FREDHOLM PERMUTATIONS AND STABLE HOMOTOPY

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Let $\mathscr{S}_\infty$ denote the infinite symmetric group; then there is a map $B\mathscr{S}_\infty \times \mathbb{Z} \to \Omega^\infty S^\infty$ inducing isomorphisms of integral homology groups [2], [3]. Because this observation has led to several new results in stable homotopy theory [5], it is of interest to find other "algebraic" models for infinite loop spaces.

Let $\Omega^{\infty-1} S^\infty = \varinjlim \Omega^{n-1} S^n$. The purpose of this note is to establish the homotopy equivalence $\Omega^{\infty-1} S^\infty \cong (BF_\infty)^+$ conjectured by J. Wagoner [12], where F_∞ is the group of Fredholm permutations (see Section 1), and where $(\cdot)^+$ is D. Quillen's construction, which in this case abelianizes the fundamental group (by adding 2- and 3-cells) without changing the homology groups [9], [12].

The proof (Section 3) uses the simplicial techniques of Quillen [10] and of M. Barratt and the author [3]. Section 1 contains the necessary preliminaries on Fredholm permutations, and Section 2 is devoted to the simplicial monoids used in Section 3.

I would like to thank J. Wagoner, whose ideas on Fredholm permutations in homotopy theory prompted this work.

1. PRELIMINARIES ON FREDHOLM PERMUTATIONS

We recall some of Wagoner's definitions and results (see [12] for details and proofs). Let $E = \mathbb{N} \times \mathbb{N} = \{e_i^k\}$ $(1 \leq i, k < \infty)$. By a *proper* map $\alpha: E \to E$ we mean a map α for which $\{\alpha^{-1}(e_i^k)\}$ is finite for each i, k. Two proper maps α and β are in the same *germ* class (notation: $\alpha \sim \beta$) if α and β agree except on a finite set. A proper map α is a *Fredholm permutation* if $\alpha \cdot \beta \sim \mathrm{id}$ and $\beta \cdot \alpha \sim \mathrm{id}$ for some proper map β.

Let P_n be the group of all bijections α of E with $\alpha(e_i^k) = e_i^k$ $(i > n)$. Let $\Sigma_n \subset P_n$ be the subgroup of finite permutations α $(\alpha \sim \mathrm{id})$. Let F_n be the group of all germ classes of Fredholm permutations α with $\alpha(e_i^k) = e_i^k$ $(i > n)$. Let $\Sigma_\infty = \bigcup \Sigma_n$, $P_\infty = \bigcup P_n$, $F_\infty = \bigcup F_n$. Then there is an exact sequence [12, Section 7.1]

$$1 \to \Sigma_\infty \to P_\infty \xrightarrow{\rho} F_\infty, \tag{1.1}$$

where ρ associates each permutation with its germ class, and where moreover $\mathrm{im}(\rho) = [F_\infty, F_\infty]$ and $F_\infty/[F_\infty, F_\infty] = \mathbb{Z}$.

(1.2) Now BP_∞ is acyclic [12, Corollary 2.2], and hence BP_∞^+ is contractible; hence, using (1.1), one obtains the following result [12, Proposition 7.3].

THEOREM 1.3 (Wagoner). $\mathbb{Z} \times (B\Sigma_\infty)^+ \cong \Omega(BF_\infty^+)$.

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2. THREE SIMPLICIAL MONOIDS

In this section, we discuss three simplicial monoids used in the proof of Wagoner's conjecture. Define

$$\Sigma = * \amalg \coprod_{n\geq 1} \overline{W}\Sigma_n,$$

$$P = * \amalg \coprod_{n\geq 1} \overline{W} P_n,$$

$$F^c = * \amalg \coprod_{n\geq 1} \overline{W} F^c_n,$$

where $F^c_n = [F_n, F_n]$. Multiplication is induced by the juxtaposition homomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ ($P_n \times P_m \to P_{n+m}$, $F^c_n \times F^c_m \to F^c_{n+m}$) defined by

$$(\alpha \times \beta)(e^k_i) = \begin{cases} \alpha(e^k_i) & (1 \leq i \leq n), \\ e^{\ell}_{j+n} & (n < i \leq n+m), \quad \text{where } e^{\ell}_j = \beta(e^k_{i-n}), \\ e^k_i & (n+m < i) \end{cases}$$

for $\alpha \times \beta \in \Sigma_n \times \Sigma_m (P_n \times P_m,\ F^c_n \times F^c_m)$. The identity element in each dimension is the degeneracy of the basepoint $*$. This multiplication is clearly associative, and therefore Σ, P, and F^c are simplicial monoids.

LEMMA 2.1. *The homology algebras* $H_*\Sigma$, H_*P, *and* H_*F^c *are commutative.*

Proof. Let $\alpha \in P_{n+m}$ be the permutation of E defined by the rule

$$\alpha(e^k_i) = \begin{cases} e^k_{i+m} & (1 \leq i \leq n), \\ e^k_{i-n} & (n < i \leq n+m), \\ e^k_i & (n+m < i). \end{cases}$$

Let $\sigma\colon F^c_{n+m} \to F^c_{n+m}$ denote conjugation by the element of F^c_{n+m} represented by α (see (1.1)). Then the diagram

$$\begin{array}{ccc} F^c_n \times F^c_m & \longrightarrow & F^c_{n+m} \\ \downarrow \tau & & \sigma \downarrow \\ F^c_m \times F^c_n & \longrightarrow & F^c_{n+m} \end{array}$$

(where τ is transposition) commutes. Now, since $\overline{W}\sigma \simeq \mathrm{id}$ [11], the induced multiplication $\overline{W}F^c_n \times \overline{W}F^c_m \to \overline{W}F^c_{n+m}$ is homotopy commutative, and hence the result follows for F^c. A similar argument works for P and Σ.

Let $\mathscr{S}_n$ denote the symmetric group acting on $\{1, 2, \cdots, n\}$. Barratt [1] has defined a simplicial monoid $\Gamma^+S^0 = * \amalg \coprod_{n\geq 1} \overline{W}\mathscr{S}_n$ with multiplication induced by $\mathscr{S}_n \times \mathscr{S}_m \to \mathscr{S}_{n+m}$. If $U(\cdot)$ denotes the universal group functor [3] (or group completion functor [9]) and $\Gamma S^0 = U(\Gamma^+S^0)$, then $|\Gamma S^0| \cong \Omega^\infty S^\infty$. We shall compare ΓS^0 and $U\Sigma$. Let $i_n\colon \mathscr{S}_n \to \Sigma_n$ be the monomorphism defined by

$$i_n(\alpha)(e_i^k) = \begin{cases} e^k_{\alpha(i)} & (k = 1,\ 1 \leq i \leq n), \\ e_i^k & \text{otherwise}, \end{cases}$$

and let $i_\infty = \varinjlim \overline{W} i_n \colon \overline{W} \mathscr{S}_\infty \to \overline{W} \Sigma_\infty$.

LEMMA 2.2. *The induced homomorphism*

$$i_{\infty *} \colon H_*(\overline{W} \mathscr{S}_\infty) \to H_*(\overline{W} \Sigma_\infty)$$

is an isomorphism.

Proof. Let $E_n = \{e_i^k\}$ $(1 \leq i \leq n)$, and define a bijection $f \colon E_n \to \mathbb{N}$ by $f(e_i^k) = (k-1)n + i$. Define an isomorphism $\phi_n \colon \mathscr{S}_\infty \to \Sigma_n$ by

$$\phi_n(\alpha)(e_i^k) = \begin{cases} f^{-1} \alpha f & (e_i^k \in E_n), \\ e_i^k & \text{otherwise}. \end{cases}$$

Since the inclusion $\mathscr{S}_n \to \mathscr{S}_\infty$ induces an isomorphism in integral homology in the range $k < \frac{n+1}{2}$ [8], and since ϕ_n extends i_n in the sense of the diagram

$$\begin{array}{ccc} & \mathscr{S}_\infty & \\ \cup\nearrow & & \searrow \phi_n \\ \mathscr{S}_n & \xrightarrow{\ i_n\ } & \Sigma_n, \end{array}$$

the result follows upon passage to direct limits.

Let q, r, s denote the basepoints of $\overline{W}(\Sigma_1)$, $\overline{W}(P_1)$, $\overline{W}(F_1^c)$, respectively. We shall consider q, r, and s as the generator of $\pi_0 \Sigma$, $\pi_0 P$, and $\pi_0 F^c$, respectively, since all these monoids are isomorphic to $\mathbb{Z}^+$. Define a map of simplicial sets $q^{-\infty} \colon \overline{W} \Sigma_\infty \to (U\Sigma)_0$ by $q^{-\infty} = \varinjlim (xq^{-n})$, where $xq^{-n} \colon \overline{W} \Sigma_n \to (U\Sigma)_0$ denotes multiplication by q^{-n} in $U\Sigma$, and where $(U\Sigma)_0$ is the component of the basepoint $*$. Similarly, define the map $r^{-\infty} \colon \overline{W} P_\infty \to (UP)_0$ (respectively, the map $s^{-\infty} \colon \overline{W} F_\infty^c \to (UF^c)_0$). By Lemma 2.1, the algebras $H_* \Sigma$, $H_* P$, and $H_* F^c$ are commutative; further Σ, P, and F^c are clearly free monoids in each dimension. Hence Quillen's theorem [9] implies the following result.

LEMMA 2.3. *The induced homomorphisms*

$$q_*^{-\infty} \colon H_* \overline{W} \Sigma \to H_*(U\Sigma)_0,$$

$$r_*^{-\infty} \colon H_* \overline{W} P \to H_*(UP)_0,$$

$$s_*^{-\infty} \colon H_* \overline{W} F^c \to H_*(UF^c)_0$$

are isomorphisms.

By [3], there is an isomorphism $p_*^{-\infty} \colon H_* \overline{W} \mathscr{S}_\infty \to H_*(\Gamma S^0)_0$, where $p^{-\infty} = \varinjlim (xp^{-n})$ and p is the basepoint of $\overline{W} \mathscr{S}_1$. Let $i \colon \Gamma S^0 \to U\Sigma$ be the homomorphism of simplicial groups induced by the maps $\overline{W} i_n \colon \overline{W} \mathscr{S}_n \to \overline{W} \Sigma_n$.

COROLLARY 2.4. *The map* $i: \Gamma S^0 \to U\Sigma$ *is a homotopy equivalence.*

Proof. Let $j: (\Gamma S^0)_0 \to (U\Sigma)_0$ be the map restricted to the component of the basepoint $*$. It suffices to show that j is a homotopy equivalence. Consider the commutative diagram

$$\begin{array}{ccc} \overline{W}\mathscr{S}_\infty & \xrightarrow{i_\infty} & \overline{W}\Sigma_\infty \\ \big\downarrow p^{-\infty} & & \big\downarrow q^{-\infty} \\ (\Gamma S^0)_0 & \xrightarrow{j} & (U\Sigma)_0 \end{array} .$$

The induced diagram of homology groups shows (by Lemmas 2.2 and 2.3) that $j_*: H_*(\Gamma S^0) \to H_*(U\Sigma)_0$ is an isomorphism. Since $(\Gamma S^0)_0$ and $(U\Sigma)_0$ are connected simplicial groups (H-space objects), the H-space version of the Whitehead theorem implies j is a homotopy equivalence.

3. THE HOMOTOPY EQUIVALENCE

The following is our main result.

THEOREM 3.1. *There is a homotopy equivalence* $BF_\infty^+ \cong \Omega^{\infty-1} S^\infty$.

The proof is given in paragraph (3.7); first we establish the following.

PROPOSITION 3.2. $|(UF^c)_0| \cong B(\Omega^\infty S^\infty)_0$.

Proof. The epimorphisms $\rho_n: P_n \to F_n^c$ (1.1) induce surjective maps $\overline{W}\rho_n: \overline{W}P_n \to \overline{W}F_n^c$ and hence an epimorphism $\rho': (UP)_0 \to (UF^c)_0$ of simplicial groups. Let $K = \ker \rho'$; then the short exact sequence

$$K \to (UP)_0 \to (UF^c)_0$$

of simplicial groups becomes a short exact sequence

$$|K| \to |(UP)_0| \to |(UF^c)_0| \qquad (3.3)$$

of topological groups, by the exactness properties of the geometric realization functor (with values in the category of compactly generated spaces, see [4, Chapter III, Section 3.3]).

Now $|\overline{W}P_\infty|$ is acyclic, by (1.2); hence $(UP)_0$ is acyclic, by Lemma 2.3. Thus, by simplicity, $(UP)_0$ is contractible, and therefore (3.3) is a universal $|K|$-bundle; that is, $|(UF^c)_0| \cong B|K|$.

Consider the commutative diagram of simplicial Kan fibrations

$$\begin{array}{ccccc} \overline{W}\Sigma_\infty & \longrightarrow & \overline{W}P_\infty & \xrightarrow{\overline{W}\rho} & \overline{W}F_\infty^c \\ \big\downarrow \phi & & \big\downarrow r^{-\infty} & & \big\downarrow s^{-\infty} \\ K & \longrightarrow & (UP)_0 & \xrightarrow{\rho'} & (UF^c)_0 \end{array}$$

where $\phi = r^{-\infty} | \overline{W}\Sigma_\infty$. By Lemma 2.3, $s^{-\infty}$ and $r^{-\infty}$ induce isomorphisms in homology, and

$$(3.4)\qquad \pi_1(UF^c)_0 \approx H_1(UF^c)_0 \approx H_1(\overline{W}F^c_\infty).$$

Since F^c_∞ is a perfect group (see (1.1), (1.2)),

$$(3.5)\qquad H_1(\overline{W}F^c_\infty) = F^c_\infty/[F^c_\infty, F^c_\infty] = 0.$$

By the argument of [12, Lemma 3.1], $\pi_1 \overline{W}F^c_\infty = F^c_\infty$ acts trivially on $H_*\overline{W}\Sigma_\infty$. Hence, by the Comparison Theorem [6, p. 355],

$$\phi_*: H_*(\overline{W}\Sigma_\infty) \to H_*(K)$$

is an isomorphism.

Now ϕ has the factorization $\phi = i \cdot q^{-\infty}$ indicated by the diagram

$$\begin{array}{ccc} \overline{W}\Sigma_\infty & \xrightarrow{\ \phi\ } & K \\ & {\scriptstyle q^{-\infty}}\searrow \quad \nearrow{\scriptstyle i} \cup & \\ & (U\Sigma)_0 & \end{array}$$

and hence, by Lemma 2.3, i induces an isomorphism in homology. By simplicity, i is a homotopy equivalence. Hence $|K|$ and $|(U\Sigma)_0|$ are homotopy equivalent via a map of topological groups. By Corollary 2.4, $(U\Sigma)_0 \cong (\Gamma S^0)_0$; thus

$$|(UF^c)_0| \cong B|K| \cong B|(U\Sigma)_0| \cong B|(\Gamma S^0)_0| \cong B(\Omega^\infty S^\infty)_0.$$

COROLLARY 3.6. $(BF^c_\infty)^+ \cong B(\Omega^\infty S^\infty)_0$.

Proof. By Lemma 2.3, $s_*^{-\infty}: H_*(\overline{W}F^c_\infty) \to H_*((UF^c)_0)$ is an isomorphism, and hence $|s^{-\infty}|_*: H_*(|\overline{W}F^c_\infty|) \to H_*(|(UF^c)_0|)$ is also an isomorphism [7, Proposition 16.2]. By the $(\cdot)^+$ construction [12, Section 1],

$$(|s^{-\infty}|^+)_*: H_*(|\overline{W}F^c_\infty|^+) \to H_*(|(UF^c)_0|)$$

is therefore an isomorphism. Now $\pi_1(|\overline{W}F^c_\infty|^+) = 0$ (3.5) and $(UF^c)_0$ is simply connected (see (3.4), (3.5)), and therefore $|\overline{W}F^c_\infty|^+ \cong |(UF^c)_0|$. By [3, Lemma 2.3.3], $|\overline{W}F^c_\infty| \cong B|F^c_\infty|$; thus it follows that $(B|F^c_\infty|)^+ \cong |(UF^c)_0| \cong B(\Omega^\infty S^\infty)_0$.

(3.7) *Proof of Theorem* 3.1. Let $f: S^1 \to BF^+_\infty$ represent a generator of $\pi_1 BF^+_\infty = \mathbb{Z}$ (1.2). Since F^c_∞ is normal in F_∞, the space $(BF^c_\infty)^+$ is homotopy equivalent to the universal cover of BF^+_∞. Since BF^+_∞ is an H-- space [12; 1.2], the composite map

$$(BF^c_\infty)^+ \times S^1 \xrightarrow{\text{incl} \times f} BF^+_\infty \times BF^+_\infty \xrightarrow{\text{multipl.}} BF^+_\infty$$

is a homotopy equivalence, hence by Corollary 3.6, $B(\Omega^\infty S^\infty)_0 \times S^1 \cong BF^+_\infty$. Now the natural multiplication $(\Omega^\infty S^\infty)_0 \times \mathbb{Z} \to \Omega^\infty S^\infty$ is strongly homotopy multiplicative, and therefore $B((\Omega^\infty S^\infty)_0 \times \mathbb{Z}) \cong B(\Omega^\infty S^\infty)$ [2, Lemma 7.1]. Combining these equivalences, we obtain the desired result $BF^+_\infty \cong B(\Omega^\infty S^\infty) \cong \Omega^{\infty-1} S^\infty$.

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