AN EXTENSION THAT NOWHERE HAS THE FRÉCHET PROPERTY

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In this note, we describe for every T_1-space an extension that does not have the Fréchet property anywhere; as a by-product, we obtain an answer to a question posed by A. V. Arhangel'skiĭ and S. P. Franklin [2, p. 319].

The main corollary (Corollary 2.8) of this note was announced in a slightly weaker form in [10], and it was proved in an entirely different way in [11].

In the construction of progressively more pathological topologies, countable spaces satisfying the first axiom of countability at no point have been considered by J. Novak [14], M. Bebutoff and V. Schneider [3], and R. Engelking [6, pp. 108-109], among others. In [2], A. V. Arhangel'skiĭ and S. P. Franklin construct a countable, homogeneous, sequential Hausdorff space S_ω that is not first-countable. We describe here a method of embedding a prescribed space in a nowhere-first-countable space, and we use it to answer a problem posed in [2] concerning such spaces.

The construction described in this paper is also useful in other directions; several applications of the construction to homogeneous extensions can be seen in [12].

1. CONSTRUCTION AND PROPERTIES OF X^*

Let X be a nondiscrete T_1-space, let x_0 be a nonisolated point of X, and let X^* denote the set of all finite sequences of elements in $X \setminus \{x_0\}$. (The null sequence with no term is also considered as a finite sequence.) Clearly, X and X^* have the same cardinality.

To define a topology on X^*, we first note that each element t of X^* gives rise to a function $f_t: X \to X^*$ in the following natural way.

Let $t = (x_1, x_2, \cdots, x_n)$. Then

$$f_t(x) = \begin{cases} (x_1, x_2, \cdots, x_n, x) & \text{if } x \neq x_0, \\ (x_1, x_2, \cdots, x_n) & \text{if } x = x_0. \end{cases}$$

Thus we have a family of X^*-valued functions on the space X, namely $\{f_t \mid t \in X^*\}$. We give to X^* the strongest (largest, finest) topology in which all these functions are continuous. In this topology, a set $U \subset X^*$ is open if and only if $f_t^{-1}(U)$ is open in X for each t in X^*. (Some topologists call this the *initial topology* given by the family $\{f_t\}$ of functions. According to [4, p. 595], X^* is inductively generated by the family $\{f_t\}$.)

PROPOSITION 1.1. *X^* is a quotient of a sum of copies of* X. (Later, we shall prove that X is also a quotient of X^*.)

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Proof. For each t in X^*, let X_t be a homeomorphic copy of X, and let $h_t: X_t \to X$ be a fixed homeomorphism. Let $\sum_{t \in X^*} X_t$ be the disjoint topological sum of all these copies of X. Let $\phi: \sum_{t \in X^*} X_t \to X^*$ be defined by $\phi \mid X_t = f_t \circ h_t$. Then it is easily seen that ϕ is a quotient map onto X^*.

The following proposition gives a method of directly describing the topology of X^*.

PROPOSITION 1.2. *A set* $U \subset X^*$ *is open if and only if for each* $(x_1, x_2, \cdots, x_n) \in U$ *there exist an open neighbourhood* V *of* x_n *and an open neighbourhood* W *of* x_0 *in* X *such that*

(i) $(x_1, x_2, \cdots, x_{n-1}, v) \in U$ *for every* v *in* V,

(ii) $(x_1, x_2, \cdots, x_{n-1}, x_n, w) \in U$ *for every* w *in* $W \setminus \{x_0\}$.

(*When* n = 0, *condition* (i) *is omitted.*)

Proof. Let $U \subset X^*$ be open, and let $t = (x_1, x_2, \cdots, x_n)$ belong to U. Since $f_t: X \to X^*$ is continuous, $f_t^{-1}(U)$ is open in X. Let $W = f_t^{-1}(U)$. Now $f_t(x_0) = t \in U$. Therefore $x_0 \in W$. Thus W is an open neighbourhood of x_0 such that $f_t(W) \subset U$ and therefore (ii) is satisfied. If $n \geq 1$ (that is, if t is not the null sequence), consider $s = (x_1, x_2, \cdots, x_{n-1})$. Since $f_s: X \to X^*$ is continuous, $f_s^{-1}(U) \setminus \{x_0\} = V$ is open in X. Now $f_s(x_n) = t \in U$, so that $x_n \in V$. Thus V is an open neighbourhood of x_n such that $f_s(V) \subset U$, and (i) is satisfied.

Conversely, let $U \subset X^*$ be as stated in the proposition. We show that then, for each t in X^*, $f_t^{-1}(U)$ is a neighbourhood of each of its points. Let $t = (x_1, x_2, \cdots, x_n)$. Let $x \in f_t^{-1}(U)$ and $x \neq x_0$. Then $(x_1, x_2, \cdots, x_n, x) \in U$. Therefore, by our assumption, there exists an open neighbourhood V of x in X such that $(x_1, x_2, \cdots, x_n, v) \in U$ for every v in V. This implies that $f_t(V) \subset U$, so that $V \subset f_t^{-1}(U)$. Thus $f_t^{-1}(U)$ is a neighbourhood of $x \neq x_0$. Now let $x_0 \in f_t^{-1}(U)$. Then $t \in U$, and therefore, by our assumption, there exists an open neighbourhood W of x_0 in X such that $f_t(W) \subset U$. Therefore $f_t^{-1}(U)$ is a neighbourhood of x_0. Thus $f_t^{-1}(U)$ is open in X for each $t \in X^*$, and this proves that U is open in X^*.

Next we describe the topology of X^* in a nicer way, using the notion of weak topology of [5].

PROPOSITION 1.3. *The topology of* X^* *is the weak topology determined by the family* $\{f_t(X) \mid t \in X^*\}$ *of subsets of* X. *In other words,* $U \subset X^*$ *is open if and only if* $U \cap f_t(X)$ *is open in* $f_t(X)$ *for each* t *in* X^*.

Proof. Let $U \subset X^*$ be such that $U \cap f_t(X)$ is open in $f_t(X)$ for each t in X^*. Then, for every t in X^*, the set $f_t^{-1}(U) = f_t^{-1}(U \cap f_t(X))$ is open in X, since $f_t: X \to f_t(X)$ is continuous. Therefore U is open in X^*.

Note. $F \subset X^*$ is closed if and only if $F \cap f_t(X)$ is closed in $f_t(X)$ for each t in X^*.

PROPOSITION 1.4. X *is homeomorphic to a closed subspace of* X^*.

Proof. In fact, we show that each f_t is a homeomorphism of X onto a closed subspace of X^*. Let $t \in X^*$. First we show that the set $f_t(F)$ is closed in X^* for each closed $F \subset X$. Now, whatever $s \in X^*$ may be, the set $f_s^{-1}(f_t(F))$ is either empty, or a singleton, or F. Therefore, $f_s^{-1}(f_t(F))$ is closed in X, for each s in X^*. This implies that $f_t(F)$ is closed in X^*. In particular, the range of f_t is closed in

X^*, and f_t: $X \to f_t(X)$ is a closed map. We already know that it is continuous. By virtue of its definition, it is one-to-one. Thus f_t is a homeomorphism of X onto a closed subspace of X^*.

PROPOSITION 1.5. *X is a quotient of* X^*.

Proof. Consider the function p: $X^* \to X$ that maps each nonnull sequence onto its first term and maps the null sequence onto x_0. First we prove that p is continuous. Let $U \subset X$ be open, and let $t \in p^{-1}(U)$. If t is the null sequence, then U is a neighbourhood of x_0 and $(u) \in p^{-1}(U)$ for each $u \in U$. Hence (ii) of Proposition 1.2 is satisfied in this case. If t is a nonnull sequence, then its first term belongs to U. We see that the neighbourhood X of x_0 and the neighbourhood V of the last term of t satisfy the criterion of Proposition 1.2, where $V = X \setminus \{x_0\}$ or $V = U \setminus \{x_0\}$ according as t has more than one term or not. Therefore, by Proposition 1.2, $p^{-1}(U)$ is open. Thus p is continuous. Clearly, it is surjective. To show that p is a quotient map, let U be a subset of X such that $p^{-1}(U)$ is open. Then $U = f_o^{-1}(p^{-1}(U) \cap f_o(X))$, where o denotes the null sequence. Since f_o is a homeomorphism from X into X^* (by virtue of Proposition 1.4), it follows that U is open in X. Thus p: $X^* \to X$ is a quotient map.

PROPOSITION 1.6. X^* *is a* T_1*-space.*

Proof. In the proof of Proposition 1.1, X^* is exhibited as a quotient of a T_1-space under a finite-to-one mapping. Therefore, X^* is a T_1-space.

Definition 1.7. Let X be a topological space, and let $x \in X$. Then X is said to satisfy the *Fréchet axiom* at x if, whenever x belongs to the closure of a subset A of X, there exists a sequence in A that converges to x. The space X is called a *Fréchet space* if it satisfies the Fréchet axiom at each of its points.

Note. Let X be a topological space, and let $x \in X$. Let there exist a countable neighbourhood base at x in X. Then X satisfies the Fréchet axiom at x. In particular, if X is first-countable, then X is a Fréchet-space.

PROPOSITION 1.8. *There is no point of* X^* *where the Fréchet axiom is satisfied.*

Proof. We want to show that, for every t in X^*, there exists a subset S_t of X^* such that

(1) t is in the closure of S_t and

(2) no sequence from S_t converges to t.

Let $t = (x_1, x_2, \cdots, x_n)$. We let S_t be the set of all sequences having exactly $n + 2$ terms, of the form $(y_1, y_2, \cdots, y_{n+2})$, where $y_i = x_i$ for $1 \leq i \leq n$ and $y_{n+1}, y_{n+2} \in X \setminus \{x_0\}$.

Now we prove (1). If U is an open set containing t, then, by virtue of Proposition 1.2, there exists a neighbourhood W of x_0 such that $(x_1, x_2, \cdots, x_n, w) \in U$ for every w in $W \setminus \{x_0\}$. Since x_0 is not an isolated point of X, there exists a w_1 in W such that $w_1 \neq x_0$. Now $(x_1, x_2, \cdots, x_n, w_1) \in U$. Repeating the argument with this element, we find that there exists an element $w_2 \neq x_0$ in X such that $(x_1, x_2, \cdots, x_n, w_1, w_2) \in U$. Thus $U \cap S_t$ is not empty. Thus t belongs to the closure of S_t.

Now we prove (2). Let $\{t_1, t_2, \cdots, t_r, \cdots\}$ be a sequence in S_t. For $r = 1, 2, \cdots$, let t_r' denote the $(n + 1)$st term of t_r (where n is the number of terms in t). We consider the sequence $\{t_1', t_2', \cdots\}$ of elements in X. Two cases are possible: (i) The sequence contains a constant sequence as a subsequence, or (ii) no constant sequence is a subsequence. We treat the two cases separately.

Case (i). Let $\{t'_{r_1} = t'_{r_2} = \cdots = t'_{r_m} = \cdots\}$ be a constant subsequence of the sequence $\{t'_r\}$, consider the subsequence $\{t_{r_1}, t_{r_2}, \cdots, t_{r_s}, \cdots\}$ of the sequence $\{t_r\}$ in S_t, and let A be this subsequence considered as a subset of X^*. Then clearly A is contained in $f_s(X)$, where $s = (x_1, x_2, \cdots, x_n, t'_{r_1})$. Since $f_s(X)$ is a closed set (see the proof of Proposition 1.4), it follows that t is not in the closure of A. This implies that the sequence $\{t_r\}$ does not converge to t.

Case (ii). Let the sequence $\{t'_r\}$ possess no constant subsequence. Then it possesses a subsequence with distinct terms. That is, the sequence $\{t_r\}$ in S_t possesses a subsequence the $(n+1)$st terms of whose terms are mutually distinct. Now, if A is such a subsequence considered as a subset of X^*, then A meets each $f_s(X)$ in at most one point, and consequently A is closed (by virtue of the note at the end of Proposition 1.3). This in turn implies that the sequence $\{t_r\}$ cannot converge to t.

Thus no sequence in S_t can converge to t. This proves Proposition 1.8.

COROLLARY 1.9. X^* *is nowhere first-countable.*

PROPOSITION 1.10. X^* *is a Hausdorff space if and only if* X *is a Hausdorff space.*

Proof. Let X be a Hausdorff space, and let $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_n)$ be two distinct points of X^*. We assume without loss of generality that $n \geq m$.

Case 1. Let $x_i = y_i$ for $1 \leq i \leq m$ (in this case, we say that the second point extends the first point). Consider the set $f_t(X)$, where $t = (x_1, x_2, \cdots, x_m)$. This subspace of X^* is a Hausdorff space, since it is homeomorphic to X, by Proposition 1.4. Hence the points $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_{m+1})$ can be separated by disjoint open sets, say U and V, in $f_t(X)$. Let $W = U \cup f_{t'}(X)$, where $t' = (x_1, x_2, \cdots, x_{m-1})$. We let $W = U$ if t is the null sequence. Let V^* be the set of all elements of X^* that extend some element of V. The set $(W \setminus \{t, t'\})^*$ is defined similarly. Now we can prove, by using Proposition 1.2, that V^* and $(W \setminus \{t, t'\})^* \cup \{t\}$ are disjoint open sets in X^* separating the two given points. (If t is the null sequence, we replace $W \setminus \{t, t'\}$ by $W(\{t\})$.)

Case 2. Let there exist an integer $k \leq m$ such that $x_k \neq y_k$. Let us assume that k is the smallest such integer. Let U and V be disjoint open neighbourhoods of x_k and y_k in X. Let U_1 be the subset of X^* consisting of the sequences of length at least m with kth terms in U. Defining V_1 analogously, we see that (by Proposition 1.2) U_1 and V_1 are disjoint open sets in X^* separating the two given points.

Thus we have proved that if X is a Hausdorff space then so is X^*. The converse follows trivially from Proposition 1.4.

PROPOSITION 1.11. X^* *is connected if and only if* X *is connected.*

Proof. Let X be connected, and let $t = (x_1, x_2, \cdots, x_m)$ be any element of X^*. Consider the finite sequence

$$f_o(X),\ f_{t_1}(X),\ f_{t_2}(X),\ \cdots,\ f_{t_m}(X)$$

of subsets of X^*, where $t_i = (x_1, x_2, \cdots, x_i)$ for $1 \leq i \leq m$, and where o denotes the null sequence. This sequence has the following properties:

(i) the null sequence o is an element of $f_o(x)$,

(ii) $t \in f_{t_m}(X)$,

(iii) no two successive sets are disjoint,

(iv) each set is connected (being a continuous image of X).

This shows that t belongs to the connected component of the null sequence. Since this is true for each t in X^*, we see that X^* is connected.

Conversely, if X^* is connected, it follows from Proposition 1.5 that X is connected.

PROPOSITION 1.12. X^* *is sequential if and only if* X *is sequential.*

Proof. The family of all sequential spaces is closed under the formation of sums, quotients, and closed subspaces (see [7]). Hence Proposition 1.12 follows from Propositions 1.1 and 1.4.

Remark 1.13. From Propositions 1.1 and 1.5 it follows that if P is a topological property invariant under sums and quotients, then X has P if and only if X^* has P. Such properties are sometimes referred to as coreflexive properties. The following are examples of such properties: local connectedness, local path connectedness, m-sequentialness, α-sequentialness, being a k-space, a c-space, a P_α-space, or a chain-net space (see [13] for the definitions).

PROPOSITION 1.14. X^* *is zero-dimensional if and only if* X *is zero-dimensional.* (Here dimension is understood to mean the "small inductive" dimension of Menger and Urysohn.)

Proof. Let X be zero-dimensional, let $t \in X^*$, and let U be an open neighbourhood of t in X^*. First suppose the length m of t is at least 1, and put $t' = (t_1, \cdots, t_{m-1})$. By Proposition 1.2 (i), there exists a neighbourhood V of t_m in X such that $(t_1, \cdots, t_{m-1}, v) \in U$ for all v in V. There is an open-closed neighbourhood V_0 of t_m in X such that $V_0 \subset V$. Write $F_0 = f_{t'}(V_0)$; then F_0 is an open-closed neighbourhood of t relative to $f_{t'}(X)$, contained in U. For each x in U, since f_x is a homeomorphism, there is an open-closed neighbourhood F(x) of x relative to $f_x(X)$, contained in $U \cap f_x(X)$. Define $F_1 = \bigcup_{x \in F_0} F(x)$. Then $U \supset F_1$. When the set $F_n \subset U$ has been defined, define $F_{n+1} = \bigcup_{x \in F_n} F(x)$. This defines F_n recursively; put $F = \bigcup_{n=0}^{\infty} F_n$. Then $t \in F \subset U$, and it remains to be shown that F is open-closed in X^*. By Proposition 1.3 and the remark following it, it suffices to show that, for each p in X^*, $F \cap f_p(X)$ is open-closed relative to $f_p(X)$. Now an easy induction over n shows that each point of $F_{n+1} \setminus F_n$ has length exactly $m + n + 1$. Using this fact, we can easily verify that

$$F \cap f_p(X) = \begin{cases} F(p) & \text{if } p \in F, \\ F_0 & \text{if } p = t', \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus the result follows in this case.

If $m = 0$, so that t is the null sequence o, we choose as before for each x in U an open-closed neighbourhood $F(x)$ of x relative to $f_x(x)$ contained in U, and define

$$F_1 = F(t), \qquad F_{n+1} = \bigcup_{x \in F_n} F(x), \qquad F = \bigcup_{n=1}^{\infty} F_n .$$

Again, $t \in F \subset U$, and F is open-closed; for it is easy to verify that for each p in X^*,

$$F \cap f_p(X) = \begin{cases} \emptyset & \text{unless } p \in F, \\ F(p) & \text{if } p \in F . \end{cases}$$

In carrying out this verification, we use the fact that each element of $F_1 \setminus \{t\}$ has length 1 and each element of $F_{n+1} \setminus F_n$ has length $n + 1$.

Thus we have shown that if X is zero-dimensional, then so is X^*. The converse follows from Proposition 1.4.

Remark 1.15. Similar arguments show that several other nice properties are also preserved in the passage from X to X^*. For example, if P is any one of the following properties, then X^* has P if and only if X has P: regularity, normality, path-connectedness, total disconnectedness. Thus we have shown that each non-discrete T_1-space can be embedded as a closed subspace of a nowhere-Fréchet space, with preservation of many of the standard topological properties.

2. ANSWER TO A QUESTION OF ARHANGEL'SKIǏ AND FRANKLIN

The extension studied in the previous section can be applied to several problems concerning homogeneous spaces. Of these, one application can be described now, since it involves no new notions. We show in this section that for a special class of spaces X the extension X^* is always homogeneous, and then we use the properties of this class to give the promised solution to a problem of [2]. We start with two short lemmas. Throughout this section, X is a T_1-space with a *unique* accumulation point, which we take to be x_0. It follows immediately that X is a Hausdorff space, and in fact, that it is zero-dimensional (hence regular).

Notation 2.1. Let $t = (x_1, x_2, \cdots, x_n) \in X^*$. Then U_t denotes the set of all elements of X^* that extend t; that is,

$$U_t = \{(y_1, y_2, \cdots, y_m) \in X^* \mid m \geq n;\ y_i = x_i \text{ if } 1 \leq i \leq n\} .$$

Also, let $V_t = U_t \setminus \{t\}$.

LEMMA 2.2. *For each* t *in* X^*, *the set* U_t *is both open and closed in* X^*.

Proof. Let t' be a point of X^*. Then it is easily seen that

(i) $f_{t'}(X) \subset U_t$, if $t' \in U_t$;

(ii) $f_{t'}(X) \cap U_t$ is a singleton (which is isolated in $f_{t'}(X)$), if $t \in f_{t'}(X) \setminus \{t'\}$; and

(iii) $f_{t'}(X)$ is disjoint from U_t, otherwise.

Thus $f_{t'}(X) \cap U_t$ is either the whole of $f_{t'}(X)$, or it is empty, or it consists of a single isolated point; in any case, it is both open and closed in $f_{t'}(X)$. This is true for each t' in X^*. Therefore we deduce from Proposition 1.3 and the note below it that U_t is both open and closed in X^*.

LEMMA 2.3. (i) *For each* t *in* X^*, *there is a homeomorphism* h_t *from* X^* *onto* U_t *that takes the null sequence to* t.

(ii) *For each nonnull* $t \in X^*$, *there is a homeomorphism* g_t *from* $X^* \setminus U_t$ *onto* X^* *that takes the null sequence to itself.*

Proof. Let $t = (x_1, x_2, \cdots, x_n)$.

(i) Consider the map $h_t: X^* \to U_t$ defined by

$$h_t(y_1, y_2, \cdots, y_n) = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m).$$

Clearly, h_t is a bijection. Observe that h_t takes the null sequence to the element t. The openness of h_t and h_t^{-1} can be proved by using Proposition 1.2.

(ii) Let t be nonnull. Then its last term x_n is an isolated point in X. We first note that there exists a homeomorphism $p: X \setminus \{x_n\} \to X$. (For if the complement in X of every neighbourhood of x_0 is finite, then an arbitrary bijection of $X \setminus \{x_n\}$ onto X, keeping x_0 fixed, will automatically be homeomorphism; if some neighbourhood W of x_0 has infinite complement, take p to be the identity map on $W \setminus \{x_n\}$ combined with an arbitrary bijection of $X \setminus (W \cup \{x_n\})$ onto $(X \setminus W) \cup \{x_n\}$.)

To define g_t, note first that $U_t \subset V_{t'}$, and that $X^* \setminus U_t$ is the union of the disjoint sets $X^* \setminus V_{t'}$ and $V_{t'} \setminus U_t$. Now (a) if $y = (y_1, \cdots, y_m) \in V_{t'} \setminus U_t$, note that $m \geq n$ and $y_n \neq x_n$; thus we can define

$$g_t(y) = (y_1, \cdots, y_{n-1}, p(y_n), y_{n+1}, \cdots, y_m).$$

And (b) if $y \in X^* \setminus V_{t'}$, define $g_t(y) = y$. Thus (a) gives a bijection of $V_{t'} \setminus U_t$ onto $V_{t'}$, and (b) gives a bijection of $X^* \setminus V_{t'}$ onto $X^* \setminus V_{t'}$. Therefore g_t is a bijection of $X^* \setminus U_t$ onto X^*. Also, the null sequence is a fixed point of g_t. The openness of g_t and $g_{t'}$ can be proved by means of Proposition 1.2 and the fact that x_0 is the only accumulation point of X.

THEOREM 2.4. *Let* X *be a Hausdorff space with a unique accumulation point. Then* X^* *is homogeneous.*

Proof. Let t and t' be any two elements of X^*.

Case (i). Let neither t nor t' be null. Let the maps h_t, $h_{t'}$, $g_{t'}$ be the homeomorphisms of Lemma 2.3. Define a map $g_{t,t'}$ from X^* onto X^* by

$$g_{t,t'} \mid U_t = h_{t'} \circ h_t^{-1}, \qquad g_{t,t'} \mid (X^* \setminus U_t) = g_{t'}^{-1} \circ g_t.$$

Then $g_{t,t'}$ maps U_t homeomorphically onto $U_{t'}$, and it maps $X^* \setminus U_t$ homeomorphically onto $X^* \setminus U_{t'}$. By Lemma 2.2, it is a homeomorphism of X^* onto itself. Also, by its definition, it takes t to t'.

Case (ii). Let $t = o \neq t'$. Using Lemma 2.3, define

$$g_{t,t'} = \begin{cases} h_{t'} \circ g_{t'} & \text{on } X^* \setminus U_{t'}, \\ g_{t'}^{-1} \circ h_{t'}^{-1} & \text{on } U_{t'}. \end{cases}$$

Then $g_{t,t'}$ is a homeomorphism of X^* onto X^* that takes t to t' and t' to t.

Case (iii). Let $t \neq o = t'$. Then, as in case (ii), we get a map $g_{t',t}$ that takes t to t' and t' to t.

Thus, in all cases, there is a homeomorphism of X^* that takes t to t'. Thus X^* is homogeneous.

The following theorem asserts that the extension X^* almost determines the space X.

THEOREM 2.5. *Let* X *and* Y *be two Hausdorff spaces, each having a unique accumulation point. Further, let* Y *be a Fréchet space. If* Y *is a subspace of* X^*, *then* Y *is homeomorphic to the disjoint sum of a discrete space and a closed subspace of* X.

Proof. Let t be the unique accumulation point of Y. As in the proof of Proposition 1.8, we can show that no sequence in $U_t \setminus f_t(X)$ converges to t. Also, by Lemma 2.2, U_t is an open neighbourhood of t. This implies that no sequence in $X^* \setminus f_t(X)$ converges to t. Consequently, the set $Y \setminus f_t(X)$ is closed in Y, since Y is a Fréchet space. But we have already noted that $f_t(X)$ is closed in X^*, so that $Y \setminus f_t(X)$ is open in Y. Thus $Y \setminus f_t(X)$ is both open and closed in Y, and therefore Y is the disjoint sum of $Y \setminus f_t(X)$ and $Y \cap f_t(X)$. Since t is the unique accumulation point of Y, it follows that $Y \setminus f_t(X)$ is discrete. Also, $Y \cap f_t(X)$ is homeomorphic to a closed subspace of X, since $f_t(X)$ is homeomorphic to X and since every subspace of X containing the unique accumulation point is closed. This proves the theorem.

COROLLARY 2.6. *Under the hypotheses of Theorem* 2.5, *if* X *is locally compact, so is* Y; *if* X *is first-countable, so is* Y.

COROLLARY 2.7. *Let* X_1 *be the one-point-compactification of a countably infinite discrete space. Let* X_2 *be the sequential fan defined in* [8]. *Let* X_3 *be the set of all rational numbers with the topology in which zero is the only nonisolated point and its neighbourhoods are the usual ones. Then the three spaces* X_1^*, X_2^*, *and* X_3^* *are mutually nonhomeomorphic.*

Proof. We recall that we obtain the sequential fan X_2 from the sum of a countably infinite number of copies of X_1 by identifying all nonisolated points to a single point. X_2 is both a Hausdorff space and a Fréchet space. It has a unique accumulation point, at which the first axiom of countability is not satisfied.

X_1 and X_3 are Hausdorff spaces with unique accumulation points satisfying the first axiom of countability. The space X_1 is compact, but X_2 and X_3 are not locally compact.

Therefore, by Corollary 2.7, neither X_2 nor X_3 is homeomorphic to a closed subspace of X_1^*; further, X_2 is not homeomorphic to a closed subspace of X_1^* or X_3^*.

It follows from Proposition 1.4 that X_1^*, X_2^*, and X_3^* are mutually nonhomeomorphic.

COROLLARY 2.8. *There exist at least three mutually nonhomeomorphic, countable, zero-dimensional, homogeneous sequential Hausdorff spaces that are not Fréchet spaces.*

Proof. This is easily proved by combining Corollary 2.7 with Propositions 1.10, 1.12, and 1.14 and Theorem 2.4.

3. CONCLUDING REMARKS

Remark 3.1. The space S_ω constructed in [2] is a countable, zero-dimensional, homogeneous sequential Hausdorff space that is not a Fréchet space. In [2], it was asked whether there exist other countable homogeneous sequential Hausdorff spaces that are not first-countable. Our Corollary 2.8 answers this question in the affirmative. (The question can be answered more easily by means of connected spaces with the stated properties; but it was presumably intended that the spaces be also zero-dimensional; in a letter to the author, S. P. Franklin made it clear that he wanted a zero-dimensional example.) One example is given in [8], but it is a Fréchet space. Corollary 2.8 above gives examples of at least two spaces that share with S_ω all the properties listed above but are not homeomorphic to S_ω. Note that these spaces also have sequential order ω_1.

Remark 3.2. The considerations in this paper arose from our desire to have a satisfactory characterisation of S_ω. One such characterisation is given in [12].

Remark 3.3. It can be shown that when X is a Hausdorff space with a unique accumulation point, then X^* satisfies unusually strong conditions of homogeneity. (Every homeomorphism between two compact subspaces of X^* can be extended to a homeomorphism of X^* onto itself.) The properties of X^* can be used to prove the abundance of distinct topological types of countable homogeneous subspaces of βN, where βN denotes the Stone-Čech compactification of a countably infinite, discrete space. For the proof, see [12].

Remark 3.4. It can be shown that in several special cases, X^* can be viewed as the smallest homogeneous extension of the space X. The precise statement of this result requires categorical notions, and it can be seen in [12]. There are also cases where X^* is far from being homogeneous.

Remark 3.5. By means of the theorems of this paper, it is easy to exhibit a functor from the category of all Hausdorff spaces with unique accumulation points into the category of all homogeneous Hausdorff spaces. Our results prove that this functor behaves well with respect to several topological properties.

Remark 3.6. M. Shimrat [15] has described a method of embedding an arbitrary space in a homogeneous space. In this paper, we have described another method of embedding a Hausdorff space, with a unique accumulation point, in a homogeneous space. When it applies, our approach has considerable advantages, as shown, for instance, by Theorem 2.5 and Remarks 3.3 and 3.4; here is a further illustration. If X is an extremally disconnected Hausdorff space with a unique accumulation point x (that is, if the neighbourhoods of x form an ultrafilter) we can show (see [12]) that X^* is also extremally disconnected; this result is to be contrasted with the fact (whose proof has not appeared in the literature) that the homogeneous extension described in [15] is not extremally disconnected, except in trivial cases. However, Shimrat's method can be applied to arbitrary spaces, whereas the present method applies only to very special ones. It is worth noting that homogeneous extremally disconnected spaces have attracted the attention of several topologists (see [1], [9] for example) and that our method yields a large collection of new examples of such spaces.

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