# RATIONAL EXPRESSIONS OF CERTAIN AUTOMORPHIC FORMS

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1. Let G denote the group of fractional linear transformations of the upper half-plane $H^+ = \{x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}$. For $\Gamma \subset G$, let f(z) be a Γ-automorphic form of weight k, and let (Γ, k) denote the vector space of all such forms. By considering generators of G, R. A. Rankin [1] and H. L. Resnikoff [3] have defined differential operators $D^m$ (m is an integer exceeding 1) such that for all subgroups $\Gamma \subset G$, the relation

(1) $$D^m: (\Gamma, k) \to (\Gamma, m(k+2))$$

holds. Set $f_i(z) = \frac{d^i}{dz^i} f(z)$. It has been shown that if $f \in (\Gamma, h)$ and if

$$P \in \mathbb{C}[f, f_1, f_2, \cdots, f_m] \cap (\Gamma, k),$$

then P is a quotient of some Q in $\mathbb{C}[f, D^2 f, \cdots, D^m f]$ and an appropriate power of f(z).

We shall show that $D^2$ and $D^3$ are sufficient for a rational representation, if operator composition is admitted. Let "o" denote composition. If we define

(2) $$D^{r,s} f \equiv (D^3 \circ)^r \circ (D^2 \circ)^s f,$$

where r and s are integers, then it will be enough to show that $D^m f$ is a rational function of $\{D^{r,s} f\}$, for all pairs (r, s) such that

(3) $$r \in \{0, 1\}, \quad s \in \{0, 1, 2, \cdots, [m/2]\}, \quad \text{and } 3r + 2s \leq m.$$

Here, [x] denotes the greatest integer not exceeding x. The denominator of our expression will assume a convenient form.

2. Let $f(z) \in (\Gamma, k)$ $(k > 0)$, and denote $d^m/dz^m$ by $L^m$. It is known [3] that

(4) $$L^m: (\Gamma, 1-m) \to (\Gamma, 1+m) \quad (m > 1).$$

An easy calculation shows that

$$f^{((k+1)m-1)/k} L^m(f^{(1-m)/k}) \in \mathbb{R}[f, f_1, \cdots, f_m].$$

That is, regardless of the branches chosen for $f^{((k+1)m-1)/k}$ and $f^{(1-m)/k}$, the final result is uniquely determined. Moreover, the resulting expression is in (Γ, m(k + 2)). Now define

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(5) $$D^m f \equiv f^{((k+1)m-1)/k} L^m(f^{(1-m)/k}).$$

We give explicit expressions for $D^2 f$, $D^3 f$, and $D^4 f$ in Table I.

TABLE I

$$D^2 f = \frac{k+1}{k^2} f_1^2 - \frac{1}{k} ff_2,$$

$$D^3 f = -\frac{4(k+1)(k+2)}{k^3} f_1^3 + 3\,\frac{2(k+2)}{k^2} ff_1 f_2 - \frac{2}{k} f^2 f_3,$$

$$D^4 f = \frac{9(k+1)(2k+3)(k+3)}{k^4} f_1^4 - 6\,\frac{3(2k+3)(k+3)}{k^3} ff_1^2 f_2 + 3\,\frac{3(k+3)}{k^2} f^2 f_2^2$$
$$+ 4\,\frac{3(k+3)}{k^2} f^2 f_1 f_3 - \frac{3}{k} f^3 f_4.$$

The basic theorem and lemma used to prove our result follow.

LEMMA 1. *Let* $f \in (\Gamma, k)$ $(k \neq 0)$ *and* $\beta \in (\Gamma, h) \cap \mathbb{C}[f, f_1, \cdots, f_m]$. *Let a typical monomial term of* $\beta$ *be of the form*

$$c(f)^{e_0} (f_1)^{e_1} \cdots (f_m)^{e_m},$$

*where* $c \in \mathbb{C}$ *and the* $e_i$ $(0 \leq i \leq m)$ *are nonnegative integers. Then*

(6) $$u = \sum_{j=0}^{m} e_j$$

*and*

(7) $$u^+ = \sum_{j=0}^{m} j(e_j)$$

*are constant, for all monomial terms of* $\beta$.

For a proof, see Rankin [1, p. 104].

THEOREM 1. *Let* $f$, $\beta$, $u$, *and* $u^+$ *be as in Lemma* 1. *Then* $\beta$ *is a polynomial in* $\mathbb{C}[f, D^2 f, \cdots, D^m f]$, *divided by* $f^{u^+ - u}$.

*Proof.* Resnikoff [3] adapted the proof of Rankin [1], [2]. Note that in the statement of this theorem in [3], the power of $f(z)$ is given as $u - u^+$. The proof clearly shows this to be a printing error. The correct power is $u^+ - u$, as we have given it.

Our basic result is the following.

THEOREM 2. *Let* $f \in (\Gamma, k)$ $(k > 0)$. *For all* $m > 1$, *we have that*

$$D^m f = \beta_1 / \beta_2,$$

*where* $\beta_i \in \mathbb{R}[f, D^{r,s} f]$ $(i = 1, 2)$. *The function* $D^{r,s} f$ *is given by* (2) *and* $\{(r, s)\}$ *by* (3).

*Proof.* Writing $D^m f$ as a polynomial in $f_m$, we find that

$$D^m f = \frac{1-m}{k}(f)^{m-1} f_m + E(m), \tag{8}$$

where $E(m) \in \mathbb{R}[f, f_1, \cdots, f_{m-1}]$; hence $D^m f$ is linear with respect to $f_m$. Therefore, $D^n \circ D^m f$ is linear as a polynomial in $f_{n+m}$, for all $n > 1$. To see this, note that $D^n \circ D^m f$ is linear in $(D^m f)_n$. But by (8),

$$(D^m f)_n = \frac{1-m}{k}(f)^{m-1} f_{m+n} + F,$$

where $F$ is in $\mathbb{R}[f, f_1, \cdots, f_{m+n-1}]$. Repeated application of this result implies that the function $D^{r,s} f$ is linear in $f_{3r+2s}$. Now, there exists a pair of integers $(r_0, s_0)$, satisfying (3), such that $m = 3r_0 + 2s_0$. If we set $\beta = D^{r_0,s_0} f$, then $\beta \in (\Gamma, h)$ $(h \neq 0)$ and $\beta \in \mathbb{R}[f, f_1, \cdots, f_m]$. By Theorem 1,

$$f^{u^+ - u}\beta \in \mathbb{R}[f, D^2 f, \cdots, D^m f].$$

Since $\beta$ is linear in $f_{3r_0+2s_0} = f_m$, it follows that $f^{u^+ - u}\beta$ is linear in $f_m$. This, together with equation (8), implies that $f^{u^+ - u}\beta$ is linear in $D^m f$. That is,

$$f^{u^+ - u}\beta = G_1 D^m f + G_2,$$

where $G_1$ and $G_2$ are in $\mathbb{R}[f, D^2 f, \cdots, D^{m-1} f]$. Therefore,

$$D^m f = F_1/F_2, \tag{9}$$

where

$$F_1 = f^{u^+ - u}\beta - G_2 \in \mathbb{R}[f, D^2 f, \cdots, D^{m-1} f, D^{r_0,s_0} f]$$

and $F_2 = G_1$.

To put this in the form required for the theorem, we shall use induction. For $m = 2$ and $m = 3$, the result is trivially true, since $(r_0, s_0)$ is just $(0, 1)$ or $(1, 0)$. Assume the result holds for $D^i f$ $(1 < i < m)$. Then, by the induction hypothesis, we can substitute into equation (9) for each $D^i f$ $(1 < i < m)$. An algebraic manipulation reducing the resulting expression to a simple fraction completes the proof.

*Remark.* We have required that if $D^3$ appears in some term, then it must appear as the final operator to be applied. The method of proof used above could easily be adapted for any other reordering of the operators. Indeed, any set $S = \{(a, b)\}$ such that $\{3a + 2b \mid (a, b) \in S\} = \mathbb{Z}$ could be used to define the form $\beta$, since under these conditions there exists a pair $(a_0, b_0)$ such that $m = 3a_0 + 2b_0$. Thus there is no "canonical" way to express $D^m f$ as a rational function of f, $D^2 \circ$, and $D^3 \circ$. Instead, there are many ways, and different expressions may be useful to determine different properties of the functions $D^m f$.

3. Before we state explicitly the rational expression, let us develop more machinery.

LEMMA 2. *If* $r \in \{0, 1\}$, $s > 0$, *and* $3r + 2s = m$, *then*

(10) $$D^{r,s}f = c[D^{0,s}f]^{2r}\left[\prod_{j=1}^{s-1} D^{0,s-j}f\right]f^{-m+2}D^m f + E(r, s),$$

*where* $c \in \mathbb{R}$ *and* $E(r, s) \in \mathbb{R}[f, f_1, \cdots, f_{m-1}]$.

*Remark.* For convenience, we set a product over the empty set equal to 1, and we define $D^{0,0}f \equiv f$.

*Proof.* For $r = 0$, we can construct a straightforward induction argument on $s$, using the results found in Table I. For $s = 1$, we find that $c = 1$ and $E(0, 1) = 0$.

For $r = 1$, we can again employ induction on $s$. For $s > 0$, we note that

$$D^{1,s}f = D^3 \circ D^{0,s}f;$$

therefore, we can apply the results of the case where $r = 0$ along with the results of Table I. This completes the proof.

The weight of $D^m f$ is $m(k + 2)$. Therefore, for $m = 3r + 2s$, the weight of $D^{r,s}f$ is given by the expression

$$W = \begin{cases} 2(2(\cdots(k+2)\cdots+2)+2) = 2^s k + \sum_{j=2}^{s+1} 2^j & (r = 0), \\ 3(2 + \text{weight of } D^{0,s}f) = 3(2^s k) + 3\sum_{j=1}^{s+1} 2^j & (r = 1), \end{cases}$$

where $f \in (\Gamma, k)$. Equation (10) becomes

(11) $$D^{r,s}f = c[D^{0,s}f]^{2r}\left[\prod_{j=1}^{s-1} D^{0,s-j}f\right]f^{-m+2}D^m f - \sum_{\omega\in\Omega} b_\omega f^{\omega_1}(D^2 f)^{\omega_2}\cdots(D^{m-1}f)^{\omega_{m-1}},$$

where

$$\Omega = \left\{(\omega_1, \cdots, \omega_{m-1}) \mid \omega_i \text{ is a nonnegative integer } (1 \le i \le m-1) \text{ satisfying the condition } W = \omega_1 k + \sum_{j=2}^{m-1} \omega_j(j(k+2))\right\}.$$

We shall denote the elements of $\Omega$ by $\omega$. Note that $j(k + 2)$ is the weight of $D^j f$.

Equation (11) can be rewritten as

(12) $$c[D^{0,s}f]^{2r}\left[\prod_{j=1}^{s-1} D^{0,s-j}f\right]f^{-m+2}D^m f = D^{r,s}f + \sum_{\omega\in\Omega} b_\omega f^{\omega_1}\prod_{j=2}^{m-1}(D^j f)^{\omega_j}.$$

If we consider W and $4(k+2) =$ (weight of $D^4 f$) as polynomials in k, then the inequalities

$$\sum_{j=4}^{m-1} \omega_j \leq \frac{\text{constant term of } W}{\text{constant term of } 4(k+2)} \tag{13}$$
$$\leq 3 \sum_{j=1}^{s+1} 2^j/2^3 = 3 \sum_{j=-2}^{s-2} 2^j$$

hold for all $\omega \in \Omega$.

Define $M(m) = \left(3 \sum_{j=0}^{s-2} 2^j\right) + 3$, where $m = 3r + 2s$, and where a sum over the empty set is taken to be zero. It follows that

$$\sum_{j=4}^{m-1} \omega_j < M(m). \tag{14}$$

4. Now we can define inductively the denominator of the rational expression we seek. That is, we wish to express the operator $D^m$ as a rational function of the $D^{r,s}$, where the r, s are pairs of integers satisfying (3). Denote the denominator by $\delta_m$. Clearly, it suffices to begin with $m = 4$, since for $m = 2$ and $m = 3$ the results of Theorem 2 are trivial. We shall show that

$$\delta_m D^j f = \text{poly}(f, D^{r,s} f) \qquad (4 \leq j \leq m), \tag{15}$$

where r, s satisfy (3). Note that we no longer require that $3r + 2s = m$, as in Section 3. If equation (15) holds, then, in particular, the relation

$$\delta_m D^m f = \text{poly}(f, D^{r,s} f)$$

holds, so that

$$D^m f = \text{poly}(f, D^{r,s} f)/\delta_m .$$

This means that by our choice of $\delta_m$, we shall have generated $D^m f$ as a rational function of $D^2$ and $D^3$ by using operator composition.

Set $\delta_4 = D^2 f$. By Lemma 2,

$$D^2 \circ D^2 f = c\left[\prod_{j=1}^{1} D^{0,2-j} f\right] f^{-4+2} D^4 f + E(0, 2), \tag{16}$$

where $E(0, 2) \in \mathbb{R}[f, f_1, f_2, f_3]$ and $c \in \mathbb{R}$. Then

$$D^2 \circ D^2 f = c(D^2 f)(f^{-2})(D^4 f) + E(0, 2). \tag{17}$$

An easy calculation shows that one term of $D^2 \circ D^2 f$ is $\hat{c} f^2 f_2 f_4$; hence by Lemma 1, $f^2 E(0, 2)$ is in $\mathbb{R}[f, D^2 f, D^3 f]$. Therefore, equation (17) becomes

$$(D^2 f)(D^4 f) = f^2[D^2 \circ D^2 f] - f^2 E(0, 2). \tag{18}$$

Hence $\delta_4 D^4 f \in \mathbb{R}[f, D^2 f, D^3 f, D^{0,2} f]$. It follows that $\delta_4$ satisfies the requirements of equation (15), since $D^2 f = D^{0,1} f$ and $D^3 f = D^{1,0} f$.

Assume we know that $\delta_\alpha$ $(4 \leq \alpha \leq m - 1)$ satisfies (15). Set

$$\delta_m = [D^{0,s_0} f]^{2r_0} \left[ \prod_{i=1}^{s_0-1} (D^{0,s_0-i} f) \right] (\delta_{m-1})^{M(m)}, \tag{19}$$

where $m = 3r_0 + 2s_0$. We need to verify (15). For $j < m$, we have the equation

$$\delta_m D^j f = [D^{0,s_0} f]^{2r_0} \left[ \prod_{i=1}^{s_0-1} (D^{0,s_0-i} f) \right] (\delta_{m-1})^{M(m)-1} (\delta_{m-1} D^j f).$$

By the induction hypothesis, $\delta_{m-1} D^j f = \text{poly}(f, D^{r,s} f)$, and since $\delta_n = \text{poly}(D^{0,t} f)$ for all $n \geq 4$, we have that $\delta_m D^j f = \text{poly}(f, D^{r,s} f)$, where r, s are as in (3).

Equations (12) and (19) imply that

$$\begin{aligned} \delta_m D^m f &= \frac{1}{c} f^{m-2} (\delta_{m-1})^{M(m)} (D^{r_0,s_0} f) \\ &\quad + \frac{1}{c} f^{m-2} (\delta_{m-1})^e \sum_{\omega \in \Omega} b_\omega f^{\omega_1} (D^2 f)^{\omega_2} (D^3 f)^{\omega_3} \left[ \prod_{j=4}^{m-1} (\delta_{m-1} D^j f)^{\omega_j} \right], \end{aligned} \tag{20}$$

where $e = M(m) - \sum_{j=4}^{m-1} \omega_j$. Equation (14) says that e is positive. By the induction hypothesis again, we have that $\delta_{m-1} D^j f = \text{poly}(f, D^{r,s} f)$ $(4 \leq j \leq m - 1)$. Therefore, equation (20) says that $\delta_m D^m f = \text{poly}(f, D^{r,s} f)$. Equivalently,

$$D^m f = \text{poly}(f, D^{r,s} f)/\delta_m. \tag{21}$$

The polynomial in (21) assumes a particularly convenient form. For notational ease, let W(f) denote the weight of f, if $f \in (\Gamma, k)$. Then

$$\delta_m D^m f = \sum_{\lambda \in \Lambda} c_\lambda f^{\lambda_1} \left[ \prod_{j=1}^{s_0} (D^{0,j} f)^{\lambda_{2j}} \right] \left[ \prod_{\substack{1 \leq j \leq s_0 \\ 3+2j \leq m}} (D^{1,j} f)^{\lambda_{2j+1}} \right], \tag{22}$$

where

$$\Lambda = \left\{ (\lambda_1, \cdots, \lambda_m) \,\middle|\, \text{the nonnegative integers } \lambda_i \ (1 \leq i \leq m) \text{ satisfy the} \right.$$
$$\left. \text{condition } W(\delta_m D^m f) = k\lambda_1 + \sum_{j=1}^{s_0} \lambda_{2j} W(D^{0,j} f) + \sum_{\substack{1 \leq j \leq s_0 \\ 3+2j \leq m}} \lambda_{2j+1} W(D^{1,j} f) \right\}.$$

We shall denote the elements of $\Lambda$ by $\lambda$.

It only remains to find a method of determining the coefficients $\{c_\lambda\}_{\lambda \in \Lambda}$. To develop such a procedure, we shall no longer restrict the domain of the differential operators $D^m$ to $\Gamma$-automorphic forms. This is permissible, because by equation (5), for each fixed k and for each differentiable function f(z), the expression $D^m f$ is calculated through formal manipulation. That is, the proof that

$$D^m f \in \mathbb{R}[f, f_1, \cdots, f_m]$$

requires no properties unique to $\Gamma$-automorphic forms. The rules of differentiation are sufficient, since for each fixed k, (5) is a formal identity, valid for every differentiable function f(z).

To calculate $\{c_\lambda\}_{\lambda\in\Lambda}$, we shall choose N (N = card($\Lambda$)) functions, substitute them into (22), and then apply Cramer's rule to the resulting system. The only condition we must satisfy is that the determinant of the coefficients for $\{c_\lambda\}_{\lambda\in\Lambda}$ is nonzero. For notational convenience, let

$$\mathscr{A} = \det\left( f^{\lambda_1}\left[\prod_{j=1}^{s_0} (D^{0,j}f)^{\lambda_{2j}}\right]\left[\prod_{\substack{1\le j\le s_0\\ 3+2j\le m}} (D^{1,j}f)^{\lambda_{2j+1}}\right]\right), \tag{23}$$

where $\lambda$ ranges over $\Lambda$ and f(z) belongs to the set of functions we shall determine. The determinant $\mathscr{A}$ is a polynomial in these functions and their first m derivatives. We shall denote both the determinant and the underlying matrix by $\mathscr{A}$.

In order to show that $\mathscr{A}$ is nonzero, we need the following facts.

*Definition.* A finite set of functions $\{g_1, \cdots, g_n\}$ is called P-*product independent* over $\mathbb{C}$ if

$$\{(g_1)^{v_1}(g_2)^{v_2}\cdots(g_n)^{v_n}\}_{v=(v_1,\cdots,v_n)}$$

is a linearly independent set for all choices of the $v_i$ $(0\le v_i\le P)$.

Clearly, for $m\ge 4$, there exists $P = P(m) > 0$ such that P is an upper bound for the largest exponent of f(z) or each of its derivatives in (22).

LEMMA 3. *For* $m\ge 4$, P(m) *as above, and* $n\ge 0$, *there exists a differentiable, complex-valued function* F(z) *such that* $\{F, F_1, \cdots, F_n\}$ *is* P(m)-*product independent, where* $F_q = \dfrac{d^qF}{dz^q}$.

*Remark.* This result is intuitively possible, since the condition that the product set

$$\{(F)^{v_1}(F_1)^{v_2}\cdots(F_n)^{v_{n+1}}\}$$

be linearly dependent determines a subset of $\mathbb{C}[z]$ whose dimension is less than the dimension of $\mathbb{C}[z]$. Thus, a function yielding a linearly independent set surely exists.

*Proof.* Let $F(z) = \sum_{t=0}^{\infty} \alpha_t z^t$ denote a formal power series on an open subset of $\mathbb{C}$. We shall determine the $\alpha_t$ so that F(z) will be a polynomial. Each derivative of F(z) is a formal power series over the same open subset of $\mathbb{C}$. Let

$$V = \{(v_1, \cdots, v_{n+1}) \mid 0\le v_i\le P(m)\}.$$

We shall denote the elements of V by v. Suppose there exist elements of $\mathbb{R}$, say $\{\gamma_v\}$, such that

(24) $$\sum_{v \in V} \gamma_v (F)^{v_1} (F_1)^{v_2} \cdots (F_n)^{v_{n+1}} = 0 .$$

This is a strictly finite sum, since card $(V) < \infty$. The left-hand side of (24) is a power series in z; hence the coefficient of each $z^t$ must be zero. Each choice of the $\alpha_t$ such that $\alpha_t \neq 0$ for all t imposes infinitely many linear conditions on the $\gamma_v$. This is possible only if $\gamma_v$ is zero for each $v \in V$, since there exist only finitely many $\gamma_v$. In fact for this same reason, there exists a choice of the $\alpha_t$ such that $\alpha_t = 0$ for all t greater than some integer M and such that the conditions imposed on the $\gamma_v$ require that $\gamma_v = 0$ for all $v \in V$. (Since the set of linear equations satisfied by the $\gamma_v$ is homogeneous, we know that the trivial solution always exists.) Such a choice of the $\alpha_t$ with these two properties makes F(z) a polynomial. Hence, F(z) converges in the entire finite complex plane. This completes the proof.

Let $N = \text{card}(\Lambda)$, where $\Lambda$ is as in equation (22). Then Lemma 3 says that for each $m \geq 4$, there exists a complex polynomial F(z) such that $\{F, F_1, \cdots, F_n\}$ is P(m)-product independent, for $n = N^2$. This gives rise to the N functions we need to evaluate (23). For $1 \leq j \leq N$, set the Nth function equal to

(25) $$F_{(j-1)N}(z)$$

((25) denotes the ((j - 1)N)th derivative of F(z)). We need to show that for these choices, $\mathscr{A}$ does not vanish.

LEMMA 4. *Every square submatrix of* $\mathscr{A}$ *has nonzero determinant, as a polynomial in the derivatives of* F(z).

*Proof.* We shall use induction. The result is true for single entries of $\mathscr{A}$ since for each j, each of the expressions $F_{(j-1)N}$, $D^{0,i}(F_{(j-1)N})$, and $D^{1,i}(F_{(j-1)N})$ is nonzero.

Suppose the result holds for all $p \times p$ submatrices of $\mathscr{A}$, where $p < \text{card}(\Lambda)$. Consider a $(p+1) \times (p+1)$ submatrix, and consider its determinant, evaluated by expansion by minors along its first row. The first row is generated by $F_{(j_0-1)N}$, for some $j_0$. Let g denote $F_{(j_0-1)N}$, and let u denote the order of the highest derivative of g(z) appearing in any entry of this first row. We shall show that in the determinant of the $(p+1) \times (p+1)$ minor, the coefficient of $(g_u)^Q$ is nonzero, where Q is the largest exponent of $g_u$ appearing in the first row of the $(p+1) \times (p+1)$ minor.

Consider all entries of the first row that contain a factor $(g_u)^Q$. The coefficient of $(g_u)^Q$ in each of these terms is a nonzero polynomial in g, $g_1, \cdots, g_{u-1}$. Moreover, none of these polynomials is a constant multiple of any other polynomial. This follows from the form of the general term in equation (23), since $\lambda$ changes for each entry in every row of $\mathscr{A}$. By the induction hypothesis, the minor of each term containing $(g_u)^Q$ is a nonzero polynomial in the $F_{(j-1)N}$, for some values of j different from $j_0$. Further, neither g(z) nor any of its first N - 1 derivatives appears in the determinants of these minors, since by the definition of our set of N ($N = \text{card}(\Lambda)$) functions, we have that $u < N$. Therefore, the coefficient of $(g_u)^Q$ is of the form

(26) $$\sum_{i \in I} d_i P_i ,$$

where $\operatorname{card}(I) < \infty$, $P_i = \operatorname{poly}(g, g_1, \cdots, g_{u-1})$, and none of the functions $g, g_1, \cdots, g_u$ appears in any $d_i$. By the remarks following Lemma 3, the set $\{g, g_1, \cdots, g_u\}$ is P(m)-product independent. Therefore, the sum in (26) cannot be zero, because of the P(m)-independence and the form of the $d_i$. Hence the $(p+1) \times (p+1)$ submatrix has a nonzero determinant, and the proof is complete.

COROLLARY. *If* $\mathscr{A}$ *is defined by* (23) *and the functions used to evaluate* $\mathscr{A}$ *are defined by* (25), *then* $\mathscr{A} \neq 0$.

The corollary follows immediately, since the matrix $\mathscr{A}$ is a square submatrix of itself.

We have shown that there exists a set of polynomials that can be substituted into (23) to yield a system from which the $c_\lambda$ can be calculated by Cramer's rule. There is no guarantee that the set generated from the function F(z) of Lemma 3 is the simplest. Any P(m)-product independent set will suffice to prove Lemma 4. The pleasing feature of the computational method outlined here is its mechanical nature.

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