ON DECOMPOSITION OF OPERATORS

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Throughout this paper, T denotes a bounded linear operator defined on a Banach space X, and R(T) and N(T) denote the range and the null space of T, respectively.

The following theorem, due to J. G. Stampfli, may be regarded as a generalization of the decomposition theorem of F. Riesz.

THEOREM S (see [11, Theorems 1 and 1']). *Let* D_1 *and* D_2 *be two Cauchy domains, let* f_1 *and* f_2 *be two analytic functions, and suppose that*

(1) $\overline{D}_1 \cap \overline{D}_2 = \{0\} \subseteq \sigma(T) \subseteq D_1 \cup D_2 \cup \{0\}$ *and* $\sigma(T) \cap D_j \neq \emptyset$ $(j = 1, 2)$,

(2) $\operatorname{dist}(z, \sigma(T)) \leq K \operatorname{dist}(z, D_j)$ *if* $z \in D_k$ $(k \neq j;\ k, j = 1, 2)$,

(3) $f_j(z)$ *is a nonzero function analytic on* D_j *and continuous on* $\overline{D}_j$, *and*

$$\sup\{\|f_j(z)(z - T)^{-1}\| : z \in \partial D_j \setminus \{0\}\} < \infty \quad \textit{for } j = 1, 2.$$

Then the expressions

$$S_j = \frac{1}{2\pi}\int_{+\partial D_j} f_j(z)(z - T)^{-1}\,dz \qquad (j = 1, 2)$$

define two nonzero bounded linear operators on X, *and*

(a) $S_1 S_2 = S_2 S_1 = 0$,

(b) $\lambda \in \sigma(T \mid \overline{R(S_j)})$ *if* $\lambda \in D_k$ *and* $f_k(\lambda) \neq 0$ $(k \neq j;\ k, j = 1, 2)$.

(In this restatement, we have changed Jordan domains to Cauchy domains; this is immaterial. For the definition of a Cauchy domain, see [13, page 288].)

In the present paper, we answer the following questions:

(i) What is the spectrum $\sigma(T \mid \overline{R(S_j)})$ $(j = 1, 2)$?

(ii) If $\sigma(T) \cap D_1$ and $\sigma(T) \cap D_2$ are fixed, must S_1 and S_2 be unique?

(iii) If f_1 and f_2 have a common analytic extension f to $D_1 \cup D_2$, must $f(T) = S_1 + S_2$?

(iv) If, in Question (iii), $f(T) = S_1 + S_2$, must $\overline{R(S_1)} + \overline{R(S_2)}$ be closed?

The answer to (i) is that

$$\sigma(T \mid \overline{R(S_j)}) \cup \{\lambda \in D_j \cap \sigma(T) : f_j(\lambda) = 0\} = \overline{D}_j \cap \sigma(T) \qquad (j = 1, 2)$$

(Theorem 1); the answers to (ii), (iii), and (iv) are negative (see Examples 1, 2, and 3, respectively). However, in Theorem 2 we prove that under a slight extension of the domain of condition (3), the answer to Question (iii) is in the affirmative.

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1. MAIN THEOREMS

We begin with some lemmas. Recall that a (closed) subspace Y is a hyperinvariant subspace of T if it is an invariant subspace of every operator commuting with T.

LEMMA 1. *Let* Y *be a hyperinvariant subspace of* T *containing* R(g(T)) *for some analytic function* g *defined on a neighbourhood of* $\sigma(T)$. *Then*

$$\sigma(T) = \sigma(T \mid Y) \cup \{\lambda \in \sigma(T): g(\lambda) = 0\}.$$

Proof. Let S be the operator induced on X/Y by T. Since Y is a hyperinvariant subspace of T, it follows from [1, Lemma I.3.1 (page 1487)] that $\sigma(T) = \sigma(T \mid Y) \cup \sigma(S)$. Therefore g(S) is the operator induced on X/Y by g(T), and it equals zero. Thus $\sigma(S) \subseteq \{\lambda \in \sigma(T): g(\lambda) = 0\}$, and hence

$$\sigma(T) = \sigma(T \mid Y) \cup \{\lambda \in \sigma(T): g(\lambda) = 0\}.$$

We say that λ belongs to the *approximate point spectrum* $\sigma_\pi(T)$ of T if there exists a sequence $\{x_n\}$ in X such that $x_n \not\to 0$ and $(\lambda - T)x_n \to 0$ (as $n \to \infty$). A number λ belongs to $\sigma(T) \setminus \sigma_\pi(T)$ if and only if $N(\lambda - T) = \{0\}$ and $R(\lambda - T)$ is a proper closed subspace of X [13, Theorems 4.2 - B, E, H, I (pages 177-181)].

LEMMA 2. *Let* $0 \in \sigma(T) \setminus \sigma_\pi(T)$. *Let* $Y = \bigcap_{n\geq 0} T^n X$, *and let* S *be the operator induced on* X/Y *by* T. *Then* $\sigma(T) = \sigma(T \mid Y) \cup \sigma(S)$ *and* $0 \in \sigma(S) \setminus \sigma(T \mid Y)$.

Proof. Since Y is a hyperinvariant subspace of T, we see that $\sigma(T) = \sigma(T \mid Y) \cup \sigma(S)$ [1, Lemma I.3.1]. Let $x \in Y$. For each positive integer n, there exists $y_n \in X$ such that $x = T^n y_n$. Therefore $T(y_1 - T^n y_{n+1}) = 0$, and this implies that $y_1 = T^n y_{n+1}$ for all n. Thus $x = Ty_1 \in TY$, and hence $R(T \mid Y) = Y$. This shows that $0 \notin \sigma(T \mid Y)$, and since $\sigma(T) = \sigma(T \mid Y) \cup \sigma(S)$, we deduce that $0 \in \sigma(S) \setminus \sigma(T \mid Y)$.

The following lemma is extracted from the proof of Theorem 1 of [11].

LEMMA 3. *Let* D *be a Cauchy domain whose boundary intersects* $\sigma(T)$ *in at most finitely many points. Let* f *be a function analytic on* D *and continuous on* $\overline{D}$. *Assume* $\|f(z)(z - T)^{-1}\| \leq K$ *for all* $z \in (\partial D) \setminus \sigma(T)$, *where* K *is a positive constant. Let* $A = (2\pi i)^{-1} \int_{\partial D} f(z)(z - T)^{-1}\, dz$. *Let* μ *be a point in* $\sigma(T)$ *such that* $(\mu - T)x_n \to 0$ *for some sequence* $\{x_n\}$ *in* X *with* $\|x_n\| = 1$ $(n = 1, 2, \cdots)$. *Then*

(a) A *is a bounded operator defined on* X,

(b) $\lim \|(A - f(\mu))x_n\| = 0$ *if* $\mu \in D$,

(c) $\lim \|Ax_n\| = 0$ *if* $\mu \notin D$.

In particular, if f *is not identically zero on* $\sigma(T) \cap D$ *and if* $\sigma(T) \cap D \neq \emptyset$ $(\sigma(T) \setminus \overline{D} \neq \emptyset)$, *then* $A \neq 0$ $(0 \in \sigma(A))$.

The next lemma can be found in [2, page 1] or [6] in different cases. For the sake of completeness, we include an indication of the proof.

LEMMA 4. *Let* F *be a closed subset of the plane, and let* S *be a bounded linear operator defined on some Banach space* Y. *Define*

$$X_S(F) = \{x \in Y: \textit{there exists an analytic function } g_x: \mathbb{C} \setminus F \to Y \quad \textit{such that } (\lambda - S)\, g_x(\lambda) \equiv x\}.$$

Let A *be a (bounded linear) operator commuting with* S. *Let* $x \in X_S(F)$, *and let* g_x *be as in the definition of* $X_S(F)$. *Then* $X_S(F) = X_S(F \cap \sigma(S))$, $Ax \in X_S(F)$, *and* $g_x(z) \in X_S(F)$ *for all* $z \notin F$.

The proof follows from the facts that $g_x(\lambda) = (\lambda - S)^{-1}x$ for $\lambda \in \rho(S)$, $(\lambda - S)(Ag_x(\lambda)) \equiv Ax$ for $\lambda \notin F$, and $(\lambda - S)[(g_x(\lambda) - g_x(z))/(z - \lambda)] \equiv g_x(z)$ for $\lambda \notin F$ and $\lambda \neq z$.

The following theorem is an improvement on Theorem S.

THEOREM 1. *Let* T, D_1, D_2, f_1, *and* f_2 *satisfy conditions* (1), (2), *and* (3) *of Theorem* S. *Then*

(a) $\overline{R(S_j)} \subseteq X_T(\overline{D}_j)$ (j = 1, 2),

(b) $\sigma(T \mid Y_j) \cup \{\lambda \in \overline{D}_j \cap \sigma(T): f_j(\lambda) = 0\} = \overline{D}_j \cap \sigma(T)$ *for all hyperinvariant subspaces* Y_j *of* T *such that* $R(S_j) \subseteq Y_j \subseteq X_T(\overline{D}_j)$ (j = 1, 2).

Proof. (a) Let $\lambda \in D_2$. Since a change in $f_2(\lambda)$ has no effect on $\overline{R(S_1)}$, we can assume without loss of generality that $f_2(\lambda) \neq 0$. Thus $\lambda \notin \sigma(T \mid \overline{R(S_1)})$, and hence $\sigma(T \mid \overline{R(S_1)}) \subseteq \sigma(T) \cap \overline{D}_1$. This shows that $R(S_1) \subseteq X_T(\overline{D}_1)$. A similar argument for $R(S_2)$ completes the proof of (a).

(b) Let Y_j be a hyperinvariant subspace of T such that $R(S_j) \subseteq Y_j \subseteq X_T(\overline{D}_j)$ (j = 1, 2). Suppose, if possible, that $\lambda \in D_2$ is in the boundary of $\sigma(A)$, where $A = T \mid Y_1$. Let $x \in Y_1$, and let $g_x(z)$ be an analytic function such that $(z - T)\, g_x(z) = x$ for $z \in D_2$. Since each connected component of D_2 contains uncountably many points of $\rho(T)$, it follows that $g_x(z) = (z - A)^{-1}x \in Y_1$ for all $z \in \rho(A) \cap D_1$. Thus $g_x(\lambda) \in Y_1$, and hence

$$x = (\lambda - T)\, g_x(\lambda) = (\lambda - A)\, g_x(\lambda) \in R(\lambda - A).$$

Also, $(\lambda - A)x \neq 0$, because $(z - T)^{-1}x$ has an analytic extension to a neighbourhood of λ. Since x is arbitrary, we conclude that $\lambda - A$ is a bijective operator on Y_1 and thus $\lambda \notin \sigma(A)$, a contradiction. Hence $\sigma(T \mid Y_1) \subseteq \sigma(T) \cap \overline{D}_1$, and by a similar proof, $\sigma(T \mid Y_2) \subseteq \sigma(T) \cap \overline{D}_2$. So far, we have shown that

$$\sigma(T \mid Y_j) \cup \{\lambda \in \overline{D}_j \cap \sigma(T): f_j(\lambda) = 0\} \subseteq \overline{D}_j \cap \sigma(T) \qquad (j = 1, 2).$$

Now we prove the inverse inclusions.

Let $\lambda \in D_1 \cap \sigma(T)$ be such that $f_1(\lambda) \neq 0$. (Note that $f_1(0) = 0$ because $\|(\lambda_m - T)^{-1}\| \to \infty$ whenever $\lambda_m \to 0$.) We consider two cases.

Case (i) $\lambda \in \sigma_\pi(T)$. Let $\{x_n\}$ be a sequence in X such that $\|x_n\| = 1$ (n = 1, 2, $\cdots$) and $\lim (\lambda - T)x_n = 0$. In view of Lemma 3, $\{S_1 x_n\}$ is a sequence in

Y_1 such that $S_1 x_n \nrightarrow 0$ but $(\lambda - T)S_1 x_n = S_1(\lambda - T)x_n \to 0$ (as $n \to \infty$). Hence $\lambda \in \sigma_\pi(T \mid Y_1)$.

Case (ii) $\lambda \notin \sigma_\pi(T)$. Assume, if possible, that $\lambda \notin \sigma(T \mid Y_1)$. We show that this leads to a contradiction. The assumption implies that $Y_1 \subseteq Y = \bigcap_{n\geq 0} (\lambda - T)^n X$. Let $T^\wedge$ and $S_1^\wedge$ be the operators induced on X/Y by T and S_1, respectively. Obviously, $S_1^\wedge = 0$. Since $\lambda \in D_1 \cap \sigma(T^\wedge) \subseteq \sigma(T^\wedge) \subseteq \sigma(T)$ (Lemma 2), and since

$$\|(z - T^\wedge)^{-1}\| \leq \|(z - T)^{-1}\| \qquad \text{for } z \in \rho(T),$$

it follows that $T^\wedge$, D_1, and f_1 satisfy the conditions of Lemma 3 and that $S_1^\wedge = (2\pi i)^{-1} \int_{+\partial D_1} f(z)(z - T^\wedge)^{-1}\,dz \neq 0$, a contradiction.

In summary, we have shown that

$$\sigma(T \mid Y_1) \cup \{\lambda \in \overline{D}_1 \cap \sigma(T): f_1(\lambda) = 0\} = \overline{D}_1 \cap \sigma(T).$$

A similar verification for $\sigma(T \mid Y_2)$ completes the proof of the theorem.

Remark. Let T, D, f, and A be as in Lemma 3. If $f(z) \neq 0$ for all $z \in \sigma(T) \cap D$, it follows from the proof of Theorem 1 (cases (i) and (ii)) that $\sigma(T) \cap D \subseteq \sigma(T \mid Y)$ for all hyperinvariant subspaces Y of T such that $R(A) \subseteq Y \subseteq X_T(\overline{D})$. In a future paper, we shall use this, together with the next proposition, to show that a Hilbert space operator with compact imaginary part in a Schatten class C_p $(1 \leq p < \infty)$ is decomposable and that this statement is false if $p = \infty$ [2, Problem 5(e), p. 218].

The following proposition shows that the manifolds $X_T(\overline{D}_j)$ $(j = 1, 2)$ of Theorem 1 need not be closed.

Recall that a closed set Δ is called a *spectral set* for T if

$$\|u(T)\| \leq \sup\{|u(z)|: z \in \Delta\}$$

for all rational functions $u(z)$ with poles off Δ.

PROPOSITION 1. *There exists an operator* T *on a Hilbert space* X *satisfying the conditions of Theorem* S *for which* $X_T(\overline{D}_1)$ *is not closed. Moreover,* T *can be chosen so that* $\sigma(T) \subseteq \mathbb{R}$.

Proof. Let V be a nonunitary contraction operator on a Hilbert space K with $\sigma(B) = \{1\}$ (see [5, Problem 150] for existence). Let ϕ be a conformal mapping from the unit disc onto the set $\Delta_1 = \{re^{i\theta}: 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/4\}$ such that $\phi(1) = 0$. Let $A = \phi(V)$. Then $\sigma(A) = \{0\}$ and Δ_1 is a spectral set for A [4, Section 1.1], [10, proof of Theorem 8 (page 143)]. Similarly, since $z^{1/n}$ is a conformal mapping from Δ_1 onto $\Delta_n = \{re^{i\theta}: 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/4\}$, it follows that $A^{1/n}$ is well-defined, Δ_n is a spectral set for $A^{1/n}$, and $\|A^{1/n}\| \leq 1$ $(n = 1, 2, \cdots)$. Let $T = -I \oplus A \oplus A^{1/n} \oplus \cdots$ on $X = K \oplus K \oplus K \oplus \cdots$. Since Δ_n is a spectral set for all $A^{1/k}$ with $k \geq n$, we see that $\|(z - A^{1/k})^{-1}\| \leq 1/\operatorname{dist}(z, \Delta_n)$, and thus $\sigma(A^{1/n} \oplus A^{1/(n+1)} \oplus \cdots) \subseteq \Delta_n$ for all n. Hence $\sigma(T) \subseteq \{-1\} \cup \Delta_n$ for $n \geq 1$; this implies that $\sigma(T) = \{-1\} \cup E$, where E is a subset of the nonnegative numbers. We show that $E \neq \{0\}$. Let $B = A \oplus A^{1/2} \oplus \cdots$. Since $\|(A^{1/n})^n\| = \|A\|$, it follows that $\|B^n\| \geq \|A\|$ $(n = 1, 2, \cdots)$, and thus $\lim \|B^n\|^{1/n} \geq \lim \|A\|^{1/n} = 1$. (Note that $A \neq 0$.) Hence $E = \sigma(B) \neq \{0\}$. (Actually, $1 \in \sigma(B) \subseteq [0, 1]$, because Δ is a spectral set for B.)

Let

$$D_1 = \{re^{i\theta}: 0 < r < 2,\ 2\pi/3 < \theta < 4\pi/3\},$$

$$D_2 = \{re^{i\theta}: 0 < r < 2,\ -\pi/3 < \theta < \pi/3\}.$$

It is easy to see that $\sup\{\|z(z - T)^{-1}\|: z \in \partial(D_1 \cup D_2) \setminus \{0\}\} < \infty$ and that T, D_1, and D_2 satisfy the conditions of Theorem S with $f_j(z) = z$ $(j = 1, 2)$. We claim that $X_T(\overline{D}_1)$ is not closed. Since $\sigma(A^{1/n}) = \{0\}$, each direct summand K is a subset of $X_T(\overline{D}_1)$, and thus $X_T(\overline{D}_1)$ is dense in X. Therefore, since T has the single-valued-extension property [3, Lemma XVI. 5.1 (page 2149)] and $\sigma(T) \not\subseteq \overline{D}_1 \cap \sigma(T)$, it follows from [2, Theorem 1.5 (page 31)] that $X_T(\overline{D}_1)$ is not closed (T is said to have the single-valued-extension property if there exists no nonzero X-valued analytic function f such that $(\lambda - T) f(\lambda) \equiv 0$). The proof of the proposition is complete.

In the remainder of this paper, we assume that the functions f_1 and f_2 of Theorem S have a common analytic extension f to a neighbourhood of $\sigma(T)$. It follows at once from the analyticity of f in a neighbourhood of the origin that condition (3) in Theorem S is equivalent to the condition

$$\text{(3*)} \qquad \sup\{\|z^n(z - T)^{-1}\|: z \in \partial(D_1 \cup D_2) \setminus \{0\}\} < \infty,$$

where n is a positive integer. (Write $f(z) = z^n g(z)$ with $g(0) \neq 0$ in a neighbourhood of 0.) Therefore we can assume, without loss of generality, that $f(z) = z^n$ for some positive integer n (see also the conclusions (d) and (e) of Theorem 2.)

Since $\sigma(T) \subseteq D_1 \cup D_2 \cup \{0\}$, it is attractive to conjecture that $f(T) = S_1 + S_2$ (compare Theorem R below). Example 2 of the next section reveals that such hopes are ill-founded. However, with a slight modification of condition (3*), we can state the following theorem.

THEOREM 2. *Let* T *satisfy conditions* (1) *and* (2) *and the following stronger form of condition* (3) *of Theorem* S:

$$\text{(3**)} \qquad \|z^n(z - T)^{-1}\| \leq M \qquad (z \in \Delta \setminus (D_1 \cup D_2)),$$

where M *is a positive constant,* n *is a positive integer, and* Δ *is a deleted neighbourhood of the origin. Let*

$$S_j = \frac{1}{2\pi} \int_{+\partial D_j} z^n(z - T)^{-1}\, dz \qquad (j = 1, 2).$$

Then

(a) $T^n = S_1 + S_2$ *and* $S_1 S_2 = S_2 S_1 = 0$,

(b) $N(S_1) \cap N(S_2) = N(T^n)$ *and* $N(T^n) \vee R(T^n) \subseteq X_T(D_1) \vee X_T(D_2)$,

(c) $\sigma(T \mid Y_j) \cup \{0\} = \overline{D}_j \cap \sigma(T)$ *for all hyperinvariant subspaces* Y_j *of* T *such that* $R(S_j) \subseteq Y_j \subseteq X_T(\overline{D}_j)$ $(j = 1, 2)$.

Moreover, if g *is an analytic function defined in a neighbourhood of* $\overline{D}_1 \cup \overline{D}_2$ *and if* $f(z) = z^n g(z)$, *then*

(d) $f(T) = U_1 + U_2$, *where*

$$U_j = \frac{1}{2\pi} \int_{+\partial D_j} f(z)(z - T)^{-1}\, dz = g(T) S_j \qquad (j = 1, 2),$$

(e) $\sigma(U_j) = f(\overline{D}_j \cap \sigma(T))$ $(j = 1, 2)$.

Note. We can alter the Cauchy domains D_1 and D_2 so that they lie in any prescribed neighbourhood of $\sigma(T)$, without affecting the operators S_1, S_2, U_1, and U_2 and the manifolds $X_T(\overline{D}_1)$ and $X_T(\overline{D}_2)$.

Proof of Theorem 2. That S_1 and S_2 are well-defined and $S_1 S_2 = S_2 S_1 = 0$ is proved in Theorem S. We show that $T^n = S_1 + S_2$. For each $\eta > 0$, let $D(\eta) = D_1 \cup D_2 \cup \{z: |z| < \eta\}$. When η is small enough, $D(\eta)$ is a Cauchy domain containing $\sigma(T)$, and

$$\begin{aligned} 2\pi \| T^n - (S_1 + S_2) \| &= \left\| \int_{+\partial D(\eta)} z^n (z - T)^{-1}\, dz - \sum_{j=1,2} \int_{+\partial D_j} z^n (z - T)^{-1}\, dz \right\| \\ &= \left\| \int_{+\Gamma(\eta)} z^n (z - T)^{-1}\, dz \right\| \leq M \, |\Gamma(\eta)|, \end{aligned}$$

where $\Gamma(\eta)$ is a curve consisting of two subarcs of the circle $|z| = \eta$ and the portion of $\partial(D_1 \cup D_2)$ lying in the disc $|z| \leq \eta$, and where $|\Gamma(\eta)|$ denotes the length of $\Gamma(\eta)$. Letting $\eta \to 0$, we find that $\lim |\Gamma(\eta)| = 0$, and thus $T^n = S_1 + S_2$. This proves (a).

In (b), the inclusion $N(S_1) \cap N(S_2) \subseteq N(T^n)$ is obvious from the relation $T^n = S_1 + S_2$. Conversely, if $T^n x = 0$ for some $x \in X$, then

$$S_j x = \frac{1}{2\pi} \sum_{k=0}^{n-1} \int_{+\partial D_j} z^{n-k-1}\, T^k x\, dz = 0 \qquad (j = 1, 2).$$

Thus $N(T^n) = N(S_1) \cap N(S_2)$. It is easy to see that $N(T^n) \subseteq X_T(\{0\})$, and thus, by Theorem 1, $N(T^n) \vee R(T^n) \subseteq X_T(\overline{D}_1) \vee X_T(\overline{D}_2)$. Statement (b) is proved.

Statement (c) is established in the proof of Theorem 1.

Finally, for (d) and (e) we observe that

$$S_j\, g(T) = g(T) S_j = \frac{1}{2\pi} \int_{+\partial D_j} f(z)(z - T)^{-1}\, dz = U_j \qquad (j = 1, 2).$$

(For more detail, see a similar calculation in the proof of Theorem 1' of [11].) Thus $f(T) = T^n g(T) = U_1 + U_2$. Also, since $\sigma(T \mid \overline{R(S_j)}) \subseteq \sigma(T)$, we see that $f(T) \mid \overline{R(S_j)} = f(T \mid \overline{R(S_j)})$ $(j = 1, 2)$ [9, Theorem 2.12 (page 32)]. Thus, in view of Lemma 1 and the fact that $0 \in \sigma(U_j)$ (Lemma 3), we have the equations

$$\sigma(U_j) = \sigma(U_j \mid \overline{R(S_j)}) \cup \{0\} = \sigma(f(T \mid \overline{R(S_j)})) \cup \{0\} = f(\overline{D}_j \cap \sigma(T)) \qquad (j = 1, 2)$$

(apply (c), the spectral mapping theorem, and the fact that $R(S_j) \supseteq R(U_j)$ for $j = 1, 2$). The proof of the theorem is complete.

COROLLARY 1. *Let* T *be as in Theorem* 2. *Assume* X *is reflexive and* $n = 1$. *Then* $X = X_T(\overline{D}_1) \vee X_T(\overline{D}_2)$.

Proof. The growth condition $\|z(z - T)^{-1}\| \leq M$, together with the reflexivity of X, implies that $X = N(T) \oplus \overline{R(T)}$ [7, Lemma 3.1 (page 62)]. Therefore, in the light of Theorem 2(b), $X \supseteq X_T(\overline{D}_1) \vee X_T(\overline{D}_2) \supseteq N(T) \oplus \overline{R(T)} = X$; this completes the proof.

2. EXAMPLES

In this section we illustrate some of the differences between Theorem 2 and the Riesz decomposition theorem. First we restate the decomposition theorem in a form suitable to our investigations.

THEOREM R (Riesz, Dunford, $\cdots$). *Let* $\sigma(T)$ *be the disjoint union of two nonempty closed sets* σ_1 *and* σ_2. *Let* f *be an analytic function defined in a neighborhood of* $\sigma(T)$, *and let* D_1 *and* D_2 *be two Cauchy domains in the domain of* f *containing* σ_1 *and* σ_2, *respectively. Let* $P_j = (2\pi i)^{-1} \int_{+\partial D_j} (z - T)^{-1}\, dz$ $(j = 1, 2)$. *Then*

(a) $I = P_1 + P_2$ *and* $P_j^2 = P_j$ $(j = 1, 2)$.

(b) $R(P_j) = X_T(\sigma_j)$ $(j = 1, 2)$ *and* $X = X_T(\sigma_1) \oplus X_T(\sigma_2)$,

(c) $\sigma(T \mid X_T(\sigma_j)) = \sigma_j$ $(j = 1, 2)$,

(d) $f(T) = U_1 + U_2$, *where* $U_j = (2\pi i)^{-1} \int_{+\partial D_j} f(z)(z - T)^{-1}\, dz = f(T) P_j$ $(j = 1, 2)$,

(e) $\sigma(U_j) = f(\sigma_j) \cup \{0\}$ $(j = 1, 2)$.

In Theorem R, the operators P_1 and P_2 and (consequently) U_1 and U_2 are unique as long as the sets σ_1 and σ_2 are fixed. This is not the case in our decomposition, as the following example shows.

Example 1. Let V be a completely nonunitary contraction operator on a Hilbert space H with $\sigma(V) = \{1\}$ (see the proof of Proposition 1). Let ϕ be a conformal mapping from the unit disc onto the triangular plate δ with vertices 0, 1, $1 + i$ such that $\phi(1) = 0$. Let $W = \phi(V)$; then $\sigma(W) = \{0\}$ and δ is a spectral set for W. In particular,

$$(*) \qquad \|(z - W)^{-1}\| \leq 1/\mathrm{dist}(z, \delta)$$

for z outside δ. Let $X = H \oplus H \oplus H \oplus H \oplus H$ and $T = W \oplus iW \oplus - W \oplus I \oplus - I$, and let D_1 and D_2 be as in one of the following cases:

Case (i). $D_1 = \{re^{i\theta}: 0 < r < 2,\ 2\pi/5 < \theta < 3\pi/2\}$,

$D_2 = \{re^{i\theta}: 0 < r < 2,\ -\pi/4 < \theta < \pi/3\}$.

Case (ii). $D_1 = \{re^{i\theta}: 0 < r < 2,\ 9\pi/10 < \theta < 3\pi/2\}$,

$D_2 = \{re^{i\theta}: 0 < r < 2,\ -\pi/4 < \theta < 5\pi/6\}$.

Since ∂D_1 and ∂D_2 are not tangent to the edges of the triangles δ, $i\delta$, and $-\delta$, it follows from the relation (*) that there exists a constant M such that $\|z(z - T)^{-1}\| \leq M$ for z outside $D_1 \cup D_2$. Therefore, in each case, T, D_1, and

D_2 satisfy the conditions of Theorem 2 (for $n = 1$), and $\overline{D}_1 \cap \sigma(T) = \{-1, 0\}$, $\overline{D}_2 \cap \sigma(T) = \{0, 1\}$. By a proof similar to that of Theorem 2(a), one can see that

$$(**) \qquad \int_C z(z - W)^{-1} dz = 0$$

for all closed paths C such that $C \subseteq \{re^{i\theta}: r \geq 0,\ \alpha \leq \theta \leq \beta\}$ for some α, β in $(\pi/4, 2\pi)$. Thus, in Case (i) $S_1 = 0 \oplus iW \oplus -W \oplus 0 \oplus -I$, and in Case (ii) $S_1 = 0 \oplus 0 \oplus -W \oplus 0 \oplus -I$. This disproves the uniqueness of our decomposition.

The next example shows that, in Theorem 2, one cannot replace condition (3**) by its weaker form (3*); more precisely, the condition $\|z^n(z - T)^{-1}\| \leq M$ along the boundaries of D_1 and D_2 alone does not guarantee the equality of T^n and $S_1 + S_2$.

Example 2. Let T be as in Example 1, but choose D_1 and D_2 as follows:

Case (iii). $D_1 = \{re^{i\theta}: 0 < r < 2,\ 5\pi/6 < \theta < 3\pi/2\}$,

$$D_2 = \{re^{i\theta}: 0 < r < 2,\ -\pi/4 < \theta < \pi/3\}.$$

It is easy to see that T, D_1, and D_2 satisfy conditions (1), (2), and (3*) (for $n = 1$). In view of the formula (**) obtained in Example 1, we see that

$$S_1 = 0 \oplus 0 \oplus -W \oplus 0 \oplus -I \quad \text{and} \quad S_2 = W \oplus 0 \oplus 0 \oplus I \oplus 0.$$

Thus $S_1 + S_2 = W \oplus 0 \oplus -W \oplus I \oplus -I$, and hence $T \neq S_1 + S_2$.

Theorem 2 does not apply here, because the growth condition $\|z(z - T)^{-1}\| \leq M$ is not satisfied for $z \in i\delta$.

Another difference between Theorem R and Theorem 2 is in the decomposition of the underlying Banach spaces. In Theorem R, the Banach space X is the direct sum of the closed subspaces $X_T(\overline{D}_j)$ ($j = 1, 2$), whereas the manifolds $X_T(\overline{D}_j)$ of Theorem 2 may not even be closed (see Proposition 1 above). The following example shows that the manifolds $X_T(\overline{D}_j)$ ($j = 1, 2$) of Theorem 2 can be asymptotic even if they are closed and have trivial intersection. First we need a definition.

Definition. An invariant subspace Y of T is called a *maximal spectral subspace* of T if $M \subseteq Y$ for all invariant subspaces M of T such that $\sigma(T \mid M) \subseteq \sigma(T \mid Y)$. The operator T is called *decomposable* if for each finite open covering G_i ($i = 1, 2, \cdots, n$) of $\sigma(T)$ there exist maximal spectral subspaces Y_i ($i = 1, 2, \cdots, n$) of T such that

(a) $\sigma(T \mid Y_i) \subseteq G_i$ ($i = 1, 2, \cdots, n$) and (b) $X = Y_1 + Y_2 + \cdots + Y_n$.

If Y is decomposable, then $X_T(F)$ is a maximal spectral subspace of T and $\sigma(T \mid X_T(F)) \subseteq F \cap \sigma(T)$ [2, Theorem 1.5 (page 31)].

Example 3. Let X be a Hilbert space with an orthonormal basis $\{e_n\}$ ($n = \pm 1, \pm 2, \cdots$), and let $\{z_n\}$ ($n = \pm 1, \pm 2, \cdots$) be a sequence of real numbers such that

(i) $-1 < z_{-n} < z_{-n-1} < 0 < z_{n+1} < z_n < 1$ ($n = 1, 2, \cdots$),

(ii) $\lim z_n = 0$ as $|n| \to \infty$.

Find a sequence $\{\theta_n\}$ $(n = 1, 2, \cdots)$ such that $0 < \theta_n < \pi/2$, $\lim \theta_n = \pi/2$, and $(z_n - z_{-n})\tan\theta_n \leq 1$ $(n = 1, 2, \cdots)$. Define T on X by the rule

$$Te_n = z_n e_n + \begin{cases} (z_n - z_{-n})\tan\theta_{-n}\, e_{-n} & \text{if } n < 0, \\ 0 & \text{if } n > 0. \end{cases}$$

It is easy to see that $N(T) = \{0\}$ and that for $\Im z \neq 0$ (in fact, for $z \neq z_n$)

$$(z - T)^{-1} e_n = (z - z_n)^{-1} e_n + \begin{cases} [(z - z_n)^{-1} - (z - z_{-n})^{-1}]\tan\theta_{-n}\, e_{-n} & \text{if } n < 0, \\ 0 & \text{if } n > 0. \end{cases}$$

These formulas show that $\sigma(T) = \{z_n\} \cup \{0\}$, and that $\|(z - T)^{-1}\| \leq M/|\Im z|^2$ for $\Im z \neq 0$, where M is a positive constant. Thus, in view of [2, Theorem 4.3 (page 159)], T is a decomposable operator. Let D_1 and D_2 be the interiors of the triangles $(0, -1+i, -1-i)$ and $(0, 1+i, 1-i)$, respectively. It is easy to verify that T, D_1, D_2 satisfy the conditions of Theorem 2 for $n = 2$, that the manifolds $X_T(\overline{D}_j)$ $(j = 1, 2)$ are closed, and that $X_T(\overline{D}_1) \cap X_T(\overline{D}_2) = X_T(\{0\})$. Since

$$(T \mid X_T(\{0\})) \subseteq \{0\},$$

it follows from the proof of Lemma 4 of [10, page 138] that $(T \mid X_T(\{0\}))^2 = 0$; since $N(T) = \{0\}$, we have the relations $X_T(\overline{D}_1) \cap X_T(\overline{D}_2) = X_T(\{0\}) = \{0\}$.

To show that $X_T(\overline{D}_1)$ and $X_T(\overline{D}_2)$ are asymptotic, we note that $Te_n = z_n e_n$ for $n > 0$; this implies that $e_n \in X_T(\overline{D}_2)$ for $n > 0$. Also,

$$T(\sin\theta_n\, e_n + \cos\theta_n\, e_{-n}) = z_{-n}(\sin\theta_n\, e_n + \cos\theta_n\, e_{-n}) \qquad \text{for } n > 0,$$

and this implies that $\sin\theta_n\, e_n + \cos\theta_n\, e_{-n} \in X_T(\overline{D}_1)$ for $n > 0$. Now, since

$$\lim (e_n \mid \sin\theta_n\, e_n + \cos\theta_n\, e_{-n}) = 1 = \|e_n\| = \|\sin\theta_n\, e_n + \cos\theta_n\, e_{-n}\|,$$

it follows that $X_T(\overline{D}_1)$ and $X_T(\overline{D}_2)$ are asymptotic.

The ideas for constructing $X_T(\overline{D}_1)$ and $X_T(\overline{D}_2)$ are borrowed from [12, pages 21-22].

Remark. In Example 3, $\overline{N(T^*)} = \{0\}$, and therefore $R(S_1) \vee R(S_2) = \overline{R(T)} = X$; thus, in view of Theorem 2, $\overline{R(S_1)} + \overline{R(S_2)}$ is not closed. This answers Question (iv).

In [8, Proposition 1] we have shown that if T is a decomposable operator whose spectrum lies on a Jordan curve J, then $X_T(F_1 \cup F_2) = X_T(F_1) + X_T(F_2)$ for any two closed subarcs F_1 and F_2 of J such that $F_1 \cap F_2$ contains no isolated point. The following corollary shows that this assertion is not true if $F_1 \cap F_2$ contains an isolated point.

COROLLARY 2. *The operator* T *of Example* 3 *is a decomposable operator with the properties*

(a) $\sigma(T)$ *is a countable subset of* $[-1, 1]$,

(b) $X_T([-1, 0]) + X_T([0, 1])$ *is not closed.*

Note that the operator T of Example 3 is even an $\mathfrak{A}$-self-adjoint operator [2, Theorem 4.3, page 159].

3. PROBLEMS

Problem 1. Let T be as in Theorem 2. Is $X = X_T(\overline{D}_1) \vee X_T(\overline{D}_2)$?

Problem 2. Let T be as in Corollary 1. Is $X = X_T(\overline{D}_1) + X_T(\overline{D}_2)$?

Let T satisfy the growth condition

$$\sup\{\|(\Im z)(z - T)^{-1}\| : \Im z \neq 0\} < \infty . \tag{***}$$

It follows from [2, Theorem 4.3, page 159] that T is an $\mathfrak{A}$-self-adjoint operator that resembles a self-adjoint operator in many aspects. A. S. Markus [7, page 71] has constructed a Hilbert-space operator T satisfying (***) that is not similar to a self-adjoint operator (that is, $T \neq SAS^{-1}$ for all (boundedly) invertible operators S and all self-adjoint operators A). However the Markus example, like any operator similar to a self-adjoint operator, has the property that

$$X_T([a, b] \cup [c, d]) = X_T([a, b]) + X_T([c, d]) \tag{****}$$

for $a \leq b$ and $c \leq d$. Therefore it is reasonable to conjecture that an operator T satisfying (***) will (at least in a Hilbert space) have the property (****). It is easy to see that a counter-example to this conjecture will contain a negative answer to Problem 2 (note that, in view of Corollary 1 and the properties of decomposable operators, $X_T([a, b] \cup [c, d]) = X_T(a, b]) \vee X_T([c, d])$).

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