HARDY CLASSES AND RANGES OF FUNCTIONS

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I. INTRODUCTION

1. Let D be a region (that is, a connected, nonempty, open set) in the complex plane $\mathscr{C}$. Following M. Parreau [6] and W. Rudin [8], for each positive real number p, we let $H_p(D)$ denote the collection of functions f, analytic on D, for which $|f|^p$ has a harmonic majorant. (In the case where D is the unit disk, $H_p(D)$ as just defined coincides with the usual Hardy class H_p.) We let $H_0(D)$ denote the collection of analytic functions on D. *For each fixed function* $f \in H_0(D)$, *we seek to determine, by studying* f(D), *the numbers* p *for which* $f \in H_p(D)$.

One of the first results in this direction is due to Smirnov [7, p. 64]. He showed that if f is analytic on Δ, where $\Delta = \{|z| < 1\}$, and has positive real part, then $f \in H_p(\Delta)$ $(0 < p < 1)$. It is an easy step to go from Smirnov's Theorem to the result that $f \in H_p(\Delta)$ $(0 < p < \pi/\alpha)$ if $f(\Delta)$ is contained in a sector whose angular opening is α $(0 < \alpha \leq 2\pi)$. This was pointed out by G. T. Cargo [2], who also proved the following results for a function $f \in H_0(\Delta)$:

(1) If $f(\Delta) \subseteq \Omega \subsetneq \mathscr{C}$, where Ω is simply-connected, then $f \in H_p(\Delta)$ $(0 < p < 1/2)$.

(2) If $f(\Delta)$ is contained in an infinite strip, then $f \in H_p(\Delta)$, for all positive numbers p.

Cargo proved these last two results using the principle of subordination. Thus, the existing results are limited to the case where $f(\Delta) \subseteq \Omega \subsetneq \mathscr{C}$ and Ω is simply-connected.

We begin by introducing the *Hardy number* $h(\Omega)$ of a region $\Omega \subseteq \mathscr{C}$ (Chapter II). The Hardy number $h(\Omega)$ has the property that if $f \in H_0(D)$, $f(D) \subseteq \Omega$, and $h(\Omega) > 0$, then $f \in H_p(D)$ $(0 < p < h(\Omega))$. Therefore, progress in solving the stated problem will come from a study of Hardy numbers; in particular, from lower bounds for Hardy numbers. Chapter III is a step in this direction. Whereas the existing results are limited to functions whose image lies in a proper simply-connected subregion of $\mathscr{C}$, we give a lower bound for the Hardy number of an arbitrary region (Section 3). Some theorems of M. Tsuji play the central role here. The bound in Section 3 permits us to determine exactly the Hardy number of a starlike region (Section 4). We also derive a lower bound for the Hardy number of a simply-connected region whose boundary is sufficiently regular (Section 5). In some cases this is an improvement of the bound in Section 3, since it takes into account a rotational factor. Our tool here is Ahlfors' distortion theorem [1].

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In the special case where $f \in H_0(\Delta)$, existing theorems for Hardy classes on Δ allow us to relate the Hardy number of $f(\Delta)$ with the growth of the maximum modulus and with the Taylor coefficients of f (Chapter IV).

Finally, we prove a theorem for an arbitrary region, of which the Phragmén-Lindelöf Theorem for a half-plane is a special case (Chapter V). This enables us to give a lower bound for the lower order of an entire function f in terms of the Hardy numbers of the components of the sets $\{|f| > c\}$.

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II. THE NOTION OF A HARDY NUMBER

2. Let $D \subseteq \mathscr{C}$ be a region, and let $f \in H_0(D)$. Let $\Omega \subseteq \mathscr{C}$ be a region containing $f(D)$. If $g \in H_p(\Omega)$ for some $p > 0$, then $|g|^p$ has a harmonic majorant u. Therefore $|g \circ f|^p \leq u \circ f$, and thus, since $u \circ f$ is again harmonic, we conclude that $g \circ f \in H_p(D)$. In particular, if g is the identity map on Ω, then $f \in H_p(D)$.

We are thus motivated to make the following definition.

Definition 2.1. Let $\Omega \subseteq \mathscr{C}$ be a region. The *Hardy number* of Ω is defined by the condition

$$h(\Omega) = \sup\{p \geq 0\colon I_\Omega \in H_p(\Omega)\},$$

where I_Ω is the identity map on Ω.

Let $\Omega \subseteq \mathscr{C}$ be a region with $h(\Omega) > 0$, and let $0 < p < q$. Since $|z|^p \leq 1 + |z|^q$, it follows that $I_\Omega \in H_p(\Omega)$ as long as $0 \leq p < h(\Omega)$. The significance of Definition 2.1 is that if $f \in H_0(D)$ and $f(D) \subseteq \Omega$, then $f \in H_p(D)$ for all p satisfying the inequality $0 \leq p < h(\Omega)$.

This may not always lead to a significant result. For example, if

$$F(z) = \left(\frac{1+z}{1-z}\right)^3 \qquad (|z| < 1),$$

then $F \in H_p(\Delta)$ $(0 < p < 1/3)$. However, $F(\Delta) = \mathscr{C} - \{0\}$, which has Hardy number zero (see observation 4 below).

We may make the following observations.

(1) If Ω_1 and Ω_2 are regions with $\Omega_1 \subseteq \Omega_2 \subseteq \mathscr{C}$, then $h(\Omega_2) \leq h(\Omega_1)$.

(2) Let $\Omega_1 \subseteq \mathscr{C}$ be a region, and suppose that $\Omega_2 = \{az + b\colon z \in \Omega_1\}$, where $a, b \in \mathscr{C}$ and $a \neq 0$. Then $h(\Omega_1) = h(\Omega_2)$.

(3) Let $\Omega \subseteq \mathscr{C}$ be a bounded region. Then $h(\Omega) = +\infty$. Thus, in the sequel, the only case of interest will be the case where Ω is unbounded.

(4) If $\mathscr{C} - \Omega$ is bounded, then $h(\Omega) = 0$. For, if $\mathscr{C} - \Omega \subseteq \{|z| \leq R\}$ $(R > 0)$, then $F(z) = R \exp\left(\frac{1+z}{1-z}\right)$ maps Δ into Ω, and F belongs to $H_p(\Delta)$ for no $p > 0$.

(5) Smirnov's Theorem shows that $J \in H_p(\Delta)$ $(0 < p < 1)$, where $J(z) = (1+z)/(1-z)$. Since J maps Δ conformally onto $\{\Re z > 0\}$, we have that $h[\{\Re z > 0\}] \geq 1$. However, since $J \notin H_1(\Delta)$, we must have that $h[\{\Re z > 0\}] = 1$.

Similarly, one shows that $h(S) = \pi/\alpha$, where S is a sector whose angular opening is α $(0 < \alpha \leq 2\pi)$.

III. LOWER BOUNDS FOR HARDY NUMBERS

In Section 3, we derive a lower bound for the Hardy number of an arbitrary region $\Omega \subseteq \mathscr{C}$. We show in Section 4 that if Ω is unbounded and starlike with respect to the point $z = 0$, then the lower bound of Section 3 determines $h(\Omega)$ exactly. We use the Ahlfors Distortion Theorem to give a lower bound for the Hardy number of a simply-connected region whose boundary is sufficiently regular (Section 5).

3. Let $\Omega \subseteq \mathscr{C}$ be an unbounded region. Let $\rho_0 = \inf\{|z|: z \in \Omega\}$ and $\rho_1 = 1 + \rho_0$. For $t \in (\rho_0, +\infty)$, we define

(1) $$\alpha_\Omega(t) = \max\{m(E): E \text{ is a subarc of } \Omega \cap \{|z| = t\}\},$$

where $m(E)$ denotes the angular Lebesgue measure of E. The function $1/\alpha_\Omega$ is upper-semicontinuous and hence is bounded on compact subsets of $(\rho_0, +\infty)$. For $t \geq 0$, we define

(2) $$\chi_\Omega(t) = \begin{cases} 0 & \text{if } \{|z| = t\} \subseteq \Omega, \\ 1 & \text{if } \{|z| = t\} \not\subseteq \Omega. \end{cases}$$

We note that χ_Ω is the characteristic function of the circular projection of $\mathscr{C} - \Omega$ onto the nonnegative real axis. For $t > \rho_1$, let

(3) $$B_\Omega(t) = \frac{\pi}{\log t} \int_{\rho_1}^{t} \frac{\chi_\Omega(r)\,dr}{r\alpha_\Omega(r)}.$$

THEOREM 3.1. *If $\Omega \subseteq \mathscr{C}$ is an unbounded region, then*

(4) $$h(\Omega) \geq \liminf_{t \to +\infty} B_\Omega(t).$$

Proof. Since the theorem holds trivially otherwise, we consider only the case where $\ell = \liminf_{t\to+\infty} B_\Omega(t) > 0$. We must show that if $0 < p < \ell$, then there exists a function u, harmonic on Ω, that satisfies the inequality $|z|^p \leq u(z)$ for all $z \in \Omega$. This follows from the proof of Theorem III.70 of M. Tsuji [10, pp. 118-119], if, instead of the estimate for harmonic measure that is used there, we use the stronger estimate proved by Tsuji in [9].

In general, equality does not hold in (4), as the examples at the end of Section 5 illustrate. However, we shall show in Section 4 that if $\Omega \subsetneq \mathscr{C}$ is an unbounded region that is starlike with respect to the point $z = 0$, then equality does hold in (4), and

$$h(\Omega) = \lim_{t\to\infty} \frac{\pi}{\alpha_\Omega(t)}.$$

Using the method of proof of Theorem 3.1, one could get an upper bound for $h(\Omega)$, if a lower estimate for harmonic measure were available (just as a better upper estimate for harmonic measure would give a better lower bound for $h(\Omega)$). In general, however, lower estimates for harmonic measure do not exist.

We now fix some notation for the remainder of this section. Let E be a Lebesgue-measurable set of positive real numbers. We let $m_\ell(E)$ denote its logarithmic measure: $m_\ell(E) = \int_E \frac{1}{t}\,dt$. If r is a real number $(r \geq 1)$, we put $E(r) = E \cap [1, r]$. Then the lower logarithmic density of E is given by the expression

$$\underline{d}_\ell(E) = \liminf_{r \to +\infty} \frac{m_\ell[E(r)]}{\log r}.$$

If $\Omega \subseteq \mathscr{C}$ is a region, we let $P_\Omega = \{|z|: z \in \mathscr{C} - \Omega\}$, the circular projection of $\mathscr{C} - \Omega$ onto the nonnegative real axis.

Corollaries 3.2 to 3.4 follow easily from Theorem 3.1.

COROLLARY 3.2. *Let* $\Omega \subseteq \mathscr{C}$ *be an unbounded region. Let* $E \subseteq P_\Omega$ *be a measurable set and* χ *its characteristic function. Let* $\alpha_0 = \limsup_{t \to +\infty} [\chi(t)\alpha_\Omega(t)]$. *If* $\underline{d}_\ell(E) > 0$, *then* $h(\Omega) \geq (\pi/\alpha_0)\underline{d}_\ell(E)$.

COROLLARY 3.3. *Let* $\Omega \subseteq \mathscr{C}$ *be a region. Then* $h(\Omega) \geq \underline{d}_\ell(P_\Omega)/2$.

COROLLARY 3.4. *Let* $\Omega \subsetneq \mathscr{C}$ *be a simply-connected region. Then* $h(\Omega) \geq 1/2$. (This is equivalent to a result of Cargo [2].)

We conclude this section with a theorem that is similar to the Denjoy-Carleman-Ahlfors Theorem [1].

THEOREM 3.5. *Let* $\{\Omega_k\}$ *be a collection of* n $(n \geq 2)$ *unbounded disjoint subregions of* $\mathscr{C}$. *Let* α_k, χ_k, *and* B_k *be defined relative to* Ω_k *as in* (1), (2), *and* (3). *If* $\lim_{t \to +\infty} B_k(t)$ *exists (finite or infinite) for each* k $(k = 1, 2, \cdots, n)$, *then* $h(\Omega_{k_0}) \geq n/2$ *for some* k_0.

Proof. Let R $(R \geq 1)$ be so large that $\Omega_k \cap \{|z| < R\} \neq \emptyset$ for each k $(k = 1, 2, \cdots, n)$. Then $\chi_k(t) = 1$ for $t \geq R$, and hence

$$\lim_{t \to +\infty} B_k(t) = \lim_{t \to +\infty} \frac{\pi}{\log t} \int_R^t \frac{dr}{r\,\alpha_k(r)} \qquad (k = 1, 2, \cdots, n).$$

Therefore, since

$$\sum_{k=1}^{n} \frac{1}{\alpha_k(r)} \geq n^2 \left(\sum_{k=1}^{n} \alpha_k(r) \right)^{-1} \geq \frac{n^2}{2\pi},$$

we have the inequalities

$$\frac{1}{n} \sum_{k=1}^{n} h(\Omega_k) \geq \frac{1}{n} \sum_{k=1}^{n} \lim_{t \to \infty} \frac{\pi}{\log t} \int_R^t \frac{dr}{r\alpha_k(r)}$$

$$= \lim_{t\to\infty} \frac{\pi}{n \log t} \int_R^t \frac{1}{r}\left[\sum_{k=1}^{n} \frac{1}{\alpha_k(r)}\right] dr$$

$$\geq \frac{n}{2} \lim_{t\to+\infty} \frac{\log t/R}{\log t} = \frac{n}{2}.$$

Hence, $h(\Omega_{k_0}) \geq n/2$ for some k_0.

We suspect that the theorem is true without the regularity hypothesis that each $\lim_{t\to+\infty} B_k(t)$ exist. If so, in view of Theorem 7.2, Theorem 3.5 could be used to give another proof of the Denjoy-Carleman-Ahlfors Theorem.

4. It follows from Corollary 3.2 that

$$h(\Omega) \geq \liminf_{t\to+\infty} \frac{\pi}{\alpha_\Omega(t)}, \tag{5}$$

if Ω is unbounded and $\chi_\Omega(t) = 1$ for $t \geq t_0$. In this section, we prove the following result.

THEOREM 4.1. *Let* $\Omega \subsetneq \mathscr{C}$ *be an unbounded region that is starlike with respect to the point* $z = 0$. *Then*

$$h(\Omega) = \lim_{t\to+\infty} \frac{\pi}{\alpha_\Omega(t)}.$$

Proof. Since Ω is starlike with respect to $z = 0$, the function $t \to \alpha_\Omega(t)$ is nonincreasing and positive, and therefore $A = \lim_{t\to+\infty} \alpha_\Omega(t)$ exists. Thus inequality (5) becomes

$$h(\Omega) \geq \frac{\pi}{A}. \tag{6}$$

If $A = 0$, then $h(\Omega) = \pi/A = +\infty$.

Suppose that $A > 0$. For each $t > 0$, there are only finitely many subarcs E of $\Omega \cap \{|z| = t\}$ with $m(E) > \alpha_\Omega(t)/2$, where $m(E)$ is the angular Lebesgue measure of E. We let E_t be some subarc of $\Omega \cap \{|z| = t\}$ satisfying the condition $m(E_t) = \alpha_\Omega(t)$. For each $t > 0$, let $k(t)$ have the following properties:

(i) $k(t) \in [0, 2\pi)$;

(ii) if $E_t \neq \{|z| = t\}$, the ray $\{xe^{ik(t)}: x \geq 0\}$ is the bisector of E_t. Otherwise, let $k(t) = 0$.

Noting that $\{k(n)\}_{n=1}^{\infty}$ is a bounded sequence, we conclude that there exists a subsequence $\{k(n_j)\}_{j=1}^{\infty}$ with $k(n_j) \to k_0$. Thus, for each δ $(0 < \delta < A/2)$, there exists N so large that

$$\left\{ xe^{i\theta}: 0 < x < n_j \text{ and } k_0 - \frac{A}{2} + \delta < \theta < k_0 + \frac{A}{2} - \delta \right\} \subseteq \Omega$$

for all $j \geq N$. Since $n_j \to \infty$ as $j \to \infty$, we have the inclusion

$$\left\{ xe^{i\theta}\colon x > 0 \text{ and } k_0 - \frac{A}{2} + \delta < \theta < k_0 + \frac{A}{2} - \delta \right\} \subseteq \Omega$$

for all δ $(0 < \delta < A/2)$. Letting $\delta \to 0$, we conclude that

$$S = \left\{ xe^{i\theta}\colon x > 0 \text{ and } k_0 - \frac{A}{2} < \theta < k_0 + \frac{A}{2} \right\} \subseteq \Omega .$$

Since $h(S) = \pi/A$ (by observation 5 of Section 2) and $S \subseteq \Omega$, we have that

$$h(\Omega) \leq h(S) = \frac{\pi}{A} .$$

This, together with inequality (6) above, completes the proof.

Remark I. We have determined the Hardy number of every starlike region: If Ω is bounded, then $h(\Omega) = +\infty$; if Ω is unbounded, an appropriate translation takes Ω into a region satisfying the hypotheses of Theorem 4.1, and this region has the same Hardy number as Ω.

Remark II. Let Ω satisfy the hypotheses of Theorem 4.1. Suppose that $A = \lim_{t \to +\infty} \alpha_\Omega(t) > 0$. We showed that for some k_0,

$$S = \left\{ xe^{i\theta}\colon x > 0 \text{ and } k_0 - \frac{A}{2} < \theta < k_0 + \frac{A}{2} \right\} \subseteq \Omega$$

and $h(S) = h(\Omega) = \pi/A$. Therefore, each region G with $S \subseteq G \subseteq \Omega$ has Hardy number π/A also.

5. The object of this section is to develop a lower bound for $h(\Omega)$ in the case where the region $\Omega \subseteq \mathscr{C}$ is simply-connected and has a regular boundary (satisfying the hypotheses of Theorem 5.2). We derive this bound using the Ahlfors Distortion Theorem [1], which may be stated as follows.

THEOREM 5.1. *Let* G *be a simply-connected region in the* s-*plane* $(s = x + iy)$. *Let* $\Gamma\colon (0, 1) \to G$ *be continuous, one-to-one, and suppose* Γ *satisfies the conditions*

$$\lim_{t \to 0} \Re\, \Gamma(t) = -\infty \quad \text{and} \quad \lim_{t \to 1} \Re\, \Gamma(t) = +\infty .$$

Suppose that on each line $\Re\, s = x$ *there exists a segment* θ_x *satisfying the conditions*

(1) θ_x *has finite length* $\theta(x)$;

(2) θ_x *lies, except for its endpoints, in* G;

(3) θ_x *separates* $\Gamma(t)$, *for* t *near* 0, *from* $\Gamma(t)$, *for* t *near* 1; *and,*

(4) *whenever* $x_1 < x_2$, θ_{x_2} *separates* θ_{x_1} *from* $\Gamma(t)$, *for* t *near* 1.

Suppose F *is one-to-one and maps* G *conformally onto* $T = \{w\colon |\Im\, w| < \pi/2\}$ *such that* $\lim_{t \to 0} \Re\, F[\Gamma(t)] = -\infty$ *and* $\lim_{t \to 1} \Re\, F[\Gamma(t)] = +\infty$. *Put*

$$\xi_1(x) = \inf_{s \in \theta_x} \Re\, F(s) \text{ and } \xi_2(x) = \sup_{s \in \theta_x} \Re\, F(s) .$$

Then

$$\xi_1(x_2) - \xi_2(x_1) \geq \pi \int_{x_1}^{x_2} \frac{dx}{\theta(x)} - 4\pi,$$

provided $\int_{x_1}^{x_2} \frac{dx}{\theta(x)} > 2$.

In Theorem 5.2, we follow Ahlfors by letting θ_x^j denote the cross-cut lying on $\{\Re s = x\}$, which, among those cross-cuts satisfying condition (3) in Theorem 5.1, is met first if the curve Γ_j is described in the positive direction.

THEOREM 5.2. *Let* $\Omega \subseteq \mathscr{C}$ *be a simply-connected region whose boundary contains the point* $z = 0$. *Let* $g\colon \Omega \to \mathscr{C}$ *be an analytic logarithm of the identity map on* Ω, *and let* $G = g(\Omega)$. *Suppose that* G *satisfies the following conditions:*

1. *There exist a finite collection of curves* $\{\Gamma_j\}_{j=1}^n$, *as in the Ahlfors theorem, with respective cross-cuts* $\{\theta_x^j\}_{j=1}^n$, *and a positive real number* R *so that if* $s \in G$ *and* $\Re s = x > R$, *then* $s \in \theta_x^j$, *for some* j $(j = 1, 2, \cdots, n)$.

2. *For* $x > R$, *each component of* $G \cap \{\Re s > x\}$ *intersects a unique curve* Γ_j $(1 \leq j \leq n)$.

Let $y_j(x)$ *denote the ordinate of the lower endpoint of the cross-cut* θ_x^j. *Then*

$$h(\Omega) \geq \frac{\pi(1 + \lambda^2)}{\beta},$$

where

$$\beta = \max_{1 \leq j \leq n} \{\limsup_{x \to +\infty} \theta^j(x)\}$$

and

$$\lambda = \liminf_{\substack{x \to +\infty \\ x > x_1}} \left| \frac{y_j(x) - y_j(x_1)}{x - x_1} \right|$$

for some fixed j $(1 \leq j \leq n)$.

Proof. Let F_j denote a univalent analytic map of G onto the strip $T = \{\xi + i\eta\colon |\eta| < \pi/2\}$ such that

$$\lim_{t \to 0} \Re F_j[\Gamma_j(t)] = -\infty \qquad \text{and} \qquad \lim_{t \to 1} \Re F_j[\Gamma_j(t)] = +\infty$$

(see [5, p. 19] for a theorem that can be used to establish the existence of such a map). Let $\xi_1^j(x)$ and $\xi_2^j(x)$ denote, respectively, the infimum and supremum of $\Re F_j$ on the cross-cut θ_x^j.

Following the method of Ahlfors in his proof of a sharpening of the Denjoy Conjecture [1, pp. 25-27], we put in the s-plane a new rectangular coordinate system (u, v) whose axes make an acute angle α with the positive x- and y-axes, respectively. That is, for a fixed real number α $(|\alpha| < \pi/2)$, the positive u-axis is the set $\{re^{i\alpha}\colon r > 0\}$ and the positive v-axis is the set $\{re^{i(\alpha+\pi/2)}\colon r > 0\}$. We shall dispose of α shortly.

Fix x_1 and x_2 $(x_1 < x_2)$. Choose u_1 and u_2 as follows: If $\alpha \geq 0$, let

$$u_1 = x_1 \cos \alpha + [y_j(x_1) + 2\pi] \sin \alpha$$

and

$$u_2 = x_2 \cos \alpha + y_j(x_2) \sin \alpha ;$$

if $\alpha < 0$, let

$$u_1 = x_1 \cos \alpha + y_j(x_1) \sin \alpha$$

and

$$u_2 = x_2 \cos \alpha + [y_j(x_2) + 2\pi] \sin \alpha ,$$

for some fixed j $(1 \leq j \leq n)$. When u_1 and u_2 are chosen in this manner, we have that

(i) the line $u = u_1$ passes through the point having (x, y)-coordinates $(x_1, y_j(x_1) + 2\pi)$ or $(x_1, y_j(x_1))$, according as $\alpha \geq 0$ or $\alpha < 0$, and

(ii) the line $u = u_2$ passes through the point having (x, y)-coordinates $(x_2, y_j(x_2))$ or $(x_2, y_j(x_2) + 2\pi)$, according as $\alpha \geq 0$ or $\alpha < 0$.

With u_1 and u_2 chosen as above, we have the relation

$$(u_2 - u_1) = (x_2 - x_1)\cos \alpha + [y_j(x_2) - y_j(x_1) \mp 2\pi] \sin \alpha , \tag{7}$$

where we use the upper or lower sign according as $\alpha \geq 0$ or $\alpha < 0$. Now fix α so that

$$\tan \alpha = \frac{y_j(x_2) - y_j(x_1)}{x_2 - x_1} .$$

With this value of α, we have that

$$\begin{aligned} (u_2 - u_1) &= \{(x_2 - x_1)^2 + [y_j(x_2) - y_j(x_1)]^2\}^{1/2} - 2\pi \sin |\alpha| \\ &\geq \{(x_2 - x_1)^2 + [y_j(x_2) - y_j(x_1)]^2\}^{1/2} - 2\pi . \end{aligned} \tag{8}$$

For each $u_0 \in [u_1, u_2]$, there exist a finite number of cross-cuts on the line $u = u_0$ that separate $\theta^j_{x_1}$ from $\theta^j_{x_2}$. Let δ_{u_0} be the cross-cut, among those satisfying this condition, that is first met if we describe the curve Γ_j in the positive direction. Let δ_{u_0} have length $\delta(u_0)$. For $u \in [u_1, u_2]$, we let $\xi_1(u)$ and $\xi_2(u)$ denote, respectively, the infimum and supremum of $\Re F_j$ on the cross-cut δ_u. Then, by Theorem 5.1,

$$\xi_1(u_2) - \xi_2(u_1) \geq \pi \int_{u_1}^{u_2} \frac{du}{\delta(u)} - 4\pi , \tag{9}$$

provided

$$\int_{u_1}^{u_2} \frac{du}{\delta(u)} > 2 . \tag{10}$$

An application of the Schwarz Inequality yields the inequality

$$\int_{u_1}^{u_2} \frac{du}{\delta(u)} \geq \frac{(u_2 - u_1)^2}{\int_{u_1}^{u_2} \delta(u)\,du}. \tag{11}$$

Combining inequalities (9) and (11), we see that

$$\xi_1(u_2) - \xi_2(u_1) \geq \frac{\pi(u_2 - u_1)^2}{\int_{u_1}^{u_2} \delta(u)\,du} - 4\pi, \tag{12}$$

whenever condition (10) is satisfied.

Hypotheses 1 and 2 imply that

$$\bigcup_{[u_1,u_2]} \delta_u \subseteq G \cap \{x_1 \leq \Re s \leq x_2\} \qquad (x_1 > R).$$

Hence we must have that $\int_{u_1}^{u_2} \delta(u)\,du \leq 2\pi(x_2 - x_1)$. Using this fact and inequalities (8) and (11), we find that (10) is satisfied if $x_1 > R$ and $(x_2 - x_1) > 4\pi(1 + \sqrt{3}/2)$.

As long as $x_1 > R$, we have the inclusion

$$\bigcup_{[u_1,u_2]} \delta_u \subseteq \bigcup_{[x_1,x_2]} \theta_x^j,$$

and thus

$$\int_{u_1}^{u_2} \delta(u)\,du \leq \int_{x_1}^{x_2} \theta^j(x)\,dx \leq (x_2 - x_1) \cdot \sup_{x \geq x_1} \theta^j(x).$$

Therefore, if $x_1 > R$ and $(x_2 - x_1) > 4\pi(1 + \sqrt{3}/2)$, we get the inequality

$$\xi_1(u_2) - \xi_2(u_1) \geq \frac{\pi(u_2 - u_1)^2}{(x_2 - x_1) \cdot \sup_{x \geq x_1} \theta^j(x)} - 4\pi. \tag{13}$$

We observe that $\xi_1^j(x_2) \geq \xi_1(u_2)$ and $\xi_2^j(x_1) \leq \xi_2(u_1)$. Combining inequalities (8) and (13) with this observation, we find that

$$\xi_1^j(x_2) - \xi_2^j(x_1) \geq \frac{\pi(x_2 - x_1)}{\sup_{x \geq x_1} \theta^j(x)} \left[\sqrt{1 + \left(\frac{y_j(x_2) - y_j(x_1)}{x_2 - x_1} \right)^2} - \frac{2\pi}{(x_2 - x_1)} \right]^2 - 4\pi, \tag{14}$$

as long as $x_1 > R$ and $(x_2 - x_1) > 4\pi(1 + \sqrt{3}/2)$.

Hypotheses 1 and 2 imply that λ is independent of (i) the choice of the analytic logarithm g, (ii) the choice of x_1, and (iii) the choice of j $(1 \leq j \leq n)$.

Let ε and Λ be fixed ($\varepsilon > 0$ and $0 < \Lambda < 1 + \lambda^2$). Fix x_1 $(x_1 = x_1(\varepsilon) > R)$ so large that

$$\max_{1\leq j\leq n}\ [\sup_{x\geq x_1}\ \theta^j(x)] \leq \beta + \varepsilon .$$

Then pick N $(N > x_1 + 4\pi(1 + \sqrt{3}/2))$ so large that when $x_2 > N$, we have that

$$\left[\sqrt{1+\left(\frac{y_j(x_2) - y_j(x_1)}{x_2 - x_1}\right)^2} - \frac{2\pi}{(x_2 - x_1)}\right]^2 > \Lambda$$

for each j $(1 \leq j \leq n)$.

By (14), we thus have, for x_1 as fixed above and for $x_2 > N$, the inequality

$$\xi_1^j(x_2) - \xi_2^j(x_1) \geq (x_2 - x_1)\left[\frac{\pi\Lambda}{\beta+\varepsilon}\right] - 4\pi \qquad (j = 1, 2, \cdots, n).$$

Hence for $x > N$,

$$\left[\frac{\pi\Lambda}{\beta+\varepsilon}\right]x \leq \left\{4\pi + \left[\frac{\pi\Lambda}{\beta+\varepsilon}\right]x_1 - \xi_2^j(x_1)\right\} + \xi_1^j(x) = K_j + \xi_1^j(x)$$

(K_j is hereby defined). Therefore, if $\Re s > N$ and $s \in \theta_x^j$, we have the inequalities

$$\left[\frac{\pi\Lambda}{\beta+\varepsilon}\right]\Re s \leq K_j + \xi_1^j(\Re s) \leq K_j + \Re F_j(s).$$

Recall that g is an analytic logarithm of the identity map on Ω. We conclude that if $z \in \Omega$, $\log|z| > N$, and $g(z) \in \theta_x^j$, then, for $q > 0$, we have the relation

$$q\left[\frac{\pi\Lambda}{\beta+\varepsilon}\right]\log|z| \leq q\,\{K_j + \Re F_j[g(z)]\},$$

and hence

$$|z|^{q\left[\frac{\pi\Lambda}{\beta+\varepsilon}\right]} \leq \exp\{q[K_j + \Re F_j[g(z)]]\} = |\exp\{K_j + F_j[g(z)]\}|^q .$$

Note that $\exp\{K_j + F_j[g(z)]\}$ is an analytic function on Ω with positive real part. Therefore, by Smirnov's Theorem [7, p. 64], it follows that

$$|\exp\{K_j + F_j \circ g\}|^q$$

has a harmonic majorant U_j for fixed q $(0 < q < 1)$. It is clear that $U_j \geq 0$ on Ω. Consequently, we see that

$$|z|^{q\left[\frac{\pi\Lambda}{\beta+\varepsilon}\right]} \leq \sum_{j=1}^{n} U_j(z),$$

if $|z| > e^N$ and $0 < q < 1$. Therefore,

$$|z|^{q\left[\frac{\pi\Lambda}{\beta+\varepsilon}\right]} \leq e^{\left[\frac{qN\pi\Lambda}{\beta+\varepsilon}\right]} + \sum_{j=1}^{n} U_j(z)$$

for all $z \in \Omega$, as long as q is fixed $(0 < q < 1)$. Hence $h(\Omega) \geq q\frac{\pi\Lambda}{\beta+\varepsilon}$. Letting $q \to 1$, $\varepsilon \to 0$, and $\Lambda \to 1 + \lambda^2$, we conclude that

$$h(\Omega) \geq \frac{\pi(1+\lambda^2)}{\beta}. \tag{15}$$

By way of interpretation of this conclusion, we remark that β is the upper limit of the angular measures of the components of $\Omega \cap \{|z| = r\}$, as $r \to +\infty$. The presence of the rotational factor λ in the lower bound (15) seems to indicate that $h(\Omega)$ increases as "the complement of Ω is wrapped more tightly about the origin."

Example I. Let $\Omega = \mathscr{C} - \{e^{r(\cos\alpha + i\sin\alpha)}: r \text{ real}\}$, where $|\alpha| < \pi/2$ is fixed. Then Ω satisfies the hypotheses of Theorem 5.2 with $\beta = 2\pi$ and $\lambda = |\tan\alpha|$. Hence

$$h(\Omega) \geq \frac{1+\tan^2\alpha}{2} = \frac{1}{2\cos^2\alpha}.$$

This is the exact value of $h(\Omega)$, as can be seen by considering the function mapping $\{|z| < 1\}$ conformally onto Ω. Theorem 3.1 gives only that $h(\Omega) \geq 1/2$.

Example II. Let ψ be an increasing continuous function on $[1, +\infty)$ with $\psi(1) = 0$ and $\lim_{r\to+\infty} \psi(r)/r = +\infty$. Let

$$\Omega = \mathscr{C} - ([0, e] \cup \{e^{r+i\psi(r)}: r \geq 1\}).$$

Then, in the notation of Theorem 5.2, $\lambda = +\infty$. Thus $h(\Omega) = +\infty$, whereas Theorem 3.1 again gives only $h(\Omega) \geq 1/2$.

IV. FUNCTIONS ANALYTIC ON THE UNIT DISK

6. Let f be analytic on Δ $(\Delta = \{|z| < 1\})$. Let $f(\Delta) \subseteq \Omega$, where $h(\Omega) > 0$. Then, since $f \in H_p(\Delta)$ $(0 < p < h(\Omega))$, we get the following results from known theorems about $H_p(\Delta)$.

THEOREM 6.1. *If* $M(r, f) = \max_{|z|=r} |f(z)|$, *then*

$$\lim_{r\to 1} [(1-r)^{1/p} M(r, f)] = 0 \qquad (0 < p < h(\Omega)).$$

(See Hardy and Littlewood [3].)

THEOREM 6.2. *Let* $I_1(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$. *Then if* $0 < h(\Omega) \leq 1$, *we have the relation*

$$I_1(r, f) = O[(1-r)^{1-1/p}] \qquad (0 < p < h(\Omega)).$$

(See Privalov [7, p. 108].)

THEOREM 6.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(|z| < 1)$.

(1) *If* $h(\Omega) > 1$, *then* $|a_n| = o(1)$.

(2) *If* $0 < h(\Omega) \leq 1$, *then*

$$|a_n| = o(n^{1/p-1}) \qquad (0 < p < h(\Omega)).$$

(See Privalov [7, pp. 110-114].)

Theorems 6.2 and 6.3 answer in part the questions raised by W. K. Hayman [4, pp. 28-30] concerning conditions on $f(\Delta)$ that give information on the growth of $I_1(r, f)$ and $|a_n|$.

V. HARDY NUMBERS AND ORDERS OF FUNCTIONS

7. It follows from the classical Phragmén-Lindelöf Theorem that if u is subharmonic on $\{\Re z > 0\}$ and if $\limsup_{z \to \zeta} u(z) \leq 0$ for each purely imaginary ζ, then either $u \leq 0$ or else

$$\liminf_{r \to +\infty} \frac{\log^+ M(r, u)}{\log r} \geq 1,$$

where $M(r, u) = \sup_{|\theta| < \pi/2} u(re^{i\theta})$. Since the Hardy number of $\{\Re z > 0\}$ is 1, this is a special case of the following theorem.

THEOREM 7.1. *Let* $\Omega \subsetneq \mathscr{C}$ *be an unbounded region. Let* u *be subharmonic on* Ω, *and suppose that* $\limsup_{z \to \zeta} u(z) \leq 0$ *for each finite boundary point* ζ *of* Ω. *Then either* $u \leq 0$ *or else*

$$\kappa = \kappa(u) = \liminf_{r \to +\infty} \frac{\log^+ M(r, u)}{\log r} \geq h(\Omega), \tag{16}$$

where, whenever $\Omega \cap \{|z| = r\} \neq \emptyset$, *we define* $M(r, u) = \sup\{u(z): z \in \Omega, |z| = r\}$.

Proof. (The proof arose during a conversation with Mr. John Lewis.) Since inequality (16) clearly holds when $h(\Omega) = 0$, we shall assume that $h(\Omega) > 0$. Suppose that $\kappa < h(\Omega)$, and choose ε such that $\kappa < \kappa + \varepsilon < h(\Omega)$. Then there exists a sequence $\{r_n\}_{n=1}^{\infty} \subseteq [1, +\infty)$ such that $r_n \to +\infty$ and

$$\frac{\log^+ M(r_n, u)}{\log r_n} \leq \kappa + \frac{\varepsilon}{2} \qquad (n = 1, 2, 3, \cdots).$$

That is, $M(r_n, u) \leq r_n^{\kappa + \varepsilon/2}$ $(n = 1, 2, 3, \cdots)$. Let ω_n denote the harmonic measure of $\Omega \cap \{|z| = r_n\}$ with respect to $\Omega_n = \Omega \cap \{|z| < r_n\}$ (see [10, p. 111]). We let I_n and I_Ω denote the identity map on Ω_n and Ω, respectively. Then if $z \in \Omega_{n_0}$, we have for $n \geq n_0$ the inequalities

$$u(z) \leq M(r_n, u)\omega_n(z) \leq r_n^{\kappa+\varepsilon/2}\, \omega_n(z) \leq r_n^{\kappa+\varepsilon}\, \omega_n(z)$$

$$\leq \mathrm{LHM}(|I_n|^{\kappa+\varepsilon})(z) \leq \mathrm{LHM}(|I_\Omega|^{\kappa+\varepsilon})(z) < +\infty,$$

since $\kappa + \varepsilon < h(\Omega)$. Here, LHM stands for "least harmonic majorant." Letting $n \to \infty$, we see that $\{r_n^{\kappa+\varepsilon}\omega_n(z)\}_{n=1}^{\infty}$ is bounded, and hence $r_n^{\kappa+\varepsilon/2}\omega_n(z) \to 0$. Therefore $u \leq 0$.

Let u be a nonconstant, continuous, subharmonic function on $\mathscr{C}$. Let c be a fixed positive real number, and let $\Phi_c = \{z \in \mathscr{C}\colon u(z) > c\}$. Let

$$h_c = \sup\{h(\Omega)\colon \Omega \text{ is a component of } \Phi_c\}.$$

If $c_1 < c_2$, then $\Phi_{c_2} \subseteq \Phi_{c_1}$, and hence $h_{c_1} \leq h_{c_2}$. Thus the limit

$$\lim_{c \to +\infty} h_c = h(u)$$

exists. As a corollary to Theorem 7.1, we get the following result.

THEOREM 7.2. *Let* u *be a nonconstant, continuous, subharmonic function on* $\mathscr{C}$. *Then*

$$\kappa(u) = \liminf_{r \to +\infty} \frac{\log^+ M(r, u)}{\log r} \geq h(u). \tag{17}$$

Proof. Let c be a fixed positive real number, and let Ω be a component of Φ_c. We apply Theorem 7.1 to $(u - c)|\Omega$ to get the relations

$$\kappa(u) \geq \kappa[(u - c)|\Omega] \geq h(\Omega).$$

Taking the supremum over the components Ω of Φ_c, we find that $\kappa(u) \geq h_c$. The theorem follows by letting $c \to +\infty$.

Equality may hold in (17), as the example $u(z) = \Re z$ shows. It would be of interest to know if equality always holds when u is harmonic.

In general, equality need not hold in (17), as the following example illustrates. Define

$$u(z) = \begin{cases} 1 & (\Re z \leq 1), \\ n^2 + (2n+1)(\Re z - n) & (1 \leq n \leq \Re z \leq (n+1)). \end{cases}$$

Then u is subharmonic on $\mathscr{C}$, and $M(r, u) = u(r) \geq r^2$. Thus, if $1 < n \leq r \leq n+1$, we have the inequalities

$$2 \leq \frac{\log M(r, u)}{\log r} \leq \frac{\log M(n+1, u)}{\log r} \leq \frac{\log u(n+1)}{\log n} = 2\left[\frac{\log(n+1)}{\log n}\right].$$

Therefore $\kappa(u) = 2$. However, $h_c = 1$ for each $c \geq 1$, and hence $h(u) = 1 < \kappa(u)$.

We see from Corollary 3.3 that if $\Omega \subseteq \mathscr{C}$ is a region with $\{|z| = r\} - \Omega \neq \emptyset$ for each $r > 0$, then $h(\Omega) > 1/2$. Thus a special case of Theorem 7.2 is the Wiman Theorem that the lower order of an entire function with bounded minimum modulus must be at least $1/2$.

REFERENCES

1. L. Ahlfors, *Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen.* Acta Soc. Sci. Fenn. (2) A, 1, Nr. 9 (1930), 40 pp.

2. G. T. Cargo, *Some geometric aspects of functions of Hardy class* H^p. J. Math. Anal. Appl. 7 (1963), 471-474.

3. G. H. Hardy and J. E. Littlewood, *A convergence criterion for Fourier series.* Math. Z. 28 (1928), 612-634.

4. W. K. Hayman, *Research problems in function theory.* The Athlone Press [University of London], London, 1967.

5. E. Lindelöf, *Sur un principe générale de l'analyse et ses applications à la théorie de la représentation conforme.* Acta Soc. Sci. Fenn. 46 (1920), 35 pp.

6. M. Parreau, *Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann.* Ann. Inst. Fourier (Grenoble) 3 (1951), 103-197.

7. I. I. Priwalow, *Randeigenschaften analytischer Funktionen.* VEB Deutscher Verl. Wissensch. Berlin, 1956.

8. W. Rudin, *Analytic functions of class* H_p. Lectures on functions of a complex variable, pp. 387-397. The University of Michigan Press, Ann Arbor, 1955.

9. M. Tsuji, *A theorem on the majoration of harmonic measure and its applications.* Tôhoku Math. J. (2) 3 (1951), 13-23.

10. ———, *Potential theory in modern function theory.* Maruzen Co., Ltd., Tokyo, 1959.

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