DEFICIENT POINTS OF MAPS ON MANIFOLDS

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1. INTRODUCTION

The primary goal of this paper is the proof of the following theorem.

THEOREM 1.1. *Suppose M^n and N^n are oriented connected n-manifolds and $f: M^n \to N^n$ is a proper map with degree* $\deg f \neq 0$. *Let Δ_f be the set of points $y \in N^n$ for which $f^{-1}(y)$ has less than* $|\deg f|$ *points.*

(1) *Then* $\dim \Delta_f \leq n - 1$ *and Δ_f contains no closed (in N^n) subset of dimension $n - 1$.*

(2) *If f is discrete* (i.e. *each $f^{-1}(y)$ is discrete*), *then* $\dim \bar{\Delta}_f \leq n - 2$.

Walsh (4.3) has constructed a (nondiscrete) example with $\dim \bar{\Delta}_f = n$.

Definitions 1.2. A point $y \in N^n$ is called *deficient* if the number of points $\# f^{-1}(y) < |\deg f|$, and its deficiency $\delta_f(y) = |\deg f| - \# f^{-1}(y)$; for a nondeficient point $\delta_f(y)$ is defined to be 0. The set of deficient points is denoted by Δ_f. Hopf defined deficient points and proved the following result:

THEOREM 1.3. (Hopf [11, Anhang II]; see also [12, Section 1]). *Let M^2 and N^2 be closed, connected oriented manifolds, and let $f: M^2 \to N^2$ be continuous with* $\deg f \neq 0$. *Then*

$$\sum \{\delta_f(y) : y \in N^2\} \leq |\deg f| \chi(N^2) - \chi(M^2).$$

Here $\chi(M^2)$ is the Euler characteristic, and, in particular, Δ_f is discrete for dimension $n = 2$. Earlier H. Kneser [16] had shown that

$$0 \leq |\deg f| \chi(N^2) - \chi(M^2),$$

and Hopf used Kneser's result in his proof. The Hurwitz-Riemann formula for complex analytic functions (cf. [12, p. 274]) shows that Hopf's inequality is sharp. (See also [6] and its references and [33, p. 18, (3.14)].)

Background 1.4. For dimension $n = 1$ it is easy to see that there are no deficient points (5.8), i.e. $\Delta_f = \emptyset$. For dimensions $n > 2$ Hopf restricted his attention to simplicial maps "um Komplikationen zu vermeiden" [12, p. 280]. In case f is simplicial it is easy to see that Δ_f is an at most $(n - 2)$-dimensional subcomplex and Hopf and others discussed its further structure (cf., (3.11)).

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For arbitrary continuous functions with $n > 2$ Hocking and Young in their text [13, p. 270], (cf., also [32, p. 366]) asked whether there need exist *any* nondeficient points, and Honkapohja [9] showed that the set of nondeficient points is dense; thus $\dim \Delta_f \leq n - 1$ [1, p. 14, (4.9)(a)]. In the special cases where f is a C^2 map (3.11), or is discrete open (2.9), or is a light locally sense preserving map (3.9), it follows from other work that $\dim \bar{\Delta}_f \leq n - 2$. But for an arbitrary map, $\dim \bar{\Delta}_f$ may be n (Walsh [28], cf., (4.3)).

Definition 1.5. Shepardson [23] called $y \in N^n$ *component-wise deficient* if the number $k(y)$ of components of $f^{-1}(y)$ is less than $|\deg f|$, called the number $|\deg f| - k(y)$ the deficiency $d_f(y)$ of y, and denoted the set of deficient points by D_f. He proved the following generalization of Hopf's theorem:

THEOREM 1.6. (Shepardson [23]) *Let M^2 and N^2 be closed connected oriented 2-manifolds and let $f : M^2 \to N^2$ be continuous with* $\deg f \neq 0$. *Then*

$$\sum \{d_f(y) : y \in D_f\} + \operatorname{rank} \operatorname{imag} i^* \leq |\deg f| \chi(N^2) - \chi(M^2),$$

where i^* is the homomorphism in cohomology in dimension one induced by the inclusion map $i : f^{-1}(D_f) \to M^2$.

If M^2 and N^2 are not necessarily closed, but f is proper, then D_f is still discrete [24, p. 22].

Remark 1.7. In general, $\Delta_f \subset D_f$, but for $n \geq 3$ D_f may be n-dimensional: D. C. Wilson [31, p. 107, Theorem 2] constructed for $n \geq 3$ maps $f : S^n \to S^n$ which are monotone (each $f^{-1}(y)$ is connected and nonempty) of arbitrary degree. Thus the analog of (1.1) (1) with Δ_f replaced by D_f is false, and this fact makes the proof of (1.1) more subtle. For other examples see Section 4. While Δ_f need not be closed in N^n (4.6), it is a G_δ set (3.6).

Remark 1.8. In case the manifolds in (1.1) are not necessarily orientable, an absolute degree $A(f)$ was defined by Hopf and others (cf., Section 5), and in Section 5 we prove that Theorem (1.1), Hopf's theorem (1.3) and Shepardson's theorem (1.6) are still true for $|\deg f|$ replaced by $A(f)$ ((5.13), (5.16), (5.17)).

Conventions 1.9. Except where otherwise specified, the following conventions hold throughout this paper. Manifolds are assumed to be without boundary, separable, and connected. Maps are continuous functions and are proper (i.e. the inverse image of a compact set is compact), and *f will always refer to a proper map $f : M^n \to N^n$*. Except in Section 5, *manifolds are oriented and the degree* $\deg f \neq 0$.

The cohomology used is Alexander-Spanier cohomology with compact supports and with coefficients in the integers Z (see [25, pp. 320–323] and [1, p. 1ff]). The homomorphism on cohomology induced by f is denoted by f^*, and that on the fundamental group π_1 by $f_\#$.

For a finite set B, $\# B$ is its number of elements.

Remark 1.10. For locally compact finite dimensional separable metric spaces X the usual definitions of $\dim X$ [14] agree with $\dim_Z X$, the cohomological dimension with Z (integer) coefficients: $\dim_Z X \leq n$ if and only if $H_c^{n+1}(U;Z) = 0$ for every open $U \subset X$ [1, p. 6, (1.2)]. To see this, let the one point compactification of X be denoted by X^+, let $C = X - U$, note that $H_c^{n+1}(X - C;Z) = H^{n+1}(X^+,C^+;Z)$ [25, p. 321, (11)], and apply [14, p. 152].

Outline of the paper 1.11. In Section 2 we give preliminaries on degree and local degree, define an essential point $x \in M^n$ as one at which the local degree is nonzero, and define E_f as the set of $y \in N^n$ for which $f^{-1}(y)$ has less than $|\deg f|$ essential points. Thus $\Delta_f \subset E_f$, and the lemmas in Sections 2 and 3 involve E_f rather than Δ_f. In the concluding lemma (2.6) of Section 2 we obtain by restriction of f maps f_i such that the set X_{f_i} of essential points is mapped by f_i homeomorphically onto E_{f_i}, thus (in (3.1)) reducing our problem to a more tractable special case. In Section 3 we use (2.6) to prove (1.1) for E_f in place of Δ_f, and thus deduce (1.1) as a corollary. We also note that Δ_f is a G_δ set (3.6), although it need not be closed (4.6), and we consider Δ_f for various special maps.

In Section 4 we consider various examples which show that (1.1) is generally sharp, except for one question (4.5), and we give some partial generalizations. As noted in (1.8), we prove in Section 5 generalizations to absolute degree $A(f)$ of Theorem (1.1) and Shepardson's theorem (and thus Hopf's theorem). After preliminaries, the proofs are developed in a series of lemmas, involving the three cases in which $A(f) \neq 0$, and many of these lemmas are used for both the generalized (1.1) and for the generalized Shepardson's theorem.

2. LOCAL DEGREE, ESSENTIAL POINTS, AND THE DEFICIENT SET Δ_f

Remember that M^n and N^n are connected, oriented n-manifolds and f refers to a proper map $f: M^n \to N^n$ with $\deg f \neq 0$.

Background 2.1. The orientation of M^n is a distinguished generator $\alpha_M \in H_c^n(M^n) \approx Z$, and the degree of f (written $\deg f$) is the unique integer defined by $f^*(\alpha_N) = (\deg f) \cdot \alpha_M$. Since f is proper, $f(M^n)$ is closed in N^n; if $f(M^n) \neq N^n$, then $H_c^n(f(M^m)) = 0$ [1, pp. 11–12, (4.3) (1)]. Thus if f is not onto, $\deg f = 0$.

If $U \subset N^n$ is a connected nonempty open subset, $\{V_\alpha : \alpha \in \mathscr{A}\}$ are the components of $f^{-1}(U)$, and $f_\alpha : V_\alpha \to U$ are the restrictions of f, then f_α is not onto for all but a finite number of α. Each V_α inherits an orientation from M^n [1, pp. 11–12, (413) (2)], $\deg f_\alpha = 0$ for all but a finite number of α, and $\deg f = \sum_\alpha \deg f_\alpha$.

Definitions 2.2. Consider $y \in N^n$ with $f^{-1}(y)$ discrete; since f is proper, $f^{-1}(y)$ is finite, consisting of $x_1, x_2, \ldots, x_j$. Choose a connected open neighborhood U of y sufficiently small that the x_i are in distinct components V_i of $f^{-1}(U)$. For $f_i : V_i \to U$ defined by restriction of f, $\deg f_i$ is independent of the choice of such a U, and is called the *local degree* of f at x_i. Note that $\sum_i \deg f_i = \deg f$.

If the local degree of f at x_i is nonzero, we say that x_i is an *essential point* of f, and the set of essential points of f is denoted by X_f. In the formula

$\sum_i \deg f_i = \deg f$ only the essential points x_i need be counted. A point $y \in N^n$ is called *essentially deficient* if $f^{-1}(y)$ is discrete and has fewer than $|\deg f|$ essential points. The set of essentially deficient points is denoted by E_f. Clearly $\Delta_f \subset E_f$ (1.2).

LEMMA 2.3. *Suppose T is open in N^n, $y \in T$ with $f^{-1}(y)$ discrete, and $f^{-1}(y)$ has m essential points x_i ($i = 1,2,\ldots,m$). Then there exists a connected open neighborhood $U \subset T$ of y and components V_i ($i = 1,2,\ldots,m$) of $f^{-1}(U)$ with $f^{-1}(y) \cap V_i = \{x_i\}$ such that, if $f_i : V_i \to U$ is restriction of f, then* $\deg f_i \neq 0$, *$f_i(V_i) = U$, and* $\deg f = \sum_i \deg f_i$.

The proof is immediate from (2.1) and (2.2).

LEMMA 2.4. *For each $y \in N^n$ with $f^{-1}(y)$ discrete, $f^{-1}(y)$ has at least one essential point.*

The proof is immediate from (2.3).

LEMMA 2.5. *If f is discrete, then E_f is closed.*

Proof. Let $y \in N^n - E_f$, so that $f^{-1}(y)$ has m essential points with $m \geq |\deg f|$. Let $f_i : V_i \to U$ ($i = 1,2,\ldots,m$) be as given by (2.3) for $T = N^n$, and let $z \in U$. From (2.4) applied to f_i and z, $f^{-1}(z) \cap V_i$ has at least one essential point, so that $f^{-1}(z)$ has at least m essential points. Thus $U \subset N - E_f$, so E_f is closed.

LEMMA 2.6. *Let T be open in N^n, and let $A \subset E_f \cap T$ be nonempty. Then there exist $y \in A$ such that the following properties hold for the $U \subset T$, V_i and f_i given by* (2.3):

(1) *$V_i \cap X_f \cap f^{-1}(A)$ is closed in $f^{-1}(U \cap A)$,*

(2) *$f : V_i \cap X_f \cap f^{-1}(A) \approx U \cap A$,*

(3) *$X_f \cap f^{-1}(U \cap A) \subset \bigcup_i V_i$, and*

(4) $\deg f_i \neq 0$ *($i = 1,2,\ldots,m < |\deg f|$) and* $\sum_i \deg f_i = \deg f$.

Proof. For each $y \in A$, $f^{-1}(y)$ is discrete and has $m(y)$ essential points, where $m(y) < |\deg f|$. Choose y such that $m(y)$ is maximal, say m, and let U, V_i, and f_i be as given by (2.3); thus (4) holds. Apply (2.4) to $z \in U \cap A$ and f_i to conclude that $f^{-1}(z) \cap V_i$ has at least one essential point. By the maximality property of y, each $f^{-1}(z) \cap V_i$ has exactly one essential point, i.e. point of X_f, and (3) $X_f \cap f^{-1}(A \cap U) \subset \bigcup_i V_i$. Thus (a) the function of (2) is continuous, injective, and surjective; let g be its inverse.

We will prove: (b) if z_j, $z \in U \cap A$ and $z_j \to z$, then $g(z_j) \to g(z)$. Let $W \subset V_i$ be any open neighborhood of $g(z)$, and let $D \subset U$ be an open connected neighborhood of z sufficiently small that the component V of $f^{-1}(z)$ containing $g(z)$ is contained in W [29, p. 131, (4.41)] and has no other point of $f^{-1}(z)$. Since $g(z)$ is essential, $f : V \to D$ has nonzero degree, so that $f(V) = D$. Since $z_j \in D$ for all j greater than some J, by (2.4) there is an essential point x_j in $f^{-1}(z_j) \cap V \subset V_i$; by (a) $x_j = g(z_j)$, so $g(z_j) \in V \subset W$ for all j greater than J also. Since W has arbitrary, $g(z_j) \to g(z)$, and (b) results. Conclusion (2) follows from (a) and (b).

We will now prove (1). Let $x_j \in V_i \cap X_f \cap f^{-1}(A)$, let $x \in f^{-1}(U \cap A)$, and let $x_j \to x$. Since $f(x_j) \to f(x)$, it follows from (b) that $g(f(x_j)) \to g(f(x))$. Since $g(f(x_j)) = x_j$, $x = g(f(x))$, so $x \in V_i \cap X_f \cap f^{-1}(A)$, and (1) results.

PROPOSITION 2.7. $\dim E_f \leq n - 1$, *so* $\dim \Delta_f \leq n - 1$.

Proof. Suppose $\dim E_f = n$. Then [1, p. 14, (4.9) (b)] there is a connected nonempty open $D \subset E_f$. Apply (2.6) to f with $T = A = D$; it follows that

(1) $V_i \cap X_f$ is closed in V_i for each i, and

(2) $f : V_i \cap X_f \approx U$.

From (2) and the theorem on invariance of domain [14, pp. 95–96] $V_i \cap X_f$ is open in V_i and from (1) it is closed; since V_i is connected, $V_i \cap X_f = V_i$. From (2) f_i is a homeomorphism, so $\deg f_i = \pm 1$. Thus there are at least $\deg f$ components V_i, contradicting $D \subset E_f$.

Remark 2.8. In [9] Honkapohja proved that the set of points for which $f^{-1}(y)$ has at least $|\deg f|$ points is dense in N^n, i.e. $\dim \Delta_f \leq n - 1$ [14, p. 46]. His argument yields $\dim E_f \leq n - 1$.

Actually (2.7) is a special case of (3.3), and is not used in its proof. However, all the preceding lemmas are used for (3.3), and since the proof of (2.7) is so short and instructive, we have included it.

Remark 2.9. For a not necessarily proper map f: $M^n \to N^n$ define the *branch set* $B_f \subset M^n$ to be the set of points at which f fails to be a local homeomorphism. According to [3] (based on [26] and [27]) if f is a discrete (not necessarily proper) map, then $\dim B_f = \dim f(B_f) \leq n - 1$, and $\dim B_f = \dim f(B_f) \leq n - 2$ if and only if f is open.

Now suppose f is proper and discrete. Clearly $\Delta_f \subset f(B_f)$, and since B_f is closed, $f(B_f)$ is also closed. Thus $\dim \bar{\Delta}_f \leq n - 1$ for a discrete map, and $\dim \bar{\Delta}_f \leq n - 2$ for a discrete open map. (The first conclusion also follows from (2.5) and (2.7).) Indeed, for a (proper) discrete open map, $\Delta_f = E_f = f(B_f)$. The authors were thus motivated by [3] to study deficient points of discrete maps, and to obtain (1.1) (2).

3. THE PROOF OF THEOREM (1.1)

LEMMA 3.1. *Suppose E_f contains a closed subset F of dimension $n - 1$. Then there exist a connected open set $U \subset N^n$, a component V of $f^{-1}(U)$, the restriction map $g : V \to U$, and a nonempty set $\Gamma \subset U \cap F$ and closed in U such that:*

(1) $|\deg g| > 1$;

(2) $\Gamma \subset E_g$;

(3) $g\colon \Lambda \approx \Gamma$, *where* $\Lambda = g^{-1}(\Gamma) \cap X_g$ *and is closed in* V;

(4) *there is a component* W *of* $U - \Gamma$ *with* $\emptyset \neq \Gamma = bdy_U W$ *and* $U \not\subset \bar{W}$;

(5) *for every open set* D *with* $\Gamma \cap D \neq \emptyset$, $\dim(\Gamma \cap D) = n - 1$.

Proof. By [1, p. 14, (4.9) (b)] (and (1.10)) F separates N^n locally at some point, so there exist $y \in F$ and a connected open neighborhood $T \subset N^n$ of y such that $T - F$ is not connected. Let P be a component of $T - F$, let $Q = \operatorname{int} \bar{P}$, and let $S = bdy_T Q$. Since $P \subset Q \subset \bar{P}$, Q is connected; also $\bar{Q} = \bar{P}$ and (*) $\quad \emptyset \neq S \subset bdy_T(T - \bar{Q})$.

Apply (2.6) to f, T, and $A = S$ to obtain $U \subset T$, V_i, and f_i with:

(a) $V_i \cap X_f \cap f^{-1}(S)$ closed in $f^{-1}(U \cap S)$;

(b) $f_i : V_i \cap X_f \cap f^{-1}(S) \approx U \cap S$;

(c) $X_f \cap f^{-1}(U \cap S) \subset \bigcup_i V_i$; and

(d) $\deg f_i \neq 0$ ($i = 1,2,\ldots,m < |\deg f|$) and $\deg f = \sum_i \deg f_i$.

Since $\emptyset \neq U \cap S \subset E_f$ it follows from (c) that $|\deg f_i| > 1$ for some i; from (b) it follows that $U \cap S \subset E_{f_i}$. Let $V = V_i$ and let $g = f_i$; then conclusion (1) results. Let W be a component of $U \cap Q$, and let $\Gamma = bdy_U W$. Since $\Gamma \subset U \cap S$, conclusion (2) results; conclusion (3) follows from (a) and (b); conclusion (4) follows from (*); and (5) follows from (*) and [1, p. 14, (4.9) (b)].

LEMMA 3.2. *Suppose that* g *satisfies the conclusion of* (3.1). *Then* $\dim(g^{-1}(\Gamma) - \Lambda) = n - 1$.

Proof. Suppose, on the contrary, that $\dim(g^{-1}(\Gamma) - \Lambda) \neq n - 1$. Since $\Gamma \subset E_g$ by (3.1) (2) the map of $g^{-1}(\Gamma) - \Lambda$ into Γ is discrete (2.2); since $\dim \Gamma = n - 1$ by (3.1) (5), $\dim(g^{-1}(\Gamma) - \Lambda) \leq n - 1$ [14, pp. 91–92]. From the contrary supposition, $\dim(g^{-1}(\Gamma) - \Lambda) \leq n - 2$.

Since the cohomology dimension $\dim_Z(g^{-1}(\Gamma) - \Lambda) = \dim(g^{-1}(\Gamma) - \Lambda)$ (1.10), $H_c^j(g^{-1}(\Gamma) - \Lambda) = 0$ for $j \geq n - 1$. From the cohomology sequence inclusion induces an isomorphism $H_c^{n-1}(g^{-1}(\Gamma)) \approx H_c^{n-1}(\Lambda)$, and since g maps Λ homeomorphically onto Γ, $\bar{g}^* : H_c^{n-1}(\Gamma) \approx H_c^{n-1}(g^{-1}(\Gamma))$.

Consider the diagram

$$
\begin{array}{ccccc}
H_c^{n-1}(g^{-1}(\Gamma)) & \xrightarrow{\delta_V} & H_c^n(g^{-1}(W)) & \rightarrow & H_c^n(g^{-1}(\bar{W})) \\
\bar{g}^* \uparrow \approx & & \uparrow g^* & & \uparrow \\
H_c^{n-1}(\Gamma) & \xrightarrow{\delta_U} & H_c^n(W) & \rightarrow & H_c^n(\bar{W})
\end{array}
$$

induced by g. From (3.1) (4) $\bar{W} \neq U$ and from (3.1) (1) $g(V) = U$, so $g^{-1}(\bar{W}) \neq V$. Thus $H_c^n(\bar{W}) = 0 = H_c^n(g^{-1}(\bar{W}))$ [1, p. 11, (4.3) (1)], so δ_U and δ_V are epimorphisms. Since $g^*\delta_U = \delta_V \bar{g}^*$, which is an epimorphism, g^* is an epimorphism. Since W is a nonempty connected open set in an orientable n-manifold, $H_c^n(W) = Z$ [1, pp. 11–12, (4.3) (2)], so $H_c^n(g^{-1}(W))$ is an image of Z. Since each component of $g^{-1}(W)$ is orientable, $g^{-1}(W)$ is connected and g^* is an isomorphism. It follows that $\deg g = \pm 1$, contradicting (3.1) (1). Thus $\dim(g^{-1}(\Gamma) - \Lambda) = n - 1$.

LEMMA 3.3. *There is no map f for which E_f contains a closed subset A of dimension $n - 1$.*

Proof. Suppose such an f exists, and let $g : V \to U$, Γ, and Λ be as given by (3.1).

Suppose that there exists $x \in \Lambda$ and an open neighborhood $T \subset V$ of x such that $\dim(T \cap (g^{-1}(\Gamma) - \Lambda)) \leq n - 2$. Let D be a connected open neighborhood of $f(x)$ in U sufficiently small that the component K of $g^{-1}(D)$ containing x is contained in T. It follows from (3.1) (3) that $\Gamma \cap D \subset E_{g|K}$ and from (3.1) (4) that $\dim(\Gamma \cap D) = n - 1$. Apply (3.1) again to $g|K$: $K \to D$ and $F = \Gamma \cap D$ to obtain a map $g' : V' \to U'$ and sets Γ' and Λ'. Since $V' \subset T$,

$$\dim((g')^{-1}(\Gamma') - \Lambda') \leq n - 2,$$

contradicting (3.2).

Thus (1) for every $x \in \Lambda$ and open neighborhood T of x,

$$\dim(T \cap (g^{-1}(\Gamma) - \Lambda)) = n - 1.$$

Let $g_0 = g$, $\Gamma_0 = \Gamma$, and $\Lambda_0 = \Lambda$. We will define a sequence by induction. Suppose that there exist maps $g_i : V_i \to U_i$ satisfying the conclusions of (3.1) with g_{i+1} a restriction of g_i, $\Gamma_{i+1} \subset \Gamma_i$, and $\Lambda_{i+1} \subset \Lambda_i$ $(i = 0,1,\ldots,m)$, as well as mutually disjoint compact sets ϕ_i $(i = 1,2,\ldots,m)$ with $\phi_i \subset g_{i-1}^{-1}(\Gamma_{i-1}) - \Lambda_{i-1}$ and $\Gamma_i \subset g_{i-1}(\phi_i) \cap U_i$. Thus $g(\phi_i) \subset g(\phi_{i-1})$.

Let $x \in \Lambda_m$, let $T \subset V_m$ be an open neighborhood of x sufficiently small that it is disjoint from the compact set $\bigcup_{i=1}^{m} \phi_i$, and let $D \subset U_m$ be a connected open neighborhood of $f(x)$ sufficiently small that the component K of $g_m^{-1}(D)$ containing x is contained in T. Since $K \cap (g_m^{-1}(\Gamma_m) - \Lambda_m)$ is the countable union of compact sets, it follows from (1) that it contains a compact subset ϕ_{m+1} with $\dim \phi_{m+1} = n - 1$. Since $\Gamma_m \subset E_{g_m}$ by (3.1) (2), $g_m | g_m^{-1}(\Gamma_m)$ is discrete (2.2), so $\dim(g_m(\phi_{m+1})) \geq n - 1$ [14, pp. 91–92]; since $g_m(\phi_{m+1}) \subset \Gamma_m$ and $\dim \Gamma_m = n - 1$ (by (3.1) (5)), $\dim g_m(\phi_{m+1}) = n - 1$. Apply (3.1) to g_m with $F = g_m(\phi_{m+1})$ to define a restriction map $g_{m+1} : V_{m+1} \to U_{m+1}$, with $\Gamma_{m+1} \subset \Gamma_m$ and $\Lambda_{m+1} \subset \Lambda_m$.

The inductive hypotheses are satisfied, so there exists a sequence of disjoint compact sets ϕ_m $(m = 1, 2, \ldots)$ in $g^{-1}(\Gamma) - \Lambda$ with $g(\phi_{m+1}) \subset g(\phi_m)$. Since $\bigcap_m g(\phi_m) \neq \emptyset$, it contains a point y with $f^{-1}(y)$ infinite. But each $g(\phi_m) \subset \Gamma \subset E_f$, and for each $y \in E_f$, $g^{-1}(y) \subset f^{-1}(y)$ is discrete and (since f is proper) thus finite. A contradiction results, so there is no such map.

COROLLARY 3.4. *If f is discrete, then* $\dim E_f \leq n - 2$.

Proof. Since E_f is closed (2.5), the corollary results from (3.3).

Proof of (1.1) 3.5. Since $\Delta_f \subset E_f$ (2.2), conclusion (1) follows from ((2.7) and) (3.3). In case f is discrete, E_f is closed (2.5) so $\bar{\Delta}_f \subset E_f$ and conclusion (2) results from (3.4).

PROPOSITION 3.6. *Δ_f is a G_δ set.*

A G_δ set is defined to be the countable intersection of open sets. In (4.6) we observe that Δ_f need not be closed.

Proof. Let $k = |\deg f| \neq 0$. For each m ($m = 1,2,\ldots$) let $\mathscr{A}_m$ be a countable cover of M^n by compact subsets such that each $A \in \mathscr{A}_m$ has diam $A \leq 1/m$. Let $\mathscr{B}_m$ be the collection of all those subsets $\{A_1, A_2, \ldots, A_k\} \subset \mathscr{A}_m$ consisting of precisely k mutually disjoint elements A_i. For each $\{A_1, \ldots, A_k\} \in \mathscr{B}_m$, $\bigcap_{i=1}^{k} f(A_i)$ is a compact subset of $N^n - \Delta_f$, and the union Γ over all elements of $\mathscr{B}_m$ and all m is a countable union of closed subsets of N^n, i.e. Γ is an F_σ set and $\Gamma \subset N^n - \Delta_f$.

Now let $y \in N^n - \Delta_f$. Then $f^{-1}(y)$ contains distinct points $x_1, \ldots, x_k$; choose m ($m = 1,2,\ldots$) such that $\operatorname{dist}(x_i, x_j) > 2/m$ for $i \neq j$ ($i,j = 1,2,\ldots,k$). Since $\mathscr{A}_m$ is a cover of M^n, there exist $A_i \in \mathscr{A}_m$ with $x_i \in A_i$ ($i = 1,2,\ldots,k$). Thus $y \in \bigcap_{i=1}^{k} f(A_i)$, which is one of the sets of Γ. Hence $N^n - \Delta_f \subset \Gamma$, so they are equal. Since $N^n - \Delta_f$ is an F_σ set, Δ_f is a G_δ set.

We now consider some special maps.

COROLLARY 3.7. *If f is open, then Δ_f is closed and* $\dim \Delta_f \leq n - 2$.

Proof. Let $y \in N^n - \Delta_f$, let x_i ($i = 1,2,\ldots,|\deg f|$) be some of the points of $f^{-1}(y)$, and let $V_i \subset M^n$ be mutually disjoint open sets with $x_i \in V_i$. Then $\bigcap_i f(V_i)$ is open and $y \in \bigcap_i f(V_i)$, so $\bigcap_i f(V_i) \subset N^n - \Delta_f$. Thus Δ_f is closed in N^n. That $\dim \Delta_f \leq n - 2$ follows from (1.1) (1).

In case f is discrete open this conclusion was given in (2.9), but we are not assuming each $f^{-1}(y)$ is discrete here.

Definition 3.8. The map f is *locally sense preserving* if, for every $y \in N^n$, every component Ω of $f^{-1}(y)$, every sufficiently small connected open neighborhood U of y in N^n, and V the component of $f^{-1}(U)$ containing Ω, the map $h : V \to U$ defined by restriction of f has positive degree.

Definition 3.9. A map $h : X \to Y$ is *light* if $h^{-1}(y)$ is totally disconnected for each $y \in N^n$.

Remark 3.10. Titus and Young considered locally sense preserving maps (cf. [32]) and proved: a) if f is light and locally sense preserving, then it is discrete and open. b) Moreover, if M^n and N^n are compact, then the set of $y \in N^n$ with $\# f^{-1}(y) = |\deg f|$ is dense in N^n (thus $\dim \Delta_f \leq n - 1$). From a) and (3.7) (or [26]) follows: if f is light and locally sense preserving, then $\dim \bar{\Delta}_f \leq n - 2$. However, if the hypothesis "light" is removed, $\dim \bar{\Delta}_f$ may be n [28] (cf. (4.3)).

Remark (Hopf [12]) 3.11. If f is simplicial, then Δ_f is an at most $(n - 2)$-dimensional polyhedron. Hopf studied its homology, as did others ([20], [8], and, in effect, others who studied branched coverings).

Remark 3.12. *If f is C^2 differentiable, then* $\dim \bar{\Delta}_f \le n - 2$.

Proof. Let $R_q(f)$ be the set of points at which the Jacobian matrix has rank at most q. According to [4, p. 1037] if f is C^r where r is he maximum of $n - q$ and 1, then $\dim (f(R_q)) \le q$. According to [2, p. 186, (1.3)] if f is C^2, then $\Delta_f \subset f(R_{n-2}(f))$. Since $R_{n-2}(f)$ is closed and f is proper, $f(R_{n-2}(f))$ is closed, so $\bar{\Delta}_f \subset f(R_{n-2}(f))$.

4. GENERALIZATIONS AND COUNTEREXAMPLES

First we consider some examples related to Theorem (1.1).

Examples 4.1. Generally, the conclusions in (1.1) (1) and (2) are sharp. The example of the complex analytic function z^d $(d > 1)$ shows that $n - 2$ is the best possible dimension restriction for $\dim \bar{\Delta}_f$ in (1.1) (2) and [5, p. 968, (5.6)] shows that $\dim \bar{\Delta}_f$ may be less than $n - 2$ even though $\Delta_f \ne \emptyset$ and $\Delta_f = E_f$. In general Δ_f need not be closed (4.6) even for a discrete map, although it is a G_δ set (3.6). As noted in (1.7), Δ_f in (1.1) cannot be replaced by D_f because of an example of Wilson [31, p. 107, Theorem 2]. While $\dim \Delta_f \le n - 1$, $\dim \bar{\Delta}_f$ may equal n, by (4.3) below.

Definition 4.2. The map f is *monotone* if $f^{-1}(y)$ is connected and nonempty for every $y \in N^n$.

THEOREM 4.3. (Walsh [28]) *For any k $(k = 2, 3, \ldots)$ there exists a monotone map $f : S^n \to S^n$ of degree k such that the set Δ_f of points $y \in S^n$ with $f^{-1}(y)$ a single point is dense in S^n $(n \ge 3)$.*

This result is a generalization of Wilson's example [31], and is the answer to a question Church posed to Walsh.

Remark 4.4. For every natural number n there is a subspace $X \subset \mathbf{R}^{n+1}$ such that $\dim X = n$ but X is totally disconnected, and thus every compact subset $A \subset X$ has dimension 0 ([15, p. 241], [19]). Thus (1.1) (1) does not imply that $\dim \Delta_f \le n - 2$. An affirmative answer to the following question would show that (1.1) (1) is sharp as stated.

Question 4.5. Does there exist a monotone map $f : S^n \to S^n$ $(n \ge 3)$ with $\deg f = 2, 3, \ldots$ such that the set Δ_f of points $y \in S^n$ with $f^{-1}(y)$ a single point has dimension $n - 1$? (By (1.1) (1) Δ_f can contain no *closed* subset of dimension $n - 1$.)

Example 4.6. *For $n \ge 3$ Δ_f need not be closed, even for a discrete map.*

Proof. Define $f : \mathbf{R}^3 \to \mathbf{R}^3$ by $f(u + iv, t) = ((u + iv)^2, t)$, so $\deg f = 2$ and $\Delta_f = E_f = f(B_f) = \{(0,0)\} \times \mathbf{R}$. Define $g : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$g(u, v, t) = (u^3 + tu^2 + (t/4)\, u, v, t),$$

so $\deg g = 1$, and define $h = gf$, so $\deg h = 2$. Since g is surjective,

$$\Delta_h \subset g(\Delta_f) = \{(0,0)\} \times \mathbf{R}.$$

In fact, $(0,0,t) \in \Delta_h$ if and only if $g(u,v,t) = (0,0,t)$ has at most one real solution, i.e. $u^2 + tu + (t/4) = 0$ has no real solution (other than $u = 0$ if $t = 0$). This is true precisely for $0 < t < 1$, so $\Delta_h = \{(0,0,t) \in \mathbf{R}^3 : 0 < t < 1\}$.

Now we consider generalizations of (1.1).

Definitions 4.7. A point $x \in M^n$ is *essential* if there exist arbitrarily small connected open neighborhoods U of $f(x)$ with components V of $f^{-1}(U)$ containing x such that $f|V : V \to U$ has nonzero degree. (This extends the definition of (2.2), which applied only in case $f^{-1}(y)$ is discrete.) Let $\tilde{X}_f \subset M^n$ be the set of essential points, and let $\tilde{E}_f$ be the set of $y \in N^n$ such that $f^{-1}(y)$ is totally disconnected and the number of essential points of $f^{-1}(y)$ is less than $|\deg f|$. (For E_f (2.2) we required that $f^{-1}(y)$ be discrete.)

Remark 4.8. The following results of Sections 2 and 3 are true for E_f replaced by $\tilde{E}_f$ and "$f^{-1}(y)$ discrete" replaced by "$f^{-1}(y)$ totally disconnected," with substantially the same proofs: (2.3) [but replace

$$\text{"}f^{-1}(y) \cap V_i = \{x_i\}\text{"} \quad \text{by} \quad \text{"}f^{-1}(y) \cap \tilde{X}_f \cap V_i = \{x_i\}\text{"}],$$

(2.4), (2.5) [If f is light (3.9), then E_f is closed], (2.6), (2.7), (3.1) and (3.2). The proof of (3.3) does not carry over to $\tilde{E}_f$, although statement (1) in its proof is still correct. As a result we obtain (from (2.5) and (2.7)):

Remark 4.9. *If f is light, then* $\dim \bar{\Delta}_f \leq n - 1$.

The authors do not know whether $\dim \bar{\Delta}_f \leq n - 2$ in this case.

Remark 4.10. Theorem (1.1) is true for (oriented) "n-manifold" replaced by (oriented) "cohomology n-manifold over Z" ($n - cm$, or "generalized n-manifold"; cf. [1, Chapter I]). With this change the proof of each of the lemmas used in the proof of (1.1) is still valid; in particular, invariance of domain (used in (2.7)) is valid for $n - cms$ [18, p. 110, (2.1)]. We use $n - cms$ to prove the following result about manifolds (4.12).

Definitions 4.11. A *Z-acyclic set* is one which has the cohomology of a point. Let Ω_f be the set of points $y \in N^n$ for which $f^{-1}(y)$ has less than $|\deg f|$ components and each component is Z-acyclic.

COROLLARY 4.12. (1) $\dim \Omega_f \leq n - 1$ *and Ω_f contains no closed (in N^n) subset of dimension $n - 1$.*

(2) *Assume that for each $y \in N^n$, $f^{-1}(y)$ has a finite number of components and each is Z-acyclic; then* $\dim \bar{\Omega}_f \leq n - 2$.

Proof. Let $A \subset N^n$ be a closed subset and suppose that for each $y \in A$ $f^{-1}(y)$ has a finite number of components and each component is Z-acyclic. Consider the upper semicontinuous decomposition [29, p. 122] of M^n whose elements are the points in $M^n - f^{-1}(A)$ and the components of $f^{-1}(y)$ for $y \in A$. Let the quotient space be K, let $g : M^n \to K$ be the quotient map, and define $h : K \to N$ by $h(x) = f(g^{-1}(x))$, so that $f = hg$.

Then K is a locally compact separable metric space [29, pp. 122–125]. Since $f^{-1}(A)$ is closed in M^n, $h^{-1}(A)$ is closed in K by the definition of the quotient topology. Since $h^{-1}(y)$ is discrete for each $y \in A$ and $\dim A \leq n$, $\dim h^{-1}(A) \leq n$ [14, pp. 91–92]; since

$$g : M - f^{-1}(A) \approx K - h^{-1}(A), \quad \dim(K - h^{-1}(A)) \leq n$$

also. Thus $\dim K \leq n$. Now g is an acyclic map, so K is an orientable $n-cm$ over Z by [30, pp. 21–22] (cf. also [17, pp. 138–139] and [22]), and $g : H_c^n(K) \approx H_c^n(M^n)$ (the Vietoris mapping theorem [25, p. 346]). Thus $\deg h = \deg f$.

For conclusion (1) suppose $A \subset \Omega_f$ is closed in N^n with $\dim A = n - 1$, and let $f = hg$ be the factorization given above for this A. Then $A \subset \Delta_h$, contradicting (4.10) (1) (i.e. (1.1) (1) for $n - cms$) applied to h. For conclusion (2) let $A = N^n$ and let $f = hg$ be the corresponding factorization (this is the monotone-light factorization of f [29, p. 141]). Thus $\Delta_h = \Omega_f$, and (2) results from (4.10) (2).

The acyclic hypothesis is needed: see (1.7) and (4.3). We could have obtained (4.12) by modifying the proof of (1.1): define "essential components" of $f^{-1}(y)$ in place of "essential points," and in place of the homeomorphism in (2.6) (2) use the Vietoris mapping theorem to obtain an isomorphism in cohomology.

5. ABSOLUTE DEGREE

In this section *manifolds are allowed to be nonorientable.* Hopf defined [11] the Absolutgrad of a (proper) map $f: M^n \to N^n$, Olum defined [21] the group-ring degree, and Epstein defined [7] the *absolute degree* $A(f)$. The absolute value of the group-ring degree agrees with the other two functions. In (5.3)–(5.5) we summarize Epstein's treatment and generally follow his notation, and the interested reader may consult [7] for details. In particular, $\bar N$ no longer refers to the closure of N.

The goal of this section is to prove analogs of Theorem (1.1), Hopf's theorem (1.3), and Shepardson's theorem (1.6) for maps on not necessarily orientable manifolds ((5.13), (5.16), (5.17)).

Definition 5.1. If $f_* : \pi_1(M,m) \to \pi_1(N,n)$ is the homomorphism on the fundamental group induced by f, the *index of* f [12, p. 278] is defined to be the index of the subgroup $f_*(\pi_1(M,m))$ in $\pi_1(N,n)$.

Definition 5.2. The mod 2 *degree of* f is defined to be 0 or 1 according as $f^* : H_c^n(N^n; Z_2) \to H_c^n(M^n, Z_2)$ is the zero homomorphism or is nonzero.

Definition of absolute degree $A(f)$ 5.3. We construct the commutative diagram below, and for simplicity omit reference to base points.

$$\begin{tikzcd}
\tilde M \arrow[r, "p"] \arrow[d, "\tilde f"'] & M \arrow[d, "\bar f"'] \arrow[dr, "f"] & \\
\tilde N \arrow[r, "s"] & \bar N \arrow[r, "r"] & N
\end{tikzcd}$$

Let r be the covering map (unique up to covering transformation) such that imag $r_* =$ imag $f_\#$, and let $\bar{f}$ be the unique map (lifting) with $f = r\bar{f}$. Then r is an (index f)-to-1 covering map, and $\bar{f}_*$ is onto. Let p be the identity if M is orientable, and the orientable double covering of M if M is nonorientable. Let s be the covering with imag $(\bar{f}p)_* =$ imag s_*, and let $\tilde{f}$ be the unique lifting with $\bar{f}p = s\tilde{f}$. Thus $\tilde{f}_*$ is onto.

The absolute degree $A(f)$ is defined to be 0 except in the following three cases:

(1) Index f is finite, and M and $\bar{N}$ are orientable. Then $\tilde{f} = \bar{f}$, and we define $A(f) = (\text{index } f) \cdot |\deg \tilde{f}|$.

(2) Index f is finite, M and $\bar{N}$ are nonorientable, and $\tilde{N}$ is orientable. Define $A(f) = (\text{index } f) \cdot |\deg \tilde{f}|$ again.

(3) Index f is finite, M is nonorientable, s is the identity, and the mod 2 degree of $\bar{f}$ is nonzero. Define $A(f) =$ index f.

THEOREM 5.4. (cf. [7]) *If M and N are orientable, then $A(f) = |\deg f|$. In any case $A(f)$ is congruent* mod 2 *to the* mod 2 *degree of f. If f and g are properly homotopic, $A(f) = A(g)$.*

A theorem of Hopf gives geometric content to the notion of absolute degree.

THEOREM (Hopf [10], [11]; cf. [7]) 5.5. *Given a (proper) map $f : M^n \to N^n$, there exists a properly homotopic proper map $g : M^n \to N^n$ (thus $A(f) = A(g)$) and an n-cell $D \subset N^n$ such that $g^{-1}(D)$ has $A(f)$ components, each an n-cell mapped homeomorphically by g onto D. Furthermore there is no such n-cell D' for which $g^{-1}(D')$ has fewer components.*

This suggests considering the notion of deficiency for the absolute degree.

Definition 5.6. For $f : M^n \to N^n$ with M^n and N^n not necessarily orientable, define deficient point, deficiency $\delta_f(y)$, and deficient set Δ_f as in (1.2) with $|\deg f|$ replaced by $A(f)$. Similarly the componentwise notions of (1.5) may be defined.

PROPOSITION 5.7. *If $A(f) \neq 0$, then the number of components of $f^{-1}(y)$ is at least* index f, *for each $y \in N^n$. Thus $\delta_f(y) \leq d_f(y) \leq A(f) -$* index f.

Hopf [12, p. 278] gives the orientable case.

Proof. Use the diagram in (5.3). Since r is an (index f)-to-1 covering map and $f = r\bar{f}$, it suffices to prove that $\bar{f}$ is onto in cases (1), (2) and (3) of (5.3). If $\bar{f}$ is not onto, then $\tilde{f}$ is not onto; thus in cases (1) and (2) $\deg \tilde{f} = 0$, and in case (3) the degree mod 2 of $\bar{f}$ is 0, contradicting the assumption that $A(f) \neq 0$.

While the following result for Δ_f is a special case of (1.1) (1), a direct proof is easy, and the conclusion for D_f is new.

COROLLARY 5.8. *If $f : M^1 \to N^1$ with $A(f) \neq 0$* (i.e. $n = 1$ *and* $\deg f \neq 0$), *then $\Delta_f = D_f = \emptyset$.*

Proof. Each of M^1 and N^1 is either $\mathbf{R}$ or S^1. Since f is proper, $M^1 = \mathbf{R}$ implies $N^1 = \mathbf{R}$; and since $A(f) = \deg f \neq 0$, $M^1 = S^1$ implies $N^1 = S^1$. Thus $M^1 = N^1$. If $f : \mathbf{R} \to \mathbf{R}$ is proper, there is a one-point compactification

$$g : (S^1,\infty) \to (S^1,\infty),$$

and it has the same degree. Thus we may suppose that $M^1 = N^1 = S^1$. In the diagram of (5.3) $f = r\bar{f}$, so deg r = index f is finite and $\bar{N} = S^1$; since $\bar{f}_*$ is onto, deg $\bar{f} = 1$. Thus deg f = index f, and the conclusion results from (5.7).

Examples 5.9. We follow the notation of (5.3), case (1) and case (2). Let $g_d : S^1 \to S^1$ be the d-to-1 covering map, $S(g_d) : S^2 \to S^2$ be its suspension, P^2 be the real projective plane, and $\pi : S^2 \to P^2$ be the covering map.

Case (1). Let $\bar{f} = S(g_d)$, $r = \pi$, and $f = r\bar{f}$. Then $A(f) = 2d$, Δ_f is a single point y if $d > 1$, and $\delta_f(y) = 2(d-1)$.

Case (2). Let d be odd, let $\tilde{f} = S(g_d)$, let $p = s = \pi$, and define $f = \bar{f}$ by $\bar{f}p = s\tilde{f}$. Then $A(f) = d$, Δ_f is a single point y, and $\delta_f(y) = d - 1$.

LEMMA 5.10. (i) *In case* (3) *of the diagram of* (5.3), $\Delta_f = D_f = \emptyset$.

(ii) *If, in addition,* $n = 2$ *and the manifolds are closed, then*

$$0 \leq A(f)\chi(N) - \chi(M).$$

Proof. (i) Since (f) = index f, $\Delta f = D_f = \emptyset$ results from (5.7).

(ii) Since $\bar{f}_*$ is onto (5.3), it follows that $\bar{f}_* : H_1(M) \to H_1(\bar{N})$ is onto, so the first Betti numbers satisfy $\beta_1(M) \geq \beta_1(\bar{N})$. Since M is nonorientable

$$\chi(M) = 1 - \beta_1(M) \leq 1 - \beta_1(\bar{N}) \leq \chi(\bar{N}).$$

Now $r : \bar{N} \to N$ is an (index f)-to-1 covering map, so

$$\chi(\bar{N}) = (\text{index } f)\chi(N) = A(f)\chi(N),$$

and the second conclusion results.

LEMMA 5.11. *In case* (2) *of the diagram of* (5.3) *if* $\bar{y} \in \bar{N}$ *and* Γ *is a component of* $f^{-1}(\bar{y})$, *then* $p^{-1}(\Gamma)$ *has exactly two components, each mapped homeomorphically by* p *onto* Γ.

Proof. Both $\bar{f}_*$ and $\tilde{f}_*$ are onto by (5.3). Since p and s are 2-to-1 covering maps in case (2), imag p_* has index 2 in $\pi_1(M)$ and imag s_* has index 2 in $\pi_1(\bar{N})$. We may as well suppose that the base point $m \in M$ is chosen in Γ, so that $f(m) = \bar{y}$ is the base point of $\bar{N}$.

Let $D \subset \bar{N}$ be an open n-cell about $\bar{y}$, and let E be the component of $f^{-1}(D)$ containing Γ. Each component of $p^{-1}(E)$ is mapped by the 2-to-1 covering map p onto E, so there are at most two such components. If there are exactly two, each is mapped by p homeomorphically onto E and the conclusion results. Thus we may suppose that $p^{-1}(E)$ is connected, and we will deduce a contradiction.

Let γ be a path in $p^{-1}(E)$ joining the base point $\tilde{m} \in M$ to the other point of $p^{-1}(m)$. Then $p\gamma$ is a loop whose path class $[p\gamma]$ is not in imag p_*, and since imag p_* has index 2 in $\pi_1(M)$, $[p\gamma]$ and imag p_* generate $\pi_1(M)$. Since $\bar{f}_*$ is onto, imag $(\bar{f}p)_*$ and $[\bar{f}p\gamma]$ generate $\pi_1(\bar{N})$. But $\bar{f}p\gamma$ is a loop in the n-cell D, so that imag $(\bar{f}p)_*$ alone generates $\pi_1(N)$. Now $\bar{f}p = s\tilde{f}$ and imag s_* has index 2 in $\pi_1(N)$, so a contradiction results.

LEMMA 5.12. *Let $\alpha = 1$ or 2. In case* (α) *of* (5.3),

(i) $\alpha\delta_f(y) = \sum_j [|\deg \tilde{f}| - \# \tilde{f}^{-1}(\tilde{z}_j)] \leq \sum_j \delta_{\tilde{f}}(\tilde{z}_j)$, *for any $y \in \Delta_f$ and $\tilde{z}_j$ the points of $s^{-1}(r^{-1}(y))$.*

(ii) *Thus $\Delta_f \subset r(s(\Delta_{\tilde{f}}))$.*

(iii) *The analogous statements for d_f and D_f are also true.*

Proof. (i) We may assume that $A(f) \neq 0$. In cases (1) and (2) of (5.3) $\tilde{M}$ and $\tilde{N}$ are orientable, so $|\deg \tilde{f}|$ is defined, and $A(f) = (\text{index } f) \cdot |\deg \tilde{f}|$. Now r is an (index f)-to-one covering map, so $r^{-1}(y) = \{z_k : k = 1, 2, \ldots, \text{index } f\}$. Thus

$$\begin{aligned}\delta_f(y) &= A(f) - \# f^{-1}(y)\\ &= (\text{index } f)\,|\deg \tilde{f}| - \sum_k \# \bar{f}^{-1}(z_k)\\ &= \sum_k [|\deg \tilde{f}| - \# \bar{f}^{-1}(z_k)].\end{aligned}$$

Let $\tilde{z}_j$ be the points of $(rs)^{-1}(y)$. In case (1) p and s are identity maps, so the $\tilde{z}_j$ are the z_k and conclusion (i) results. In case (2) for $\bar{y} \in \bar{N}$ and Γ a component of $\bar{f}^{-1}(\bar{y})$, $p^{-1}(\Gamma)$ has precisely two components, each mapped homeomorphically by p onto Γ (5.11). Thus $j = 1, 2, \ldots, 2$ (index f) and (i) results. Conclusion (ii) is a consequence of (i), and (iii) results from the same proof with "number of components" in place of "number of points" #.

PROPOSITION 5.13. *Theorem* (1.1) *holds for M^n and N^n possibly nonorientable and $|\deg f|$ replaced by $A(f)$.*

Proof. Since $A(f) \neq 0$, we have cases (1), (2), and (3). In case (3) $\Delta_f = \emptyset$ by (5.10), and the conclusion results. In cases (1) and (2) $\Delta_f \subset r(s(\Delta_{\tilde{f}}))$ by (5.12) and r and s are covering maps; the conclusion results from (1.1) applied to $\tilde{f}$.

LEMMA 5.14. *Let M^2 and N^2 be closed and oriented, and let $f : M^2 \to N^2$ be continuous with* $\deg f \neq 0$. *Assume that $B \subset N^2$ is finite and $f^{-1}(B)$ has a finite number h of components, and define $D(B) = (\#B)|\deg f| - h$ (this number may be negative). Let K be the union of the* 2-cell *components of $M^2 - f^{-1}(B)$ and let $L = K \cup f^{-1}(B)$. Then*

$$D(B) + \text{rank } H^1(L) \leq |\deg f|\,\chi(N^2) - \chi(M^2).$$

Proof. This is a slight generalization of [23, p. 283, (3.3)], in that the original version assumed that B was (any finite subset of) D_f. The proof is the same as that of [23, Section 3] with $\sum \{D_f(y) : y \in B\}$ replaced by $D(B)$. (In [23, bottom of p. 281, (c)] the homotopy should be chosen so that

$$g(S^1_{ijk} \times (-\epsilon, 0]) \subset U_j - y_j.)$$

LEMMA 5.15. *Consider case* (2) *of the diagram of* (5.3), *where* $n = 2$ *and* M *and* N *are closed* 2-*manifolds. Let* $B \subset \bar{N}$ *be finite, let* $\tilde{B} = s^{-1}(B)$, *let* K [*respectively* $\tilde{K}$] *be the union of the* 2-cell *components of* $M - \bar{f}^{-1}(B)$ [*respectively* $\tilde{M} - \tilde{f}^{-1}(\tilde{B})$], *and let* $L = K \cup \bar{f}^{-1}(B)$ *and* $\tilde{L} = \tilde{K} \cup \tilde{f}^{-1}(\tilde{B})$. *Assume that* $\bar{f}^{-1}(B)$ *has a finite number* m *of components. Then* $\tilde{L} = p^{-1}(L)$ *and*

$$\operatorname{rank} H^1(\tilde{L}) \geq 2 \operatorname{rank} H^1(L).$$

Proof. Let X be a 2-cell component of $\tilde{K}$; then $p(X)$ is contained in a component Y of $M - \bar{f}^{-1}(B)$, and $\bar{p} = p|X : X \to Y$ is a covering map. We will prove that (a) $\bar{p}$ is a homeomorphism. Suppose not; then $\bar{p}$ is 2-to-1, and is the orientable covering of Y. Let $S \subset Y$ be a circle defining a loop around which the orientation of Y changes. Then $Y - S$ is connected, while the circle $\bar{p}^{-1}(S)$ separates the 2-cell X into two nonhomeomorphic components. Since $\bar{p}$ maps each of these components homeomorphically onto $Y - S$, a contradiction results, and a) follows.

If Y is a 2-cell component of $M - \bar{f}^{-1}(B)$, then $p^{-1}(Y)$ consists of two 2-cells, each mapped by p homeomorphically onto Y, and from a) each 2-cell component arises in this way. In particular, b) $\tilde{K} = p^{-1}(K)$ and $\tilde{L} = p^{-1}(L)$.

For each 2-cell component X of $\tilde{K}$, there are closed 2-cells X_j such that $X_j \subset \operatorname{int} X_{j+1}$ and $\bigcup_j X_j = X$. Since $X - \operatorname{int} X_j \approx S^1 \times [0,1)$, its closure is connected, so $bdy\ X = \bigcap_j Cl\,[X\text{-int}\, X_j]$ is connected. Thus c) $bdy\ X$ is contained in a single component $\tilde{\Gamma}$ of $\tilde{f}^{-1}(\tilde{B})$, and the analog is true for $\bar{f}^{-1}(B)$.

By hypothesis $\bar{f}^{-1}(B)$ has a finite number m of components, so from (5.11) $\tilde{f}^{-1}(\tilde{B})$ has $2m$ components. From c) it follows that L has m components and $\tilde{L}$ has $2m$ components. Since $\tilde{L}$ is a subset of the orientable manifold $\tilde{M}$, and L is a subset of the nonorientable manifold M, $H^2(\tilde{L}) = 0$ or Z and $H^2(L) = 0$ or Z_2. Thus $H^j(\tilde{L})$ is finitely generated for $j \neq 1$, and d) rank $H^1(\tilde{L})$ is finite by (5.14). In the cohomology sequence

$$H^2(\tilde{M},\tilde{L}) \xleftarrow{\delta} H^1(\tilde{L}) \xleftarrow{i^*} H^1(\tilde{M})$$

imag δ is a subgroup of a free abelian group and so is free abelian and finitely generated by d), and $H^1(\tilde{M})$ is finitely generated since $\tilde{M}$ is closed, so $H^1(\tilde{L})$ is also finitely generated.

Since p: $\tilde{L} \to L$ is a 2-to-1 covering map, Z_2 acts freely on $\tilde{L}$ with orbit space L. Since $H^*(\tilde{L})$ is finitely generated, it follows from [1, p. 46, (5.3)] that $H^*(L)$ is finitely generated and $\chi(\tilde{L}) = 2\chi(L)$. Thus the Betti numbers satisfy

$$2m - \beta_1(\tilde{L}) + \beta_2(\tilde{L}) = 2(m - \beta_1(L)),$$

where $\beta_2(\tilde{L}) = 0$ or 1. Hence $2\beta_1(L) \leq \beta_1(\tilde{L})$, which (with b) above) is the desired conclusion.

THEOREM 5.16. *Let M^2 and N^2 be closed manifolds (possibly nonorientable), and let $f: M^2 \to N^2$ be continuous with $A(f) \neq 0$. Then*

$$\sum \{d_f(y) : y \in D_f\} + \text{rank imag } i^* \leq A(f)\chi(N^2) - \chi(M^2),$$

where i^* is the homomorphism in cohomology (or homology) in dimension one induced by the inclusion map $i: f^{-1}(D_f) \to M^2$.

This is the generalization of Shepardson's theorem (1.6) to possibly nonorientable manifolds, and it implies the corresponding generalization of Hopf's theorem (1.3), which states that $\sum \{\delta_f : y \in \Delta_f\} \leq A(f)\chi(N^2) - \chi(M^2)$, as well as Kneser's theorem, which states that the right side is nonnegative.

Proof. Since $A(f) \neq 0$ the only possibilities are (1), (2), and (3) of (5.3). In case (3) the conclusion results from (5.10), so we may assume case (α) ($\alpha = 1, 2$). Thus $\tilde{M}$ and $\tilde{N}$ are orientable, so $|\deg \tilde{f}|$ is defined and $A(f) = (\text{index } f)|\deg \tilde{f}|$. Since p and s are α-to-1 covering maps and r is an (index f)-to-1 covering map, $\chi(\tilde{M}) = \alpha\chi(M)$ and $\chi(\tilde{N}) = \alpha\chi(\bar{N}) = \alpha(\text{index } f)\chi(N)$. Thus

(i) $$\alpha[A(f)\chi(N) - \chi(M)] = |\deg \tilde{f}|\chi(\tilde{N}) - \chi(\tilde{M}).$$

(ii) For each $\tilde{y} \in N$ and component Γ of $\tilde{f}^{-1}(y)$, $p^{-1}(\Gamma)$ consists of α components each mapped homeomorphically by p onto Γ — for $\alpha = 1$ (case (1)) p is a homeomorphism, and for $\alpha = 2$ use (5.11). For each $y \in D_f$, $f^{-1}(y)$ has a finite number of components, so for each $\tilde{z} \in s^{-1}(r^{-1}(y))$, $\tilde{f}^{-1}(\tilde{z})$ has a finite number of components. By (1.6) $D_{\tilde{f}}$ is finite, so by (5.12) D_f is finite also. Thus $B = s^{-1}(r^{-1}(D_f))$ is finite and $\tilde{f}^{-1}(B)$ has a finite number $\tilde{h}$ of components. By (5.14)

(iii) $$[(\#B)|\deg \tilde{f}| - \tilde{h}] + \text{rank } H^1(\tilde{L}) \leq |\deg \tilde{f}|\chi(\tilde{N}) - \chi(\tilde{M}),$$

where $\tilde{L} = \tilde{K} \cup \tilde{f}^{-1}(B)$ and $\tilde{K}$ is the union of the 2-cell components of $\tilde{M} - \tilde{f}^{-1}(B)$. From (5.12)

(iv) $$\alpha\left(\sum \{d_f(y) : y \in D_f\}\right) = (\#B)|\deg \tilde{f}| - \tilde{h}.$$

Thus from (i), (iii), and (iv),

(v) $$\alpha\left(\sum \{d_f(y) : y \in D_f\}\right) + \text{rank } H^1(\tilde{L}) \leq \alpha[A(f)\chi(N) - \chi(M)].$$

In case $\alpha = 1$ p is a homeomorphism, so $\tilde{L} \approx L$, and in case $\alpha = 2$,

$$2 \text{ rank } H^1(L) \leq \text{rank } H^1(\tilde{L})$$

by (5.15), and it follows from (v) that

(vi) $$\sum \{d_f(y) : y \in D_f\} + \operatorname{rank} H^1(L) \leq A(f)\chi(N) - \chi(M).$$

From the diagram

(vii)
$$\begin{tikzcd} H^1(M) \arrow[r] \arrow[dr] & H^1(L) \arrow[d] \\ & H^1(f^{-1}(D_f)) \end{tikzcd}$$

with homomorphisms induced by inclusion, rank imag $i^* \leq$ rank $H^1(L)$, and the desired conclusion results.

Remark 5.17. The generalization of (5.16) to maps $f : (M^2, \partial M^2) \to (N^2, \partial N^2)$ on compact 2-manifolds with boundary is true, if we assume that either a) $f^{-1}(\partial N^2) = \partial M^2$ or b) $D_f \cap \partial N^2 = \emptyset$. To see this, double the domain and range to obtain a map F on closed manifolds. Verify that $A(F) = A(f)$ (this requires careful checking); in case a) $A(\partial f) = A(f)$ and by (5.8) $D_{\partial f} = \emptyset$, so $D_f \cap \partial N = \emptyset$ in both cases a) and b). Use (5.16) (vi) for F to deduce (vi) for f; then use (vii) as above (see [23, p. 283, (3.3) and (3.4)]).

The generalization of (5.13) to maps $f : (M^n, \partial M^n) \to (N^n, \partial N^n)$ results easily.

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