QUASI-COMMUTATIVITY OF H-SPACES

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Two H-space multiplications m and n on a space X are called H-*equivalent* provided there exists an H-map $f: (X, m) \to (X, n)$ that is a homotopy equivalence. H-equivalence is an equivalence relation, and the enumeration of the H-equivalence classes of multiplications is a topic of current interest. A question that arises in this connection is whether each multiplication m on an H-space is H-equivalent to its transpose, the multiplication n given by the relation $n(x, y) = m(y, x)$. This is certainly the case for each group multiplication, for we may choose the homotopy equivalence that takes each element to its inverse. A similar situation exists in loopspace multiplications. In [2], it is shown that every homotopy-Moufang H-space multiplication on a space is H-equivalent to its transpose, and it is suggested there that this relation may hold for all H-space multiplications. Problem 34 of [6] asks whether this is the case for each multiplication on a finite H-complex. In this paper, we develop an obstruction theory for this question, and we produce some counter-examples. We exhibit a multiplication on a generalized Eilenberg-MacLane space that is not H-equivalent to its transpose, and we demonstrate the existence of a multiplication on 3-dimensional real projective space that has the same property.

PROPOSITION A. *Let* $E = K(Z_2, 1) \times K(Z_2, 4)$. *The multiplication* m *on* E *characterized by making the composition*

$$K(Z_2, 1)^2 \xrightarrow{i_1^2} E^2 \xrightarrow{m} E \xrightarrow{P_2} K(Z_2, 4)$$

correspond to $x \otimes x^3 \in H^4(K(Z_2, 1) \wedge K(Z_2, 1); Z_2)$ *is not* H-*equivalent to its transpose.*

We could verify this proposition directly by computing the elements involved in the appropriate cohomology groups. Instead, we shall give an alternate verification, as an application of the general theory.

PROPOSITION B. *There exists an* H-*space multiplication on the real projective* 3-*space* P_3 *that is not* H-*equivalent to its transpose.*

We now develop the formalism necessary to handle these propositions. Corresponding to a pointed space X, we recall the space FX of unbased paths of varying lengths on X. The elements of FX are maps $\lambda: [0, \infty] \to X$ that are constant in some neighborhood of ∞. The basepoint of FX is the constant path at the basepoint of X. We shall consider the projections π_0 and π_∞ of FX to X given by evaluation at 0 and ∞, respectively. We make FX into a groupoid as follows. Let λ_1 and λ_2 in FX be such that $\pi_\infty(\lambda_1) = \pi_0(\lambda_2)$. Then define their *sum* $\lambda_1 + \lambda_2$ by the formula

$$(\lambda_1 + \lambda_2)[t] = \begin{cases} \lambda_1(t) & (0 \leq t \leq r_1), \\ \lambda_2(t - r_1) & (r_1 \leq t \leq \infty), \end{cases}$$

where

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$$r_1 = \sup\{x \in [0, \infty] \mid \text{there is a } y > x \text{ such that } \lambda_1(y) \neq \lambda_1(x)\}.$$

If X is an H-space with multiplication $(x, y) \to xy$, then FX inherits a *product* $(\lambda_1 \lambda_2)[t] = \lambda_1(t)\lambda_2(t)$. If $f\colon X \to X'$ is a map, the induced map $Ff\colon FX \to FX'$ is given by the formula $Ff(\lambda)[t] = f(\lambda(t))$. Let

$$PX = \{\lambda \in FX \mid \pi_0(\lambda) = x_0 = \text{basepoint of } X\}$$

and

$$\Omega X = \{\lambda \in PX \mid \pi_\infty(\lambda) = x_0\}.$$

Recall that if X and X' are H-spaces, a map $\theta\colon X \to X'$ is called an H-*map* provided there exists a homotopy $V\colon X^2 \to FX'$ such that

$$\pi_0 \circ V(x, y) = \theta(x)\,\theta(y) \quad \text{and} \quad \pi_\infty \circ V(x, y) = \theta(xy).$$

Suppose that $\phi\colon X \to X$ and $\phi'\colon X' \to X'$ are homotopy equivalences. Then we call a map $\theta\colon X \to X'$ a (ϕ, ϕ')-*map* provided there exists a homotopy $G\colon X \to FX'$ such that

$$\pi_0 \circ G = \phi' \circ \theta \quad \text{and} \quad \pi_\infty \circ G = \theta \circ \phi.$$

Following [2], we call an H-space *quasi-commutative* (a QC-*space*) provided there exist a homotopy equivalence $\phi\colon X \to X$ and a homotopy $Q_\phi\colon X^2 \to FX$ such that

$$\pi_0 \circ Q_\phi(x, y) = \phi(y)\,\phi(x) \quad \text{and} \quad \pi_\infty \circ Q_\phi(x, y) = \phi(xy).$$

Finally, if (X, ϕ) and (X', ϕ') are QC-spaces and $\theta\colon X \to X'$ is a map, we call θ a QC-*map* provided there exist homotopies $V\colon X^2 \to FX'$ and $G\colon X \to FX'$ that make θ an H-map and a (ϕ, ϕ')-map, respectively, and a secondary homotopy $D\colon X^2 \to F(FX')$ such that

$$\pi_0 \circ D = Q_{\phi'} \circ (\theta \times \theta), \qquad \pi_\infty \circ D = \theta \circ Q_\phi,$$

$$F\pi_0(D(x, y)) = V(\phi(y), \phi(x)) + G(y)\,G(x),$$

$$F\pi_\infty(D(x, y)) = G(xy) + F\phi'(V(x, y))$$

(see Figure 1).

$$\begin{tikzcd}
\phi'(\theta(x)\,\theta(y)) \arrow[r, "\phi' \circ V(x, y)"'] & \phi' \circ \theta(xy) \arrow[r, "G(xy)"'] & \theta(\phi(xy)) \\
\phi'(\theta(y))\,\phi'(\theta(x)) \arrow[u, "Q_{\phi'}(\theta(x),\, \theta(y))"] \arrow[r, "G(y)\,G(x)"] & \theta(\phi(y))\,\theta(\phi(x)) \arrow[r, "V(\phi(y),\, \phi(x))"] & \theta(\phi(y)\,\phi(x)) \arrow[u, "\theta \circ Q_\phi(x, y)"']
\end{tikzcd}$$

Figure 1.

We now describe the behavior of these structures in Postnikov systems. Let n be a positive integer, and let X be a space such that $\pi_k(X) = 0$ $(k \geq n)$. Let G be an abelian group, and let K(G, n) denote an Eilenberg-MacLane space of type (G, n). Let $\Omega K(G, n+1) \to E \xrightarrow{\pi} X$ be the principal fibration induced by a map

$$\theta : X \to K(G, n+1).$$

Then E consists of pairs $(x, \lambda) \in X \times PK(G, n+1)$ such that $\theta(x) = \lambda(\infty)$.

PROPOSITION 1. *If* E *is an* H-*space, then* X *can be made into an* H-*space so that* π *and* θ *are* H-*maps. Conversely, if an* H-*space multiplication for* X *is chosen for which* θ *is an* H-*map, then each* H-*space multiplication for* E *that makes* π *an* H-*map is homotopic to one of the form*

$$(x_1, \lambda_1)(x_2, \lambda_2) = (x_1 x_2, \lambda_1 \lambda_2 + V(x_1, x_2)),$$

where V *is a choice of homotopy* $V: X^2 \to FK(G, n+1)$ *as in the definition of* H-*map.*

PROPOSITION 2. *If* $\phi'': E \to E$ *is a homotopy equivalence, then there exist homotopy equivalences* $\phi: X \to X$ *and* $\phi': K(G, n+1) \to K(G, n+1)$ *such that* π *is a* (ϕ'', ϕ)-*map and* θ *is a* (ϕ, ϕ')-*map. Conversely, if* $\phi: X \to X$ *and*

$$\phi': K(G, n+1) \to K(G, n+1)$$

are homotopy equivalences such that θ *is a* (ϕ, ϕ')-*map, then each homotopy equivalence* $\phi'': E \to E$ *that makes* π *a* (ϕ'', ϕ)-*map is homotopic to one of the form*

$$\phi''(x, \lambda) = (\phi(x), F\phi''(\lambda) + G(x)),$$

where $G: X \to FK(G, n+1)$ *is a choice of map as in the definition of* (ϕ, ϕ')-*map.*

PROPOSITION 3. *Suppose that* E *and* X *are* H-*spaces such that* π *and* θ *are* H-*maps. Suppose that* $\phi'': E \to E$, $\phi: X \to X$, *and* $\phi': K(G, n+1) \to K(G, n+1)$ *are homotopy equivalences such that* π *is a* (ϕ'', ϕ)-*map and* θ *is a* (ϕ, ϕ')-*map.*

1) *If* (E, ϕ'') *is a* QC-*space, then* X *can be given a structure of* QC-*space such that* π *and* θ *are* QC-*maps.*

2) *Conversely, if* (X, ϕ) *is a* QC-*space with homotopy* $Q_\phi: X^2 \to FX$, *and* θ *is a* QC-*map, then each homotopy* $Q_{\phi''}: E^2 \to FE$ *that makes* E *into a* QC-*space such that* π *is a* QC-*map can be deformed (so as to preserve boundary conditions) into one of the form*

$$Q_{\phi''}((x_1, \lambda_1), (x_2, \lambda_2)) = (Q_\phi(x_1, x_2), FQ_{\phi'}(\lambda_1, \lambda_2) + D(x_1, x_2)),$$

where $D: X^2 \to F(FK(G, n+1))$ *is a choice of secondary homotopy as in the definition of* QC-*map.*

The proof of Proposition 1 is found in [5, pp. 127-129]. Proposition 2 is contained in the material on pp. 438-441 of [3]. Proposition 3 may be proved similarly to Theorems 16 and 17 of [5].

The homotopy classes of maps $V: X^2 \to FK(g, n+1)$ that preserve the boundary conditions for H-map are in one-to-one correspondence with the group of homotopy classes $[X \wedge X; \Omega K(G, n+1)]$, which in turn is isomorphic to $H^n(X \wedge X; G)$.

Similarly, the classes of maps $G: X \to FK(G, n+1)$ that preserve the boundary conditions for a (ϕ, ϕ')-map correspond to elements of $H^n(X; G)$, and the classes of maps $D: X^2 \to F(FK(G, n+1))$ that preserve the boundary conditions for QC-map correspond to elements of $H^{n-1}(X \wedge X; G)$.

Our objective is to be able to decide whether an H-space multiplication on E is quasi-commutative. We may write the multiplication in the form established by Proposition 1,

$$(x_1, \lambda_1)(x_2, \lambda_2) = (x_1 x_2, \lambda_1 \lambda_2 + F(x_1, x_2)).$$

A necessary and sufficient condition for the existence of a homotopy equivalence $\phi'': E \to E$ and a map $Q_{\phi''}: E^2 \to FE$ satisfying the definition of QC-space is that there exist a secondary homotopy $D: X^2 \to F(FK(G, n+1))$ satisfying the definition for θ to be a QC-map with respect to the boundary conditions determined by the G, ϕ, ϕ', Q_ϕ, and $Q_{\phi'}$ obtained from ϕ'' and $Q_{\phi''}$ as in Propositions 2 and 3. The obstruction to the existence of D may be transcribed to cohomology from Figure 1 as follows. It is the projection from $H^n(X \times X; G)$ onto $H^n(X \wedge X; G)$ of the class of

$$\begin{aligned}(Q_\phi^\# \circ \theta^\# - (\theta \times \theta)^\# \circ Q_{\phi'}^\# + T^\# \circ (\phi \times \phi)^\# \circ V^\# - V^\# \circ (\phi')^\# \\ + T^\# \circ (G \times G)^\# \circ m_1^\# - m^\# \circ G^\#)[\iota],\end{aligned}$$

where $\iota \in C^q(G, n; G)$ is a representative of the fundamental class, and where m and m_1 are the multiplications in X and $K(G, n+1)$, respectively.

Example 1. Let $X = K(Z_2, 1)$, and let $\theta: X \to K(Z_2, 5)$ be the constant map. Then $E = K(Z_2, 1) \times K(Z_2, 4)$. We regard θ as a homomorphism of loop spaces. Hence ϕ and ϕ', unique up to homotopy, may be taken to be the maps induced by reversing the parameter of loops. The maps Q_ϕ and $Q_{\phi'}$, again unique up to homotopies that preserve the boundary conditions, may be chosen to be the stationary homotopies. Consequently, for elements

$$v \in H^4(K(Z_2, 1) \wedge K(Z_2, 1); Z_2) \quad \text{and} \quad g \in H^4(K(Z_2, 1); Z_2),$$

the obstruction becomes

$$(T^* - 1^*)[v] - \overline{m}^*(g).$$

Now $H^*(K(Z_2, 1); Z_2)$ is a polynomial algebra with generator x of dimension 1. The nonzero element $x^4 \in H^4(K(Z_2, 1); Z_2)$ is primitive. If we choose the multiplication on $K(Z_2, 1) \times K(Z_2, 4)$ determined by $v = x \otimes x^3$, the obstruction is $x^3 \otimes x - x \otimes x^3$, and this is not zero. Hence we have a multiplication on $K(Z_2, 1) \times K(Z_2, 4)$ that is not quasi-commutative, and this proves Proposition A.

Example 2. We consider a Postnikov system for the 3-dimensional real projective space P_3. Since P_3 is the homotopy type of $\Omega BSO(3)$, we may choose the spaces and maps to be loop spaces and loop maps. It is easy to compute that P_3 has only two homotopy classes of homotopy self-equivalences, the identity and another, ϕ, which we may represent by reversing the parameter of loops. It was proved in [1] that no multiplication on P_3 is homotopy-commutative. Hence, to find a multiplication on P_3 that is not quasi-commutative, it is sufficient to find one, m, such that ϕ is not an H-map between m and its transpose.

The first Postnikov invariant is $\theta_1 \in H^4(K(Z_2, 1); Z)$. We obtain a fibration $\Omega K(Z, 4) \to E_2 \to K(Z_2, 1)$. If we take the loop addition on E_2 and induce

ϕ: $E_2 \to E_2$ by reversing the direction of loops, we can choose the homotopy Q_ϕ between $\phi(x+y)$ and $\phi(y)+\phi(x)$ to be the stationary homotopy. It is unique up to a homotopy that preserves boundary conditions, since $H^2(K(Z_2, 1) \wedge K(Z_2, 1); Z) = 0$. The second Postnikov invariant is $\theta_2 \in H^5(E_2; Z_2)$, and this induces a fibration $\Omega K(Z_2, 5) \to E_3 \to E_2$. The obstruction for θ_2 to be a QC-map with respect to ϕ takes the same form as that of Example 1,

$$(T^* - 1^*)[v] - \overline{m}^*(g) \in H^4(E_2 \wedge E_2; Z_2),$$

where $v \in H^4(E_2 \wedge E_2; Z_2)$ and $g \in H^4(E_2; Z_2) = H^4(P_3; Z_2) = 0$. For any choice of $v \in H^4(E_2 \wedge E_2; Z_2)$ (and hence a choice of multiplication in E_3), the quasi-commutativity obstruction is $T^*(v) - v$. In dimensions less than four, we have the relation $H^*(E_2; Z_2) = H^*(K(Z_2, 1); Z_2)$. If we again choose $v = x \otimes x^3$, we obtain a multiplication m_0 for E_3 that is not quasi-commutative. It follows from Theorem 1.1 of [4] that each multiplication on a stage of a Postnikov system for P_3 lifts to a multiplication on P_3. Any lifting of m_0 to P_3 produces a non-quasi-commutative H-space structure on P_3. This verifies Proposition B.

Remark 1. A similar (but longer) computation produces an H-space structure that is not quasi-commutative on the underlying space of SU(3), thus providing a simply connected example.

Remark 2. Since any homotopy-Moufang H-space multiplication is quasi-commutative, our examples must be H-space structures that are not homotopy-Moufang spaces.

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