bar{\phi}_{-\partial_-\bar{W}}$.

Notice that F_{i+1} is constructed from F_i by forming a connected sum with a torus, as in Section 1. We claim that $\bar{\phi}\,|_{F_{i+1}}$ is constructed from $\bar{\phi}\,|_{F_i}$ in the manner required to apply the lemma of the last section. The only observation that is not immediate here is that $\bar{\phi}\,|_{F_{i+1}}(m)=1$. This follows from the fact that m is null homotopic in $\bar{W}$, as it bounds the cocore of a 1-handle.

A similar argument applied to G_i and G_{i+1} completes the proof. □

3. Nonorientable and nonfree actions. In this section we discuss the effect of dropping the orientability assumption, or the restriction to free actions, on the stability result already obtained.

Nonorientable actions. The direct generalization of the main theorem to an unoriented setting would be the following: If ϕ_1 and ϕ_2 are free actions on connected, closed surfaces M_1 and M_2, then ϕ_1 and ϕ_2 are stably equivalent if they are freely bordant. Here bordism and equivalence would be defined without reference to orientation.

As stated, this result clearly is false: If M_1 and M_2 happen to be orientable, and ϕ_1 and ϕ_2 preserve some orientation, then ϕ_1 and ϕ_2 will be stably equivalent if and only if the classes represented by ϕ_1 and ϕ_2 in $\Omega_2^{\text{free}}(G)$, $[\phi_1]$ and $[\phi_2]$, satisfy $[\phi_1]=\pm[\phi_2]$. Using the fact that the unoriented bordism group $\mathfrak{N}_2^{\text{free}}(G)$ is isomorphic to $H_2(G,\mathbf{Z}_2)$, whereas $\Omega_2^{\text{free}}(G)\simeq H_2(G,\mathbf{Z})$, an example of unoriented (but orientable) actions which are not freely bordant via an orientable manifold, but which are freely bordant via a nonorientable manifold is readily constructed.

If one widens the notion of stability however, the generalization becomes valid. The correct notion of stabilization in the unoriented setting is to admit stabilizations which involve forming the connected sum with Klein bottles as well as with tori. Making this notion formal, and proving the generalized theorem are straightforward, following the exact lines of the previous work.

We conclude this section with the observation that in certain cases one need not widen the notion of stabilization to prove the unoriented version of the theorem. The first of these is for the case of G actions on a nonorientable surface M. The proof here depends on the observation that for some orientation reversing curve $\gamma \subseteq \bar{M}$, $\bar{\phi}(\gamma)=1\in G$ (just let γ be the projection of an orientation reversing curve on M). The second case is if two actions of G on orientable surfaces are bordant via an orientable 3-manifold, W. In this case one has information about $\bar{W}$ coming from the fact that W is orientable. Details are left to the reader.

Nonfree actions. The reader is referred to [1; 2] for the definitions and details concerning fixed point data. If ϕ is an action of a group G on a surface M, the singular set S of ϕ is the set of points on M with nontrivial isotropy subgroups. S is a finite set, a fact which depends on the compactness of M and on ϕ being orientation preserving. Associated to each orbit of a singular point in S is the conjugacy class of an element in G. These conjugacy classes form the fixed point data.

The hoped-for generalization of the main theorem would be the following: If ϕ_1 and ϕ_2 are two (not necessarily free) orientation preserving actions of G on closed, connected, oriented surfaces M_1 and M_2, then ϕ_1 and ϕ_2 are stably equivalent if (and only if) they are bordant and have the same fixed point data. This generalization is false. We will first discuss cases in which it is true, and then conclude with a counterexample.

Let W be a bordism between M_1 and M_2, and ϕ be an action of G on W providing a bordism form ϕ_1 to ϕ_2.

Our first observation is that if the singular set consists only of arcs, with each arc joining a singular point in M_1 to a singular point in M_2, the generalization holds. In this case the proof is essentially the same as before. The only change is that one uses a relative handlebody decomposition of $\bar{W}-$(projection of singular set).

A second observation is that the assumption that the singular set of ϕ consists only of arcs is sufficient to prove the generalization, no assumptions about the endpoints of the arcs are required. In this case one can perform equivariant surgery on W to arrive at a situation as in the previous paragraph.

To conclude we show by means of an example that the generalization mentioned above cannot hold for all actions. We do this by constructing two free actions (which hence have the same fixed point data) which are bordant, but not freely bordant. It follows that these actions are not stably equivalent.

Figure 1 illustrates two disjoint solid handlebodies, H_1 and H_2, in S^3. A curve C in $S^3-(H_1\cup H_2)$ is also illustrated. $\pi_1(S^3-(H_1\cup H_2\cup C))$ is normally generated by the curves γ_1, γ_2, δ_1, δ_2 and β illustrated. There is a representation ρ of $\pi_1(S^3-(H_1\cup H_2\cup C))$ to $\mathbf{Z}_2\times\mathbf{Z}_2$ defined by:

$$\rho(\gamma_1)=(1,0)\quad \rho(\gamma_2)=(0,1)$$
$$\rho(\delta_1)=(0,1)\quad \rho(\delta_2)=(0,0)$$
$$\rho(\beta)=(1,0).$$

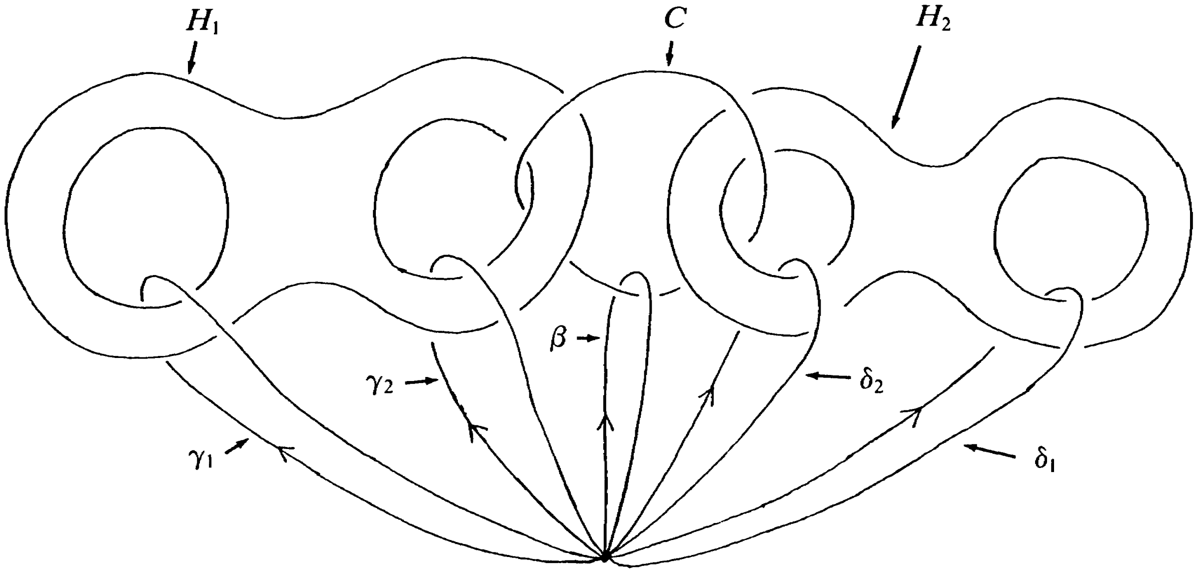

Figure 1.

This representation induces a $\mathbf{Z}_2 \times \mathbf{Z}_2$ branched cover of $S^3-(H_1 \cup H_2)$, branched over C, which we denote by W. The boundary of W consists of two genus 5 surfaces, each with a $\mathbf{Z}_2 \times \mathbf{Z}_2$ action. W provides a bordism between the surfaces, the covering translation provides a bordism between the actions.

Finally, to show that the actions are not freely bordant, we compute what each represents in $H_2(\mathbf{Z}_2 \times \mathbf{Z}_2; \mathbf{Z}) = \mathbf{Z}_2$. Let m_1, l_1, m_2, l_2 be a standard basis for $\pi_1(F_2)$ (F_2 a surface of genus 2). The action on the cover of the boundary of H_1 corresponds to the representation $\rho_1: \pi_1(F_2) \to \mathbf{Z}_2 \times \mathbf{Z}_2$ given by $\rho_1(m_1) = (1,0)$, $\rho_1(l_1) = (0,0)$, $\rho_1(m_2) = (0,1)$, $\rho_1(l_2) = (1,0)$. The action on the cover of ∂H_2 corresponds to the representation ρ_2, given by $\rho_2(m_1) = (0,1)$, $\rho_2(l_1) = (0,0)$, $\rho_2(m_2) = (0,0)$, $\rho_2(l_2) = (1,0)$.

That ρ_1 represents the generator of $H_2(\mathbf{Z}_2 \times \mathbf{Z}_2; \mathbf{Z})$ and ρ_2 represents 0 follows from the Kunneth formula used in computing $H_2(\mathbf{Z}_2 \times \mathbf{Z}_2; \mathbf{Z})$.

Added in proof: The author has constructed inequivalent freely bordant actions of the symmetric group, S_8, on a surface.

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