THE CAUCHY PROBLEM FOR CONVOLUTION OPERATORS. UNIQUENESS

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SECTION 1

In this paper we shall discuss the uniqueness of the Cauchy problem for convolution equations in $\mathbf{R}^{n+1}$. The variables in $\mathbf{R}^{n+1}$ will be denoted by $(x,t) = (x_1, \ldots, x_n, t)$. The dual variables in $\mathbf{C}^{n+1}$ will be denoted by

$$z = (\xi, \eta) = (\xi_1, \ldots, \xi_n, \eta).$$

$\operatorname{Im} \xi$ stands for $(\operatorname{Im} \xi_1, \ldots, \operatorname{Im} \xi_n)$, and similar expressions for $\operatorname{Im} z$, $\operatorname{Re} \xi$, etc. The bracket $\langle z, w \rangle$ denotes the usual bilinear product in $\mathbf{C}^n$ or $\mathbf{C}^{n+1}$, according to the context, *e.g.*, $\langle \xi, x \rangle = \xi_1 x_1 + \ldots + \xi_n x_n$. The closed half-space $\{(x,t) \in \mathbf{R}^{n+1} : t \geq 0\}$ will be denoted by $\mathbf{R}_+^{n+1}$.

All the functions or distributions considered will always depend on $n+1$ variables, unless it is explicitly stated otherwise, *e.g.*, if we write a function Φ as $\Phi(x)$ it means that it depends only on the first n variables.

Let us recall that a convolution operator in the space $\mathscr{D}'$ of distributions is a linear continuous operator that commutes with the derivations. Using the standard notations of the theory of distributions ([13], [24]), every convolution operator in $\mathscr{D}'$ is defined by an element $\mu \in \mathscr{E}'$. A particular case of convolution operators are, of course, the partial differential operators with constant coefficients $P(D)$, where P is a complex polynomial in $n+1$ variables, and D stands for the differentiation vector $D = (D_x, D_t) = \left(-i \dfrac{\partial}{\partial x_1}, \ldots, -i \dfrac{\partial}{\partial x_n}, -i \dfrac{\partial}{\partial t}\right)$.

For differential operators, the Cauchy problem can be stated in the following form [12]; [13, Chapter V]:

$$\text{(1.1)} \qquad \begin{cases} \text{Given } f \in \mathscr{D}'(\mathbf{R}^{n+1}) \text{ with } \operatorname{supp} f \subset \mathbf{R}_+^{n+1}, \\ \text{find } g \in \mathscr{D}'(\mathbf{R}^{n+1}) \text{ with } \operatorname{supp} g \subset \mathbf{R}_+^{n+1}, \\ \text{such that } P(D) g = f \text{ in } \mathbf{R}^{n+1}. \end{cases}$$

Hence, the uniqueness problem reduces to study the existence of nontrivial solutions g of the homogeneous equation $P(D)g = 0$, with $\operatorname{supp} g \subset \mathbf{R}_+^{n+1}$.

A classical theorem of Holmgren states that the necessary and sufficient condition

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for the uniqueness of (1.1) is that the degree of the polynomial P in the variable η equals the total degree of P. In this case the direction t is called noncharacteristic for $P(D)$. In case the t-axis is characteristic, one can even find functions $g \in C^\infty(\mathbf{R}^{n+1})$, $g \neq 0$, such that $P(D)g = 0$ and $\operatorname{supp} g \subset \mathbf{R}^{n+1}_+$ [13, Theorems 5.2.2 and 5.3.1.].

Following Täcklind's result on the heat equation [25], the approach for the characteristic case has been to try to prove uniqueness for a class of smooth functions restricted by growth conditions [10].

The Cauchy problem for a convolution equation given by a distribution $\mu \in \mathscr{E}'$ can be defined in a way similar to (1.1), and in general the t-axis turns out to be *characteristic,* hence we reduce ourselves to study the following uniqueness problem. Find a subspace G of $C^\infty(\mathbf{R}^{n+1})$ such that if $g \in$ G and

$$\mu * g = 0, \qquad \operatorname{supp} g \subset \mathbf{R}^{n+1}_+ \tag{1.2}$$

then $g \equiv 0$.

In fact, we are interested in finding subspaces G that do not impose vanishing conditions at infinity on their elements, that is, we are interested in a true generalization of the result of Täcklind [25]. Following the previous work of Gelfand-Shilov, Ehrenpreis et al., one of the authors gave such uniqueness classes G in the case that

$$\mu * g = D_t^q g + (L_1 * D_t^{q-1} g) + \dots + (L_q * g), \tag{1.3}$$

where the distributions of compact support L_j act only on the variable x ([2]; [4, Chapter III]).

The convolutors μ of (1.3) fail to include the important class of all difference-differential operators. In trying to deal with them, one faces immediately examples like the following. In $\mathbf{R}^2$, consider the operator

$$\begin{aligned}(\mu * g)(x,t) &= g(x,t) + g(x,t-1) \\ &\quad + \frac{1}{4\pi^2}\cdot\frac{\partial^2 g}{\partial x^2}(x-1,t) + \frac{1}{4\pi^2}\cdot\frac{\partial^2 g}{\partial x^2}(x-1,t-1),\end{aligned} \tag{1.4}$$

which has as solutions of the associated homogeneous equation the functions of the form $g(x,t) = \beta(t)\cdot\sin 2\pi x$, for any $\beta \in C^\infty(\mathbf{R})$. Therefore there is no hope of finding a uniqueness class G unless we impose some kind of vanishing condition at infinity. Let us observe that the fact that the order of the operator in the direction t is less than the total order of μ plays no role, since the functions $g(x,t) = \gamma(x)\cdot\cos \pi t$, γ arbitrary smooth function, are also solutions of the homogeneous equation.

An easy computation shows that in this case the zero set of the Fourier transform $\hat{\mu}$ contains a complex line parallel to the η-axis in $\mathbf{C}^2$. In general, it is clear that if $\mu \in \mathscr{E}'(\mathbf{R}^{n+1})$ is such that the zero set of $\hat{\mu}$ contains a complex line in $\mathbf{C}^{n+1}$ of the form

$$\{z = (\xi_0, \eta) : \eta \in \mathbf{C}\}, \tag{1.5}$$

then the functions

$$g(x,t) = \beta(t) \cdot \exp(i\langle \xi_0, x\rangle) \tag{1.6}$$

are solutions of $\mu * g = 0$, no matter which $\beta \in C^\infty(\mathbf{R})$ we pick.

It is a natural question then to find out whether the existence of solutions of the form

$$g(x,t) = \alpha(x) \cdot \beta(t) \tag{1.7}$$

for some fixed α, and β arbitrary, can be characterized in a simple manner in terms of the analytic variety $\{\hat{\mu} = 0\}$. This is the case in $\mathbf{R}^2$. In fact, one can easily prove, using the spectral synthesis theorem of L. Schwartz ([24]; see also [17]).

PROPOSITION 1.1. *Let* $\mu \in \mathscr{E}'(\mathbf{R}^2)$. *The following two conditions are equivalent:*

(i) *There exists a nontrivial function* $\alpha \in C^\infty(\mathbf{R})$, *such that for every* $\beta \in C^\infty(\mathbf{R})$, $\mu * (\alpha \otimes \beta) = 0$.

(ii) *There exists* $\xi_0 \in \mathbf{C}$, *such that for every* $\eta \in \mathbf{C}$, $\hat{\mu}(\xi_0, \eta) = 0$.

The failure of the spectral synthesis in $\mathbf{R}^n$, for $n \geq 2$, allows us to construct an example showing that Proposition 1.1 does not extend to higher dimensions. Let us fix $n \geq 2$. Consider the six distributions $\mu_1, \dots, \mu_6 \in \mathscr{E}'(\mathbf{R}^n)$ and the nontrivial $\alpha \in C^\infty(\mathbf{R}^n)$ constructed by Gurevich [11], they satisfy $\mu_1 * \alpha = \dots = \mu_6 * \alpha = 0$, and $\bigcap_{j=1}^{6} \{\xi \in \mathbf{C}^n : \hat{\mu}_j(\xi) = 0\} = \emptyset$. Pick six arbitrary distinct numbers $\eta_j \in \mathbf{C}$, and define $\mu \in \mathscr{E}'(\mathbf{R}^{n+1})$ by

$$\hat{\mu}(\xi, \eta) = \sum_{k=1}^{6} \hat{\mu}_k(\xi) \cdot \prod_{j \neq k} (\eta - \eta_j). \tag{1.8}$$

It is clear that the functions of the form (1.7) are solutions of the homogeneous equation $\mu * g = 0$, but no complex line like (1.5) is contained in the zero set of $\hat{\mu}$. Let us also observe that this μ is almost of the form (1.3), the only difference being that the coefficient of D_t^5 is not a constant.

In view of this discussion, we will only study distributions μ such that the zero set of $\hat{\mu}$ is contained in an angular region of the form $|\eta| \leq \exp \Lambda(\xi)$, for some positive function Λ. This condition automatically excludes the existence of solutions of the form (1.6), but not of the form (1.7), as the example (1.8) shows. Hence the question becomes now to find the uniqueness class G in terms of the angular opening Λ. In case of differential operators, Λ is of the form

$$\Lambda(\xi) = p \log(1 + |\xi|) + r,$$

and the condition $p > 1$ is equivalent to the t-axis being a characteristic direction [8].

In the example (1.3), $\Lambda(\xi) = p\log(1 + |\xi|) + m|\mathrm{Im}\,\xi| + r$, for some positive constants p, m, r.

The techniques used proving the results in Sections 4 and 5 are those of [2]. The new ingredients are the nontriviality of certain spaces of entire functions (*cf.* Sections 2 and 3) and a Fourier representation theorem for the solutions of the homogeneous convolution equation, more powerful than the one available in [4]. Namely, the results in [3] and [5] show that if $\hat{\mu}$ is slowly decreasing [8], a condition that we will assume from now on, and G is one of the analytically uniform (AU) spaces [8] of smooth functions considered below, then the variety V of zeros of $\hat{\mu}$ can be written as a union $V = \bigcup_j V_j$ of countably many closed sets V_j, and there exist differential operators ∂_j, such that an extension of the Fundamental Principle of Ehrenpreis [8] holds. This last principle states that if $g \in \mathrm{G}$ solves the equation $\mu * g = 0$, there are measures m_j and a function κ in the AU structure of G such that for any $S \in \mathrm{G}'$

$$(1.9) \qquad \langle g, S\rangle = S(g) = \sum_j \int_{V_j} \partial_j \hat{S}(z) \frac{dm_j(z)}{\kappa(z)}.$$

Given $\hat{S} \in \hat{\mathrm{G}}$, we sometimes write $\langle g, \hat{S}\rangle$ instead of $\langle g, S\rangle$.

It also follows from [5] and [3] that for g as in (1.9), there is a κ_0 in the AU structure, such that

$$(1.10) \qquad |\langle g, S\rangle| \le \max_{z\in V} \left\{ \frac{|\hat{S}(z)|}{\kappa_0(z)} \right\}.$$

To finish this section, we should mention that the uniqueness in classes G given by conditions of decrease at infinity has been considered in [28]. Related problems in differential equations have been considered in [19], [1], etc., and also in a very interesting paper on convolution equations by F. John [16], that is rarely mentioned in the literature.

SECTION 2

In this section we will introduce certain spaces of smooth functions with growth conditions which will be used in Section 4. The main references throughout this section are [4], [8], [9], [10].

Definition 2.1. Let $\Phi\colon \mathbf{R}^n \to [0, +\infty)$, be convex and even in each variable separately, with $\Phi(0) = 0$ and $\dfrac{\Phi(x)}{|x|} \to \infty$, when $|x| \to \infty$. We call $\mathscr{E}(\Phi)$ the space of all C^∞ functions g on $\mathbf{R}^n$ such that for every index $\alpha \in \mathbf{N}$ and $\varepsilon > 0$, there is a positive constant $c = c(\alpha, \varepsilon, g)$ for which

$$|D^{\alpha} g(x)| \leq c \exp \ \Phi(\varepsilon x), \qquad x \in \mathbf{R}^n \tag{2.1}$$

With the natural topology induced by the inequality (2.1), this space is an analytically uniform (AU) Fréchet-Montel space ([4], [8], [22], [26]). In particular, every element S of its dual space $\mathscr{E}'(\Phi)$ has a Fourier transform $\hat{S}$, that extends to an entire function in $\mathbf{C}^n$, satisfying an estimate of the form

$$|\hat{S}(z)| \leq C(1 + |z|)^N \exp \Phi^*(d \cdot \operatorname{Im} z), \tag{2.2}$$

where Φ^* is the Young conjugate of Φ,

$$\Phi^*(y) = \max_x (\langle x,y \rangle - \Phi(x)). \tag{2.3}$$

Let us recall that every entire function satisfying an estimate of the form (2.2) belongs to $\hat{\mathscr{E}}'(\Phi)$.

The AU structure of $\mathscr{E}(\Phi)$ is given by continuous positive functions κ in $\mathbf{C}^n$, that dominate the right hand side of (2.2) for any N and $d > 0$.

Definition 2.2. Corresponding to the above function Φ in $\mathbf{R}^n$, we also define a space $\mathscr{E}_0(\Phi)$ of functions in $\mathbf{R}^{n+1}$ in the following way. Let

$$\Phi_0(t) = \begin{cases} 0 & \text{for } |t| \leq 1 \\ +\infty & \quad \text{otherwise} \end{cases}$$

and $\tilde{\Phi}(x,t) = \Phi_0(t) + \Phi(x)$, for $t \in \mathbf{R}$, $x \in \mathbf{R}^n$. Then $\mathscr{E}_0(\Phi) = \mathscr{E}(\tilde{\Phi})$, where the last space is defined as above. Let us remark that

$$\tilde{\Phi}^*(y,s) = \Phi_0^*(s) + \Phi^*(y) = |s| + \Phi^*(y).$$

With this observation, it is easy to describe the AU structure of $\mathscr{E}_0(\Phi)$.

Remark 2.3. Before going any further, we would like to point out that the dependency on ε in the inequality (2.1) has been introduced only to obtain a Montel space. This condition is not necessary in what follows. In the slightly more general case, obtained by dropping the ε, one deals with spaces that are inductive limits, with no essential changes in the arguments (see [22]).

We want to introduce certain subspaces of $\hat{\mathscr{E}}'(\Phi)$. For quick references, we will follow the notation of [10], hence we will deal first with the case of a single complex variable and explain below how the general case can be reduced to this one.

Definition 2.5. Let M, Ω denote a pair of continuously differentiable, nonnegative, convex and even functions of a real variable, such that $M(0) = \Omega(0) = 0$ and they are larger at infinity than any linear function.

We denote by W_M^{Ω} the set of all entire analytic functions $\varphi(z)$ which satisfy the inequality

$$|\varphi(x + iy)| \leq C \exp[-M(bx) + \Omega(ay)], \tag{2.4}$$

where the positive constants a, b and C depend on the function $\varphi(z)$ ($z = x + iy$).

The main question concerning these spaces is whether they are trivial or not. The best known examples of nontrivial spaces are those where for $x > 0$,

$$M(x) = \Omega(x) = l\,(x) \cdot x^p,$$

$p > 1$ and l a slowly increasing function ([10], [18]).

We are here interested in the nontriviality of spaces where both Ω and M are larger than any power of x. By means of the Fourier transform one reduces this problem to the nontriviality of spaces $W_{\Omega^*}^{M^*}$ ([9], [10]). It is easy to see then that we are precisely in the limiting case of the above mentioned set of examples, namely both M^* and Ω^* are now functions dominated by $x^{1+\varepsilon}$, for every $\varepsilon > 0$, but they still grow faster than any linear function at infinity. The theorem in Section 3 gives as a corollary the nontriviality of a large class of such spaces. For the sake of simplicity we prove here the following case.

THEOREM 2.6. *The spaces $W_{\Omega^*}^{M^*}$ are nontrivial for $M^*(r) = r\,(\log r)^k$ and $\Omega^* = r(\log r)^{k-1}$, if $k > 1$. Hence, the spaces W_M^{Ω} with $\Omega(\xi) = \exp(\xi^{1/(k-1)})$ and $M(\xi) = \exp(\xi^{1/k})$ are nontrivial also. (It is understood that the above expressions hold only for large r and ξ).*

Proof. Theorem 3.7 will show that if $k > 1$, there exists a nontrivial entire function f satisfying for some positive constants a, b, c,

$$|f(z)| \leq c \exp(ar(\log(1 + r))^k),$$

where $r = |z|$, $z \in \mathbf{C}$, and $|f(x)| \leq c\exp(-b|x|(\log(1 + |x|))^{k-1})$, for $x \in \mathbf{R}$. Using the Phragmén-Lindelöf principle, one can conclude that $f \in W_{\Omega^*}^{M^*}$.

Remark 2.7. It is easy to see that the spaces $W_{\Omega^*}^{M^*}$ with $M^*(r) = O(r\log r)$ are trivial. This is due to the requirement that $\Omega^*(r)/r \to \infty$ as $r \to \infty$, imposed on us by the fact that in the applications M is given as a convex function very big at infinity and Ω is then found, $M(r) = O(\Omega(r))$. The triviality of those spaces follows from Theorem 6.3.6 of [7]. Namely, that theorem states that if F is an analytic function in the halfplane $y > 0$, continuous up to $y = 0$, such that

$$\liminf_{r\to\infty} \frac{1}{r}\log(\max\{|F(z)|: |z| \leq r, \operatorname{Im} z \geq 0\}) < \infty, \text{ and} \tag{2.5}$$

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)|}{1 + x^2}\,dx < \infty \tag{2.6}$$

hold, then it also holds

$$\int_{-\infty}^{\infty} \left|\frac{\log|F(x)|}{1 + x^2}\right| dx < \infty. \tag{2.8}$$

If $f \in W_{\Omega^*}^{M^*}$ is not identically zero, it satisfies for some $a, b, c > 0$

$$|f(x + iy)| \leq c\exp[a|y|\log|y| - \Omega^*(b|x|)],$$

hence, choosing the branch of $\operatorname{Arg} z$ in $(-\pi/2, 3\pi/2)$, we can see that $F(z) = f(z)\exp(iaz\log z)$ satisfies (2.5) and (2.6) but fails (2.8).

COROLLARY 2.8. *If $e^r = O(M(r))$ then the space W_M^Ω is trivial, no matter which Ω one chooses.*

Let us observe that there always are nontrivial entire functions satisfying $|f(x)| \le c\exp(-M(bx))$ for some c, $b > 0$, that still satisfy the growth condition $|f(z)| = O(\exp(M(b'|z|)))$ for some b' (at least when $|x|^a = O(M(x))$ for every $a > 0$). What the Corollary 2.8 states is that they cannot satisfy a bound of the type (2.4).

Remark 2.9. Theorem 2.6 also follows from the results in [21]. (The authors are indebted to Professor M. Schiffer for pointing out this paper to us.) It is shown there that if $0 \le C(r)$ is nondecreasing on $[0,\infty)$, then there is an entire function $f \not\equiv 0$ such that

$$|f(z)| \le \exp\left[\frac{4}{\log 2}\left(|y|\int_0^{|y|}\frac{C(u)}{u^2}\,du + y^2\int_{|y|}^{\infty}\frac{C(u)}{u^3}\,du\right) - C(|z|)\right],$$

which, of course, is meaningful if $\int_1^\infty C(u)\,u^{-3}\,du < \infty$. A direct computation with $C(r) = \Omega^*(r) = r(\log r)^{k-1}$, $k > 1$ (for large r) shows that $f \in W_{\Omega^*}^{M^*}$, $M^*(r) = r(\log r)^k$. The emphasis in this result of S. Mandelbrojt is that given Ω^* one can fine one M^* that works, while as pointed out in Remark 2.8 in the application in Section 4 one goes the other way around. Besides, it is not always a priori true that given M^* can find Ω^*. Nevertheless, by inverting the transform

$$C(r) \to \int_0^r C(u)\,u^{-2}\,du + r^2\int_r^\infty C(u)\,u^{-3}\,du$$

one sees that a result similar to Theorem 3.7 can also be obtained as a corollary of [21].

Now we are ready to consider the case of several variables. The conditions on the function Φ above show that we can replace it in the definition of the space $\mathscr{E}(\Phi)$ by another function of the form $\Phi_1(x_1) + \ldots + \Phi_n(x_n)$ [4, Chapter 2]. Hence, the space $W_{M_1}^{\Omega_1} \otimes \ldots \otimes W_{M_n}^{\Omega_n} \subset \hat{\mathscr{E}}'(\Phi)$ if $\Omega_j = \Phi_j^*$. As it is shown in [4], [9], [10], once the spaces W_M^Ω are nontrivial, they are dense. We collect this information in the next proposition.

PROPOSITION 2.10. *Let $\Phi(x) = |x|\log^{k-1}(1+|x|)$, $k > 1$ $(x \in \mathbf{R}^n)$, $M(r) = \exp r^{1/k}$, $\Omega(r) = \exp r^{1/(k-1)}$ $(r > 0)$. Given $b_0 > 0$, the set of linear combinations of entire functions of the form*

$$F(z) = e^{id\eta}f(\xi) \tag{2.9}$$

is dense in $\hat{\mathscr{E}}_0'(\Phi)$, where $d \in \mathbf{R}$ and f is entire in $\mathbf{C}^n$ and satisfies

$$|f(\xi)| \le C\exp[-M(b|\operatorname{Re}\xi|) + \Omega(a|\operatorname{Im}\xi|)] \tag{2.10}$$

for some $b \ge b_0$, $a, C > 0$ $(z = (\xi,\eta) \in \mathbf{C}^{n+1})$.

SECTION 3

In this section we prove the complex variable result used in Theorem 2.6, though in a more general setting for future references. This kind of result cannot be obtained from a setup of the kind in [14], Theorem 4.4.7, since the growth conditions imposed are not written in terms of subharmonic functions. It is for this reason that the only available method of proof is a direct construction, as we do here (see also Remark 2.9).

Definition 3.1. A positive differentiable function $\rho(r)$ defined on $(0, +\infty)$ is called a *proximate order* if it satisfies the following two conditions:

$$\lim_{r\to\infty} \rho(r) = 1. \tag{3.1}$$

$$\lim_{r\to\infty} r \cdot \rho'(r) \log r = 0. \tag{3.2}$$

We will restrict ourselves to proximate orders of the form

$$r^{\rho(r)} = r \cdot \log r \cdot \eta(r), \qquad r > 1. \tag{3.3}$$

where η will be a nonnegative increasing unbounded concave function with the additional restrictions (3.4) and (3.5).

$$\lim_{r\to\infty} r \cdot \frac{\eta'(r)}{\eta(r)} = 0, \tag{3.4}$$

or equivalently: for all $\varepsilon > 0$, $\eta(r) = O(r^{\varepsilon})$, when $r \to \infty$.

(3.5) The function $\mu(r) = \eta(r) + r \cdot \log r \cdot \eta'(r)$ is increasing.

We define an auxiliary function λ by the formula

$$\lambda^{-1}(s) = s \cdot \mu^{-1}(s), \tag{3.6}$$

for large values of s. This definition makes sense by condition (3.5), and λ satisfies the equation

$$\lambda(r \cdot \mu(r)) = \mu(r). \tag{3.7}$$

LEMMA 3.2. *The nonnegative function* $r \cdot \dfrac{\mu'(r)}{\mu(r)}$ *is bounded above.*

Proof. $r \cdot \mu' = 2r \cdot \eta' + r \cdot \log r \cdot \eta' + r^2 \cdot \log r \cdot \eta''$, hence

$$\frac{r \cdot \mu'}{\mu} = \text{(i)} + \text{(ii)} + \text{(iii)},$$

where

$$\text{(i)} = \frac{2r \cdot \eta'}{\eta + r \cdot \log r \cdot \eta'} \leq \frac{2r \cdot \eta'}{\eta} \longrightarrow 0 \text{ by (3.4).}$$

$$\text{(ii)} = \frac{r \cdot \log r \cdot \eta'}{\eta + r \cdot \log r \cdot \eta'} \leq 1, \text{ since } \eta \text{ is nonnegative.}$$

$$\text{(iii)} = \frac{r^2 \cdot \log r \cdot \eta''}{\eta + r \cdot \log r \cdot \eta'} \leq 0 \text{ by the concavity of } \eta.$$

COROLLARY 3.3. *There is a constant γ such that*

$$t \cdot \frac{\lambda'(t)}{\lambda(t)} \leq \gamma < 1 \; (\textit{for large } t). \tag{3.8}$$

Proof. $t \cdot \left.\dfrac{\lambda'(t)}{\lambda(t)}\right|_{t=r\cdot\mu(r)} = \dfrac{r \cdot \dfrac{\mu'(r)}{\mu(r)}}{1 + r \cdot \dfrac{\mu'(r)}{\mu(r)}}$, hence (3.8) follows from the previous lemma.

Remark 3.4. We would like to point out that the condition that η is concave can be replaced by the weaker condition (iii) *is bounded,* and the lemma and its corollary will still hold.

Remark 3.5. The function $\psi(r)$ defined by

$$\psi(r) = r \cdot \mu(r) \tag{3.9}$$

is increasing and satisfies $\dfrac{\psi(r)}{r} \longrightarrow \infty$. The additional assumption η'' increasing for large values of the argument makes $\psi(r)$ convex for large values of r.

Remark 3.6. The conditions on η can be written more simply by introducing the function φ defined by $\eta(t) = \varphi(\log t)$. In terms of this function, these conditions read as follows:

(i) $\eta \nearrow \infty \Leftrightarrow \varphi \nearrow \infty$

(ii) $t \cdot \dfrac{\eta'(t)}{\eta(t)} \longrightarrow 0 \Leftrightarrow \dfrac{\varphi'(s)}{\varphi(s)} \longrightarrow 0$

(iii) $\mu(r)$ increasing $\Leftrightarrow \varphi(s) + s \cdot \varphi'(s)$ increasing

(iv) η concave $\Leftrightarrow \varphi'(s) \cdot e^{-s}$ decreasing

(v) η'' increasing $\Leftrightarrow \varphi''' - 3\varphi'' + 2\varphi \geq 0$.

These conditions are to be satisfied for large values of s.

THEOREM 3.7. *If $\rho(r)$ is a proximate order and the function $\eta(r)$ defined by* (3.3) *satisfies the above mentioned conditions* (3.3) – (3.5), *then there exists* *an even entire function $f \not\equiv 0$ such that for some positive constants a, b, c,*

$$|f(z)| \leq c \cdot \exp(ar^{\rho(r)}) \tag{3.10}$$

where $r = |z|$, $z \in \mathbf{C}$,

$$|f(x)| \leq c \cdot \exp(-b\,\psi(|x|)), \tag{3.11}$$

$x \in \mathbf{R}$, *and* ψ *defined by* (3.9). *Moreover, we can take* $f(x)$ *real for* $x \in \mathbf{R}$.

Proof. With the notation from (3.7) we define a sequence of positive numbers (a_n),

$$a_n = n/\lambda(n). \tag{3.12}$$

It is easy to see that $\sum \dfrac{1}{a_n} = +\infty$, and $\sum \left(\dfrac{1}{a_n}\right)^{1+\varepsilon} < +\infty$, for every $\varepsilon > 0$. In fact, a_n can also be defined as the solution r of the equation $\psi(r) = n$, and then one can use that $r = O(\psi(r))$ and $\psi(r) = O(r^{1+\varepsilon})$ for every $\varepsilon > 0$. We consider the function $f(z)$ defined by the formula,

$$f(z) = \prod_{n=1}^{\infty} \frac{\sin(\pi z/a_n)}{\pi z/a_n}. \tag{3.13}$$

The fact that f is an entire function and has the right order of growth can be proved either by observing that $\dfrac{\sin z}{z} = 1 - \dfrac{z^2}{6} + \ldots$ and it is $O(\exp(|\operatorname{Im} z|))$; or by computing the density of the zeroes of f since (3.13) already gives f as a canonical product with zeroes at $\pm m\,a_n$ $(m,n = 1,2,\ldots)$ and then applying Theorem 18 of [18, Chapter I, page 45]. We will follow the second approach.

Denoting by $n(r)$ the number of positive zeroes of absolute value less than or equal to r, we see

$$n(r) \leq \sum_{a_n \leq r} \frac{r}{a_n} \leq 2 \cdot n(r), \tag{3.14}$$

hence by the above mentioned theorem of Levin, the order of f is given by the middle term of (3.14). Let us call $N = \psi(r)$, that is, $N/\lambda(N) = r$. Then

$$\sum_{a_n \leq r} \frac{r}{a_n} = r \cdot \sum_{n \leq [N]} \frac{1}{a_n} = O(r) + r \cdot \int_1^N \frac{\lambda(t)}{t} \cdot dt.$$

We want to show that this expression is $O(r^{\rho(r)})$. In fact, we have from Lemma 3.2 and the definition of μ,

$$\frac{dN}{dr} = \mu(r) + r \cdot \mu'(r) \leq C \cdot \mu(r) = C \cdot \left[\frac{d}{dr}(r^{\rho(r)}) - r^{\rho(r)-1}\right].$$

Hence

$$\frac{d}{dr}\int_l^N \frac{\lambda(t)}{t}\cdot dt \leq C\cdot\left[\frac{1}{r}\cdot\frac{d}{dr}(r^{\rho(r)}) - r^{\rho(r)-2}\right]$$

and finally, by integration by parts, it follows that

$$\int_l^N \lambda(t)/t\cdot dt \leq C\cdot r^{\rho(r)-1} + C_1.$$

From the above computations, it also follows that the estimate $r^{\rho(r)}$ for the order of f is sharp, and f is of finite type with respect to this order. Thus (3.10) has been proved.

In order to prove (3.11), it is enough to give a lower estimate for the expression $A(r) = N\cdot\log r - \sum_{n\leq N}\log a_n$, where $r > 0$ is large and N has been defined above by $N = \psi(r)$. In fact, for $x\in\mathbf{R}$, $\log|f(x)| \leq -A(\pi|x|)$. Since $t/\lambda(t)$ is an increasing function,

$$\int_l^N \log(t/\lambda(t))\cdot dt \leq \sum_{n\leq N}\log a_n \leq \int_1^N \log(t/\lambda(t))\cdot dt + \log(N/\lambda(N)).$$

Integrating by parts, we obtain

$$A(r) = N - \int_1^N t\cdot\lambda'(t)/\lambda(t)\cdot dt + O(\log r).$$

From (3.8) we have $N \geq A(r) \geq (1-\gamma)\cdot N + O(\log r)$, hence for some $\delta > 0$, we have $A(r) \geq \delta\cdot N$, r sufficiently large. Recalling that $N = \psi(r)$, we obtain (3.11) and the theorem has been proved.

SECTION 4

We are finally ready to tackle the uniqueness question for the Cauchy problem, using the material from Section 2. Let us recall from Section 1 that in dealing with the Cauchy problem (1.2) the zero set of the Fourier transform $\hat{\mu}$ of the convolutor μ lies in the angular region

(4.1) $$\{z = (\xi,\eta)\in\mathbf{C}^{n+1}: |\eta| \leq \exp\Lambda(\xi)\}.$$

Here, we study first the case where

(4.2) $$\Lambda(\xi) = \alpha|\xi|^{1/k} + \beta,$$

for some $\alpha, \beta > 0$. The value of k will be restricted in Theorem 4.1 below.

THEOREM 4.1. *Let $\hat{\mu}$ be slowly decreasing, if its zero set lies in the angle defined by* (4.1) *and* (4.2), *with $k > 1$, there is uniqueness for the Cauchy problem within the class $\mathscr{E}_0(\Phi)$, for*

$$\Phi(x) = |x| \cdot (\log(1 + |x|))^{k-1}$$

Proof. The proof follows well known lines (see [4], notes to Chapter 3, for further references). Given F as in (2.9), one defines

$$H(s; z) = e^{isd\eta} \cdot f(\xi), \tag{4.3}$$

for $0 \leq s \leq 1$. (We pick $b_0 = 2\alpha^k$). As a function of z, H is in $\mathscr{E}'_0(\Phi)$, hence if $g \in \mathscr{E}_0(\Phi)$ we can define, following Section 1,

$$h(s) = \langle g(x,t), H(s;z)\rangle. \tag{4.4}$$

The proof of the theorem consists in showing that if $\mu * g = 0$, then h is a real analytic function of s. In fact, if this were true and $g(x,t) \equiv 0$ for $t \leq 0$, then one obtains $\dfrac{d^m}{ds^m} h(0) = \left\langle g, \dfrac{\partial^m}{\partial s^m} H(0; z)\right\rangle = 0$, since from (4.3) it is obvious that $\dfrac{\partial^m}{\partial s^m} H(0; z)$ is the Fourier transform of a distribution acting only on the restriction of g and its derivatives to the hyperplane $t = 0$. By the real analyticity of h, it follows $h(1) = 0 = \langle g, F\rangle$. By Proposition 2.10, $g \equiv 0$.

From the representation formula (1.9) we can compute formally the derivatives of $h(s)$ by the formula

$$h^{(l)}(s) = \sum_j \int_{V_j} \partial_j [(id\eta)^l \cdot H(s;z)] \frac{dm_j(z)}{\kappa(z)}.$$

From the nature of this formula, it is enough to find an estimate of the form

$$|\eta|^l \cdot |e^{isd\eta}| \cdot |f(\xi)| \leq C_1 \cdot C_2^l \cdot l!\, \kappa_0(z)$$

in the angle defined by (4.1), (4.2) (see (1.10)). Now,

$$\begin{aligned} |\eta|^l \cdot |e^{isd\eta}| \cdot |f(\xi)| \leq C \cdot \exp[\alpha\, l\, |\xi|^{1/k} + \beta\, l + |d| \cdot |\mathrm{Im}\, \eta| \\ + \Omega(a|\mathrm{Im}\, \xi|) - M(b|\mathrm{Re}\, \xi|)]. \end{aligned} \tag{4.5}$$

We have to estimate the last expression in two cases. First, let us assume $|\mathrm{Im}\, \xi| \leq |\mathrm{Re}\, \xi|$, hence for some constants $C_3, C_4 > 0$, we have the right hand side of (4.5) is less or equal to:

$$C_3 C_4^l \cdot \kappa_0(z) \cdot \exp[\alpha \cdot 2^{1/k} \cdot l \cdot |\mathrm{Re}\, \xi|^{1/k} - M(b|\mathrm{Re}\, \xi|)].$$

Introducing the notation $r = 2\alpha^k \cdot |\mathrm{Re}\, \xi|$ and recalling the definition of M, we see that for large r, the expression in brackets is bounded by

$$\max_{r>>1} \left[l \cdot r^{1/k} - \exp\left(r^{1/k}\right) \right],$$

since $b \geq b_0 = 2\alpha^k$. A direct computation using the Stirling formula shows that we have obtained the correct estimation in this case.

The second case, $|\mathrm{Re}\,\xi| \leq |\mathrm{Im}\,\xi]$, reduces to the same computation as the one just done, since by adding and substracting the term $\Omega(b_0|\mathrm{Im}\,\xi|)$ to the bracketed expression in the right hand side of (4.5), we obtain the right hand side of (4.5) is less or equal to:

$$C_5 C_6^l \cdot \kappa_0(z) \cdot \exp\left[\alpha \cdot 2^{1/k} \cdot l \cdot |\mathrm{Im}\,\xi|^{1/k} - \Omega\left(b_0 |\mathrm{Im}\,\xi|\right)\right],$$

for some C_5, $C_6 > 0$.

The above estimates of $h^{(l)}(s)$ show that h can be extended to an holomorphic function in a complex neighborhood of the interval $[0,1]$, and this finishes the proof of the theorem.

Without extra effort, the proof above yields the following theorems.

THEOREM 4.2. *If condition* (4.1) *on the zero set of* $\hat{\mu}$ *is replaced by*

$$\{z = (\xi, \eta) \in \mathbf{C}^{n+1} : |\mathrm{Re}\,\eta| \leq \exp \Lambda(\xi)\} \tag{4.1'}$$

then the conclusion of Theorem 4.1 holds.

THEOREM 4.3. *If the conditions* (4.1) *and* (4.2) *on the zero set of* $\hat{\mu}$ *are replaced by* (4.1′) *and*

$$\Lambda(\xi) = \alpha |\mathrm{Re}\,\xi|^{1/k} + \beta |\mathrm{Im}\,\xi| + \gamma \tag{4.2'}$$

$(\alpha, \beta, \gamma > 0)$, *then the conclusion of Theorem 4.1 holds, when* Φ *is defined by*

$$\Phi(x) = |x| \cdot (\log(1 + |x|))^{k*},$$

where $k_* = \min\{1, k-1\}$. *(We still require* $k > 1$*).*

Remark 4.4. As pointed out in [4], Notes to Chapter III, it is not necessary to assume that μ is a distribution with compact support, but only that μ be a slowly decreasing distribution in $\mathscr{E}_0'(\Phi)$, for the Φ corresponding to the given k.

Remark 4.5. Theorem 4.3 can be regarded as an improvement on Example 1 in Chapter III of [4].

If we restrict ourselves to angular regions of the form (4.1) or (4.1′) and obtain uniqueness theorems in classes involving only growth conditions, the restriction $k > 1$ in (4.2) or (4.2′) seems to be necessary. The case where $0 < k \leq 1$ or even Λ bigger than such an exponential, will be dealt with in the next section, by imposing quasi-analytic conditions in the x-direction.

SECTION 5

In order to study the uniqueness for arbitrary functions Λ in (4.1), we introduce quasi-analytic classes $\mathscr{E}_B$. We follow essentially the notation in [8] and [4], though it should be noted that we impose restrictions only on the derivatives in the direction x. With respect to the dependency on ε below, we refer to Remark 2.3.

Definition 5.1. Let $B = \{b_l\}$ be a convex sequence, *i.e.*, $b_l = \exp G(l)$, where $G: \mathbf{R}_+ \to \mathbf{R}_+$ is a convex function such that $G(r)/r \to \infty$ when $r \to \infty$. We say that the C^∞-function g in $\mathbf{R}^{n+1}$ belongs to $\mathscr{E}_B$ if for any $\varepsilon > 0$, $R > 0$, *and* m, nonnegative integer, there exists a constant $C = C(\varepsilon, R, m; g) > 0$ such that

$$|D_t^m D_x^j g(x,t)| \le C \cdot \varepsilon^{|j|} b_{|j|}, \tag{5.1}$$

whenever $|x|, \le R$, $|t| \le R$ and j is an arbitrary n-index.

As pointed out in the references above, $\mathscr{E}_B$ is an AU-space with AU structure functions $\kappa(z)$ larger than any function of the form

$$(1 + |z|)^N \cdot \lambda(\alpha\xi) \cdot \exp(\beta|\operatorname{Im} z|), \tag{5.2}$$

where $\alpha, \beta, N > 0$ an $\lambda = \lambda_B$ is the function associated to the sequence B by the formula

$$\lambda(\xi) = \sum_l (|\xi|^l / b_l). \tag{5.3}$$

Conversely, a way of generating a sequence B as above is the following. Let $T(r)$ be a positive strictly increasing function of $r \ge 0$ such that $\log(T(r))$ is a convex function of $\log r$ and satisfying for all $l \ge 0$, $\lim_{r\to\infty} \dfrac{r^l}{T(r)} = 0$. One defines the sequence $B = \{b_l\}$ by

$$b_l = \max_{r \ge 0} \frac{r^l}{T(r)}. \tag{5.4}$$

It can be shown that the relation between λ_B and T is given by

$$\log(T(r)) - \log(1 + r) - c_1 \le \log(\lambda_B(r)) \le \log(T(2r)) + c_2, \tag{5.5}$$

for some positive constants c_1, c_2 ([20], [15]). Whenever we write $T(r)$ in this section, it is assumed that T satisfies the above conditions.

In what follows we assume that the function Λ appearing in the definition of the angular region (4.1′) is a function only of the absolute value of ξ. We furthermore assume that it is an increasing function of $|\xi|$. Hence the function $\nu^{-1}(r)$ is defined for large r if we take

$$\nu(s) = \exp(\Lambda(s)), \qquad s \geq 0. \tag{5.6}$$

THEOREM 5.2. *Assume that the zero set of $\hat{\mu}$ is contained in the angle* (4.1′) *given by a function Λ of $|\xi|$ as above. Assume further that T is such that the function F defined for large values of r by*

$$F(r) = T(\nu^{-1}(r)) \tag{5.7}$$

satisfies

(i) $\log F(r)$ *is a convex function of* $\log r$,

(ii) $r^l / F(r) \to 0$ *when* $r \to \infty$ *for all* $l > 0$, *and*

(iii) $\displaystyle\int_1^\infty \frac{\log F(r)}{r^2} \cdot dr = \infty.$

Then the space $\mathscr{E}_B$, with B associated to T via (5.4), *is a uniqueness class for the Cauchy problem* (1.2).

The proof follows the same lines as that of theorem 4.1. It is even slightly simpler, though one has to replace the real analyticity of h by a quasi-analytic condition.

Let us just compute as an example a sequence B associated to $\Lambda(\xi) = |\xi|$. In this case one has to take $T(r) = \exp(\tau(r))$, τ convex, positive function, $\tau(r)/r \to \infty$, and $\displaystyle\int_0^\infty \tau(r) \cdot e^{-r} \cdot dr = \infty$. A typical example would be given by $\tau(r) = e^r$. The sequence $\{b_l\}$ behaves essentially like $(\log l)^l$ and the smooth functions in the space $\mathscr{E}_B$ are such that in the x-direction they extend as holomorphic functions of order 1, with the bounds on the Taylor coefficients depending uniformly on t, for t bounded.

This example is not atypical, the analyticity and the order condition forced on us by the sequence B should be compared to the example (1.8), since from the construction of Gurevich, it can be seen that the function α could be chosen to be holomorphic, in this case it is the order condition that excludes such an example. Another example of nonuniqueness that just misses the conditions in Theorem 5.2 is the classical example of Tihonov on the heat equation [27]. In this case it is uniformity with respect to t of the order conditions that fails.

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