EXTREMAL LENGTH AS A CAPACITY

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1. INTRODUCTION

In Euclidean n-space E_n, the p-capacity $(1 \leq p < \infty)$ of a pair of disjoint closed sets C_0 and C_1 is defined as

$$\Gamma_p(C_0, C_1) = \inf\left(\int_{E_n} |\operatorname{grad} u|^p \, dL_n\right), \tag{1}$$

where the infimum is taken over all continuous functions u on E_n that are infinitely differentiable on $E_n - (C_0 \cup C_1)$ and assume values 0 on C_0 and 1 on C_1. Under the assumptions that C_0 contains the complement of some closed n-ball and that $1 < p < \infty$, it was shown in [14] that $\Gamma_p(C_0, C_1)$ is equal to the reciprocal of the p-dimensional extremal length of all continua in E_n that intersect both C_0 and C_1. This equality was first established by F. W. Gehring [10] in the case where p = n, and it plays an important role in the theory of quasiconformal mappings on E_n.

For an arbitrary set $E \subset E_n$, let $\psi_p(E)$ denote the reciprocal of the p-dimensional extremal length of all closed connected sets that join E to the point at infinity of E_n. By using the relationship between p-capacity and extremal length that was referred to above, we shall show that ψ_p is a capacity in the sense of Brelot.

Let W_p^1 denote the collection of distributions whose partial derivatives are functions locally in $\mathscr{L}^p$, and call a function u *p-precise* if $u \in W_p^1$ and if for every $\varepsilon > 0$, there exists an open set U such that $\psi_p(U) < \varepsilon$ and u restricted to the complement of U is continuous. For $p > 1$, we use the results of [8] to show that every function $u \in W_p^1$ is equivalent to a precise function, thus extending the result obtained by J. Deny and J. L. Lions [5] in the case p = 2. In the terminology of N. Aronszjan and K. Smith, the precise functions form a perfect functional completion whose exceptional sets are ψ_p-null sets. Finally, for every bounded Suslin set $A \subset E_n$, we shall show that

$$\psi_p(A) = \inf\left(\int_{E_n} |\operatorname{grad} u|^p\right),$$

where the infimum is taken over all precise functions u that "vanish at infinity" and for which u(x) = 1 for ψ_p-almost all $x \in A$.

2. NOTATION AND PRELIMINARIES

By L_n and H^k, we denote n-dimensional Lebesgue measure and k-dimensional Hausdorff measure in E_n (for properties of the latter, see [6]). Let $\mathscr{L}^p$ be the

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class of functions f for which $|f|^p$ is L_n-integrable, and let $\|f\|_p$ be the $\mathscr{L}^p$-norm.

2.1. A continuous, real-valued function u is said to be absolutely continuous in the sense of Tonelli (ACT) on an n-dimensional interval I if it is absolutely continuous on almost all segments in I that are parallel to the coordinate axes and if its gradient (which will be denoted by ∇u) belongs to $\mathscr{L}^1$. The function u is called ACT on an open set U if it is ACT on every interval $I \subset U$. By using integral averages, one can easily show that the infimum in (1) can be extended to the class of ACT functions (see [9]).

If Ω is a domain in E_n, $W_p^1(\Omega)$ will denote the class of distributions on Ω whose partial derivatives are in $\mathscr{L}^p$. Such distributions are functions, and each $u \in W_p^1(\Omega)$ has a representative that is absolutely continuous on almost all segments I in Ω that are parallel to the coordinate axes.

2.2. If χ is a family of closed sets in E_n, the p-dimensional module of χ $(1 \leq p < \infty)$ is defined as

$$M_p(\chi) = \inf\left(\int_{E_n} f^p dL_n : f \wedge \chi\right), \tag{2}$$

where $f \wedge \chi$ means that f is a nonnegative Borel function satisfying the condition

$$\int_\beta f\, dH^1 \geq 1$$

for every $\beta \in \chi$. By referring to [8, 2.2], one sees that f in this definition can be assumed to be lower-semicontinuous.

2.3. For $E \subset E_n$ and $1 \leq p < n$, define $\psi_p(E)$ to be the p-dimensional module of all closed connected sets that join E to the point at infinity of E_n, and let $\Psi_p(E)$ be the p-dimensional module of all nondegenerate continua that intersect E. Finally, if E is compact, let $\Gamma_p(E)$ be defined as in (1), except that the infimum is taken over all continuous functions u with compact support that are identically 1 on E and ACT in $E_n - E$. In the case where $p \geq n$, the support of each function u is required to lie in some fixed open ball S containing E, and in the definition of $\psi_p(E)$, only those continua that join E to $E_n - S$ will be considered.

The following theorem is essential in showing that ψ_p is a capacity, and while its proof is similar to that of Theorem 3.8 in [14], there are some new difficulties that deserve to be treated.

2.4. THEOREM. *If* $E \subset E_n$ *is compact, then* $\psi_p(E) = \Gamma_p(E)$ $(1 \leq p < \infty)$.

Proof. We shall consider only the case where $1 \leq p < n$ (the case where $p \geq n$ is simpler). Let B_k $(k = 1, 2, \cdots)$ denote the closed ball centered at 0 of radius k, and for simplicity of notation, assume $E \subset$ interior B_1. As in [14, Section 3], the only closed connected sets β that need to be considered in the definition are those for which $H^1(\beta \cap B_k) < \infty$, for every k. Hence, β is locally connected and therefore arcwise connected. Consequently, there exists an arc $\beta^* \subset \beta$ that joins E to ∞. Since $H^1(\beta^* \cap B_k) < \infty$ for every k, there is an arc-length parametrization of β^*, say $\gamma: [0, \infty) \to \beta^*$, such that $\gamma(0) \in E$ and $|\gamma(t)| \to \infty$, as $t \to \infty$. Now the proof proceeds precisely as in [14, Lemma 3.1] to establish the inequality $\psi_p(E) \leq \Gamma_p(E)$.

To prove the opposite inequality, let χ be the family of closed, connected sets that join E to ∞, and let f be a lower-semicontinuous function such that $f \wedge \chi$. If, for every positive integer i, we define

$$f_i(x) = \begin{cases} f(x) & (f(x) > i^{-1}\, 2^{-k}\, k^{-n/p} \text{ and } x \in \operatorname{int} B_k - B_{k-1}), \\ i^{-1}\, 2^{-k} k^{-n/p} & (f(x) \le i^{-1}\, 2^{-k} k^{-n/p} \text{ and } x \in \operatorname{int} B_k - B_{k-1}), \end{cases}$$

then f_i is lower-semicontinuous, $f_i \wedge \chi$, and $\|f_i - f\|_p \to 0$. Therefore, without loss of generality, we may assume that f is bounded away from zero by a constant C_k on each ball B_k.

For each positive integer k, let

$$f_k(x) = \begin{cases} f(x) & (f(x) \le k), \\ k & (f(x) > k), \\ 0 & (x \notin B_k), \end{cases}$$

and define

$$u_k(x) = \inf\left(\int_\beta f_k \, dH^1\right) \quad (x \in B_k),$$

where the infimum is taken over all continua β that join E to $\{x\}$. As in [14, Sections 3.4 and 3.5], the infimum is attained by some β_k, the function u_k has Lipschitz constant k, and $|\nabla u_k| \le f_k$ a.e.. Thus, to conclude the proof, it suffices to prove that

$$\liminf_{k \to \infty} m_k \ge 1,$$

where

$$m_k = \min\{u_k(x) \colon x \in \partial B_k\}.$$

To this end, let $x_k \in \partial B_k$ be such that $u_k(x_k) = m_k$, and let $\beta_k \subset B_k$ be a continuum joining $\{x_k\}$ and E such that

$$u_k(x_k) = \int_{\beta_k} f_k \, dH^1 .$$

If we assume that $\liminf_{k\to\infty} m_k < 1$, then some subsequence would satisfy the inequality

$$\int_{\beta_k} f_k \, dH^1 < 1 .$$

This implies that $H^1(\beta_k) < \infty$, since f_k is bounded away from zero on each B_k. Thus β_k may be assumed to be an arc of finite length, say a_k. Let $\gamma_k \colon [0, a_k] \to \beta_k$ be the arc-length parametrization. If each γ_k is restricted to [0, 1], then, by reasoning similar to that of [14, Lemma 3.3], there exist a subsequence (which we still denote by $\{\gamma_k\}$) and a map $\mu_1 \colon [0, 1] \to E_n$ such that $\{\gamma_k\}$ converges uniformly to μ_1 on [0, 1]. Now by restricting each γ_k of this subsequence to [0, 2], there exists another subsequence that converges to a map μ_2: $[0, 2] \to E_n$. Note that μ_2 is an extension of μ_1. By continuing in this manner and then by employing Cantor's diagonalization process, one obtains a map μ: $[0, \infty) \to E_n$ and a subsequence such that $\{\gamma_k\}$ converges uniformly to μ on compact subsets. It is easy to verify that $\beta = \mu[0, \infty)$ is a closed connected set that joins E to infinity, and consequently,

$$\int_\beta f\, dH^1 \geq 1.$$

For every $\varepsilon > 0$, there exists a positive integer m such that

$$\int_\beta f_k dH^1 > 1 - \varepsilon,$$

for $k \geq m$. Therefore,

$$\liminf_{k\to\infty} m_k = \liminf_{k\to\infty} \int_{\beta_k} f_k\, dH^1 \geq \liminf_{k\to\infty} \int_{\beta_k} f_m dH^1 \geq \int_\beta f_m\, dH^1 > 1 - \varepsilon,$$

and this concludes the proof.

3. CAPACITY AND MEASURE

From the properties of the p-dimensional module as discussed in [8, Chapter 1], it follows that the set function Ψ_p that was introduced in Section 2.3 is monotone and countably subadditive. Let A be a subset of E_n and χ the class of all continua that intersect A. If $f \in \mathscr{L}_p$ is a function such that $f \wedge \chi$, then it must be the case that

$$\int_\gamma f\, dH^1 = \infty,$$

for M_p-almost all $\gamma \in \chi$. Therefore, the following result now follows from [8, Theorem 2].

3.1. THEOREM. *For* $1 \leq p < \infty$, Ψ_p *is an outer measure on* E_n *that assumes only the values* 0 *and* ∞.

If $L_n(E) > 0$, then it follows from [13] and Theorem 2.4 that $\psi_p(E) > 0$, and therefore $\Psi_p(E) = \infty$. Consequently, in order to simplify the exposition, we shall assume in Theorems 3.5 and 3.6 below that $L_n(E) = 0$. If $L_n(E) = 0$, then there exists a function $f \in \mathscr{L}_p$ that is infinite on E. Therefore, the p-dimensional module of all continua that intersect E and that are subsets of E is zero.

3.2. LEMMA. *If* $K_1 \supset K_2 \supset \cdots$ *are compact sets, then*

$$\lim_{i\to\infty} \psi_p(K_i) = \psi_p\left(\bigcap_{i=1}^{\infty} K_i\right) \quad (1 \leq p < \infty).$$

Proof. It follows from Theorem 2.4 that we only need to show that

$$\lim_{i\to\infty} \Gamma_p(K_i) = \Gamma_p\left(\bigcap_{i=1}^{\infty} K_i\right).$$

To do this, choose $\varepsilon > 0$, and let u be a smooth function with compact support that is equal to 1 on $\bigcap_{i=1}^{\infty} K_i$ and that satisfies the inequality

$$\int_{E_n} |\nabla u|^p \, dL_n < \Gamma_p\left(\bigcap_{i=1}^{\infty} K_i\right) + \varepsilon .$$

Since u is continuous, it is no less than $1 - \varepsilon$ on K_i, for all large i. Hence, for large i,

$$\Gamma_p(K_i) \le (1-\varepsilon)^{-1} \int_{E_n} |\nabla u|^p \, dL_n < (1-\varepsilon)^{-1}\left[\Gamma_p\left(\bigcap_{i=1}^{\infty} K_i\right) + \varepsilon\right],$$

and therefore $\lim_{i\to\infty} \Gamma_p(K_i) \le \Gamma_p\left(\bigcap_{i=1}^{\infty} K_i\right)$. Since the opposite inequality is obvious, the proof is complete.

3.3. LEMMA. *If* $A_1 \subset A_2 \subset \cdots$ *are subsets of* E_n, *then*

$$\lim_{i\to\infty} \psi_p(A_i) = \psi_p\left(\bigcup_{i=1}^{\infty} A_i\right) \qquad (1 < p < n) .$$

If $p \ge n$, *the same result holds, provided the closure of each* A_i *is contained in some open ball* S.

Proof. If $1 < p < n$, let χ_i be the class of closed connected sets that join A_i to infinity, and observe that $\bigcup_{i=1}^{\infty} \chi_i$ is precisely the class of closed connected sets that join $\bigcup_{i=1}^{\infty} A_i$ to infinity. Thus the result is an immediate consequence of [14, Lemma 2.3].

In the case where $p \ge n$, the proof proceeds in a similar way, and in fact it is easier to handle.

Lemma 3.2 states that ψ_p is right-continuous on compact sets, while Lemma 3.3 implies left continuity on arbitrary sets. In the terminology of Brelot, ψ_p is a *true capacity*, and therefore the next theorem follows directly from [2, Theorem 1].

3.4. THEOREM. *If* $E \subset E_n$ *is a Suslin set, then*

$$\psi_p(E) = \sup\{\psi_p(K)\colon K \subset E,\ K \text{ compact}\} \qquad (1 < p < \infty) .$$

We shall now establish a similar result for the measure Ψ_p, and we begin with the following result.

3.5. THEOREM. *If* $E \subset E_n$, *then* $\Psi_p(E) = 0$ *if and only if* $\psi_p(E) = 0$ $(1 < p < \infty)$.

Proof. In view of the inequality $\Psi_p(E) \geq \psi_p(E)$, we need only show that $\psi_p(E) = 0$ implies $\Psi_p(E) = 0$. The assumption that $\psi_p(E) = 0$ implies that there exists some nonnegative function $f \in \mathscr{L}^p$ with the property that if λ is any ray whose end point is in E, then

$$\int_\lambda f\, dH^1 = \infty. \tag{3}$$

In case $p \geq n$, equation (3) will hold for all line segments λ one of whose end points is in E and the other in the complement of the ball S that is assumed to contain E. Therefore, by employing polar coordinates, (3) implies that the Riesz potential of order 1 has the property that

$$\infty = U_1^f(x) = \int_{E_n} |x - y|^{1-n} f(y)\, dL_n(y),$$

whenever $x \in E$. In view of Theorem 6 in [8], this leads to the conclusion that the p-dimensional module of all continua that intersect E is zero, that is, $\Psi_p(E) = 0$.

3.6. THEOREM. *If* $E \subset E_n$ *is a Suslin set, then*

$$\Psi_p(E) = \sup \{\Psi_p(K): K \subset E,\ K \text{ compact}\} \qquad (1 < p < \infty).$$

Proof. If $\Psi_p(E) = \infty$, then, by Theorem 3.5, we have that $\psi_p(E) > 0$. Therefore, Theorem 3.4 asserts the existence of a compact set $K \subset E$ with $\psi_p(K) > 0$, and hence, $\Psi_p(K) = \infty$.

We shall now consider the problem of extending the set function Γ_p from compact sets to arbitrary sets, and then we shall determine its relationship to ψ_p. We follow [2] in making this extension.

3.7. *Definition.* For an arbitrary set $A \subset E_n$, let

$${}_*\Gamma_p(A) = \sup \{\Gamma_p(K): K \subset A,\ K \text{ compact}\}$$

and

$${}^*\Gamma_p(A) = \inf \{{}_*\Gamma_p(G): G \supset A,\ G \text{ open}\}.$$

Because Γ_p is right-continuous on compact sets (see proof of Lemma 3.2), it follows from [2, Chapter II] that ${}_*\Gamma_p$ and ${}^*\Gamma_p$ agree on compact and open sets. We shall write Γ_p whenever ${}_*\Gamma_p$ and ${}^*\Gamma_p$ are equal. Moreover, it is easy to verify that for every pair of compact sets K_1 and K_2, we have the inequality

$$\Gamma_p(K_1 \cup K_2) + \Gamma_p(K_1 \cap K_2) \leq \Gamma_p(K_1) + \Gamma_p(K_2). \tag{4}$$

An equality of this type plays an important role in Choquet's general theory of capacities [4]. According to [2, Theorem 2], (4) implies that ${}^*\Gamma_p$ is a true capacity, and therefore, as in the proof of Theorem 3.4, we have that ${}^*\Gamma_p$ is an inner regular function on Suslin sets. This proves the following.

3.8. THEOREM. *If* $1 < p < \infty$, *then*

$$\psi_p(E) = {}_*\Gamma_p(E) = {}^*\Gamma_p(E) = \Gamma_p(E),$$

whenever E *is a Suslin set.*

3.9. *Remark.* As in classical capacity theory, it is possible to introduce the concept of capacitary dimension, which in our context is based on the capacity ψ_p. Corresponding to an arbitrary set $E \subset E_n$, there exists a real number α $(0 \leq \alpha \leq n)$ such that

$$\psi_{n-\beta}(E) = 0 \quad \text{for every } \beta > \alpha, \qquad \text{and} \qquad \psi_{n-\beta}(E) > 0 \quad \text{for every } \beta < \alpha.$$

The existence of the number α is obvious if one employs the following criterion to determine when the p-dimensional module of a family χ of closed sets is zero [8, p. 179]: $M_p(\chi) = 0$ if and only if there exists a function $f \in \mathscr{L}^p$ $(f \geq 0)$ such that

$$\int_\beta f\, dH^1 = \infty, \text{ for every } \beta \in \chi.$$

We call this number α the *ψ-capacitary dimension of* E. If E is a Suslin set, it follows from Theorem 3.5, [8, Theorems 6 and 7], and [8, p. 199] that the ψ-capacitary dimension of E is equal to its Hausdorff dimension, but we do not know in general under what conditions ψ_p and H^{n-p} vanish simultaneously. However, the following is known:

(i) If K is a compact set with $H^\alpha(K) = 0$ $(0 < \alpha < n - 1)$, then $\Gamma_{n-\alpha}(K) = 0$, and therefore $\psi_{n-\alpha}(K) = 0$ [13, p. 335].

(ii) If $2 < \alpha < n$, there exists a compact set K satisfying the conditions $\psi_\alpha(K) = 0$ and $H^{n-\alpha}(K) = \infty$ (see [13, p. 339] and [3, p. 28]).

(iii) Fleming has shown that $H^{n-1}(K) = 0$ if and only if $\Gamma_1(K) = 0$, whenever $K \subset E_n$ is compact [7]. By using Theorem 2.4, we obtain that $H^{n-1}(K) = 0$ if and only if $\psi_1(K) = 0$.

4. PRECISE FUNCTIONS

In this section, we introduce precise functions and show that every function in W_p^1 is equivalent to a precise function. It will then follow that the precise functions form a perfect functional completion in the sense of Aronszjan and Smith [1].

4.1. *Definition.* A function $u \in W_p^1(\Omega)$, where Ω is a bounded domain, is called p-*precise* if for every $\varepsilon > 0$, there exists an open set U such that $\psi_p(U) < \varepsilon$ and u restricted to $\Omega - U$ is continuous. B. Fuglede proved in [8, Theorem 9] that ψ_2 is equal, up to a constant factor, to Newtonian capacity. Therefore the 2-precise functions are the same as the precise functions of Deny and Lions [5, p. 354].

4.2. LEMMA. *If ϕ is a continuously differentiable function whose support is contained in Ω and if $E = \{x: |\phi(x)| > \alpha\}$, then*

$$\psi_p(E) \leq \alpha^{-p} \int_\Omega |\nabla\phi|^p\, dL_n \qquad (1 < p < \infty).$$

Proof. If K is a compact subset of E, then clearly

$$\Gamma_p(K) \leq \alpha^{-p} \int_\Omega |\nabla\phi|^p dL_n .$$

The conclusion now follows from Theorems 2.4 and 3.4.

The proof of the following theorem is similar to that of [5, Theorem 3.1].

4.3. THEOREM. *Every function in* $W_p^1(\Omega)$ *is equal almost everywhere to a* p-*precise function* $(1 < p < \infty)$.

Proof. Let $u \in W_p^1(\Omega)$, and let K be a compact subset of Ω. There exists a nonnegative C^∞-function α whose support is in Ω and that is identically 1 on K. Hence, $u^* = \alpha \cdot u \in W_p^1(\Omega)$, and the support of u^* is contained in Ω. It is well known [11, p. 64] that the mollifiers ϕ_i of u^* are of class C^∞ and that

$$(5) \qquad \|\phi_i - u^*\|_p \to 0 \quad \text{and} \quad \|\nabla\phi_i - \nabla u^*\|_p \to 0 .$$

Since the support of u^* is contained in Ω, the same may be assumed for each of the ϕ_i. By passing to a subsequence if necessary, we may assume that

$$(6) \qquad \sum_{i=1}^{\infty} 2^{ip} \|\nabla\phi_{i+1} - \nabla\phi_i\|_p < \infty .$$

Let $E_i = \{x: |\phi_{i+1}(x) - \phi_i(x)| > 2^{-i}\}$ and $W_k = \bigcup_{i=k}^{\infty} E_i$. It follows from Lemma 4.2 that

$$\psi_p(W_k) \leq \sum_{i=k}^{\infty} \psi_p(E_i) \leq \sum_{i=k}^{\infty} 2^{ip} \|\nabla\phi_{i+1} - \nabla\phi_i\|_p ,$$

and therefore (6) implies that $\psi_p(W_k) \to 0$ as $k \to \infty$. On $\Omega - W_k$, the sequence $\{\phi_i\}$ converges uniformly; hence, $\{\phi_i\}$ converges ψ_p-almost everywhere to a function v^* that is precise. Clearly, v^* is equivalent to u^* and $u^* = u$ on K. Now, by expressing Ω as the union of closed sets $K_1 \subset K_2 \subset \cdots$, where

$$K_i = \{x: d(x, \partial\Omega) \geq i^{-1}\} ,$$

it can easily be verified that there exists a precise function that is equivalent to u.

If P is a hyperplane in E_n and S a bounded subset of E_n, let S^* be the Steiner symmetrization of S with respect to P [12, Section 1.7]. It follows from [12, Section 7.3] that if S is compact, then $\Gamma_p(S) \geq \Gamma_p(S^*)$ $(1 \leq p < \infty)$, and therefore,

$$(7) \qquad \psi_p(S) \geq \psi_p(S^*) .$$

If G is a bounded open set, let $K_1 \subset K_2 \subset \cdots$ be compact sets whose union is G. Note that $G^* = \bigcup_{i=1}^{\infty} K_i^*$, and therefore Lemma 3.3 and (7) imply the inequality

$$(8) \qquad \psi_p(G) \geq \psi_p(G^*) .$$

In particular, (8) implies that

$$\psi_p(G) \geq \psi_p[\pi(G)], \tag{9}$$

where π: $E_n \rightarrow P$ is the orthogonal projection. This leads to the following theorem.

4.4. THEOREM. *If* $u \in W_p^1(\Omega)$ $(1 < p < \infty)$ *is precise and* P *is a hyperplane, then* u *is continuous on all segments in* Ω *orthogonal to* P, *except possibly for those whose projection onto* P *is a* ψ_p*-null set.*

Proof. For each positive integer i, there exists an open set $U_i \subset \Omega$ such that $\psi_p(U_i) < i^{-1}$ and u is continuous on Ω - U_i. Thus, by (8), $\psi_p[\liminf_{i \to \infty} \pi(U_i)] = 0$ and u is continuous on each segment in Ω whose projection is not in $\liminf_{i \to \infty} \pi(U_i)$.

4.5. *Remark.* We shall now employ the results of [8, Chapter III] to provide a representation for precise functions. To this end, let $R_1(x) = |x|^{1-n}$ be the Riesz kernel of order 1 and recall from [8] that if f is a nonnegative function in $\mathscr{L}^p$, then the set $E = \{x: R_1*f(x) = \infty\}$ is a Ψ_p-null set and therefore a ψ_p-null set. Here R_1*f denotes the convolution of the two functions. Moreover, if E is a Suslin set for which $\psi_p(E) = 0$, then, by Theorem 3.5, $\Psi_p(E) = 0$ and there exists a function $f \in \mathscr{L}^p$ $(f \geq 0)$ such that $R_1*f(x) = \infty$ for every $x \in E$.

Now let $u \in W_p^1(\Omega)$. According to [8, Chapter III], ∇u is an irrotational vector field, because if $\{\phi_i\}$ is a sequence of mollifiers for u, then $\phi_i \to u$ and $\nabla\phi_i \to \nabla u$ in $\mathscr{L}^p$ (see (5)). Therefore, there exists a set E with $\psi_p(E) = 0$ and with the property that if x_0 is chosen arbitrarily in Ω - E, we may define

$$u^*(x) = \int_{x_0}^{x} \nabla u + \text{const.} \qquad (x \in \Omega - E), \tag{10}$$

where the line integral refers to a curve joining x_0 to x. Fuglede showed in [8, Chapter III] that curves exist that give meaning to (10) and that (10) is independent of the choice of curve. If the constant in (10) is chosen appropriately, it follows that u^* is p-precise and equivalent to u. To see this, choose x_0 so that $\lim_{i \to \infty} \phi_i(x_0) = c$, and set the constant in (10) equal to -c. Since $\nabla\phi_i \to \nabla u$ in the $\mathscr{L}^p$-norm, we can add to E another ψ_p-null set (denote the union by E), so that for some subsequence of $\{\phi_i\}$ the following is true, for $x \in \Omega$ - E:

$$\lim_{i \to \infty} [\phi_i(x) - \phi_i(x_0)] = \lim_{i \to \infty} \int_{x_0}^{x} \nabla\phi_i = \int_{x_0}^{x} \nabla u = u^*(x) - c$$

(see [8, p. 216]). Hence $\phi_i(x) \to u^*(x)$ for $x \in \Omega$ - E, and therefore, as in the proof of Theorem 4.3, it follows that u^* is p-precise $(1 < p < \infty)$. In the terminology of Aronszajn and Smith [1], the p-precise functions form a perfect functional completion of smooth functions whose gradients are in $\mathscr{L}^p$. The exceptional class in this completion consists of precisely all subsets of Suslin sets E for which $\psi_p(E) = 0$.

Another representation of p-precise functions can be obtained in terms of Bessel potentials. Let $G_1(x)$ be the Bessel kernel of order 1 [1, p. 416]. If $f \in \mathscr{L}^p$, then G_1*f and R_1*f are simultaneously infinite. Therefore, it follows from [8] that a function $u \in W_p^1$ is p-precise $(1 < p < \infty)$ if and only if there exists a function $g \in \mathscr{L}^p$ such that $u = G_1*g$, except possibly for a ψ_p-null set.

We shall conclude this paper by proving that the infimum in (1) can be taken over the class of p-precise functions and that there is an extremal in this class.

4.6. *Definition.* For a bounded set $E \subset E_n$ and for $1 \leq p < n$, let

$$\Theta_p(E) = \inf\left(\int_{E_n} |\nabla u|^p \, dL_n\right),$$

where the infimum is taken over all p-precise functions u that are equal to 1 at ψ_p-almost every point in E and that are "admissible." A p-precise function u is admissible if there exist C^∞-functions u_n having compact support such that $u_n \to u$ at ψ_p-almost all points and $\|\nabla u_n - \nabla u\|_p \to 0$. In the case where $p \geq n$, the supports of the admissible functions are required to lie in some fixed open ball S containing E.

4.7. LEMMA. *If* $A \subset E_n$ *is compact, then*

$$\psi_p(A) = \Theta_p(A) \quad (1 \leq p < \infty).$$

Proof. We shall show that $\Theta_p(A) \geq \Gamma_p(A)$, and, in view of Theorem 2.4, this will suffice to establish the lemma. Choose $\varepsilon > 0$, and let u be an admissible function such that

$$\int_{E_n} |\nabla u|^p \, dL_n < \Theta_p(A) + \varepsilon. \tag{11}$$

Let u_i be C^∞-functions with compact support such that $u_i \to u$ at ψ_p-almost every point and such that $\|\nabla u_i - \nabla u\|_p \to 0$. As in Remark 4.5, there exists an open set U such that $\psi_p(U) < \varepsilon$ and $u_i \to u$ uniformly on $E_n - U$. Since ψ_p is a true capacity as well as a strong capacity, every ψ_p-capacitable set is outer regular [2, p. 18]. Therefore, we can assume that U contains the set $A \cap \{x: u(x) \neq 1\}$. Let χ be the class of closed connected sets β that join A to ∞. Recall that β can be taken to be of locally finite H^1-measure. Moreover, by [8, Theorem 3], we may assume, for a subsequence, that

$$\int_\beta |\nabla u_i - \nabla u| \, dH^1 \to 0 \quad (\beta \in \chi). \tag{12}$$

For every $\beta \in \chi$, there exists an arc $\beta^* \subset \beta$ that joins A to ∞, and for all such arcs β^*, u_i is of class C^∞ along β^*. Therefore, for all such arcs β^* that join A - U to ∞, we have the inequality

$$\lim_{i\to\infty} \int_{\beta^*} |\nabla u_i| \, dH^1 \geq 1. \tag{13}$$

Consequently, (11), (12), and (13) lead to the relation

$$\varepsilon + \Theta_p(A) \geq \psi_p(A - U). \tag{14}$$

Since $\psi_p(A) \leq \psi_p(A - U) + \varepsilon$ and ε is arbitrary, (14) implies that $\Theta_p(A) \geq \psi_p(A)$.

4.8. COROLLARY. Θ_p *is right-continuous on compact sets.*

We shall now show that Θ_p is left-continuous on arbitrary sets, all of which are assumed to be contained in some fixed ball B.

4.9. LEMMA. *If* $E_1 \subset E_2 \subset \cdots$ *and* $\bigcup_{i=1}^{\infty} E_i \subset B$, *then*

$$\lim_{i \to \infty} \Theta_p(E_i) = \Theta_p\left(\bigcup_{i=1}^{\infty} E_i\right) \qquad (1 < p < \infty).$$

Proof. Again, we shall only consider the case where $1 < p < n$. For each integer i, let u_i be an admissible function for E_i such that

$$\int_{E_n} |\nabla u_i|^p \, dL_n < \Theta_p(E_i) + i^{-1}.$$

Observe that $2^{-1}(u_i + u_j)$ is admissible for E_i $(j > i)$, and therefore

$$\Theta_p(E_i) \leq 2^{-p} \int_{E_n} |\nabla u_i + \nabla u_j|^p \, dL_n .$$

Without loss of generality, we may assume that the limit in Lemma 4.9 is finite, and consequently, by employing Clarkson's inequality in a manner similar to that in the proof of Lemma 2.3 in [14], it follows that

$$\lim_{i,j \to \infty} \int_{E_n} |\nabla u_i - \nabla u_j|^p \, dL_n = 0 .$$

Hence, there exists a vector field $f \in \mathscr{L}^p$ such that $\nabla u_i \to f$, in the $\mathscr{L}^p$-norm. In the terminology of [8, Chapter III], f is an irrotational field. We now proceed as in Remark 4.5 to find a set E with $\psi_p(E) = 0$ and such that if x_0 is chosen arbitrarily in $E_n - E$, then we may define

$$(15) \qquad u^*(x) = \int_{x_0}^{x} f + \text{const.} \qquad (x \in E_n - E).$$

Choose x_0 such that $\lim_{i \to \infty} u_i(x_0) = c$ exists, and set the constant in (15) equal to $-c$. Thus, as in 4.5, $u_i(x) \to u^*(x)$ for $x \in E_n - E$. Moreover, $\nabla u^* = f$ a.e., and we shall show that u^* is p-precise and admissible.

Clearly, $u^* = 1$ at ψ_p-almost all points of $\bigcup_{i=1}^{\infty} E_i$. If u^* were admissible for $\bigcup_{i=1}^{\infty} E_i$, the proof would be complete, for then

$$\lim_{i \to \infty} \Theta_p(E_i) = \lim_{i \to \infty} \int_{E_n} |\nabla u_i|^p \, dL_n = \int_{E_n} |\nabla u^*|^p \, dL_n \geq \Theta_p\left(\bigcup_{i=1}^{\infty} E_i\right).$$

To prove that u^* is admissible, observe that for each nonnegative integer i, there exist a C^∞-function v_i with compact support and an open set U_i such that $\psi_p(U_i) < 2^{-i}$, $\|\nabla v_i - \nabla u_i\|_p < i^{-1}$, and $|v_i(x) - u_i(x)| < i^{-1}$ for $x \in E_n - U_i$. If we let $V_j = \bigcup_{i=j}^{\infty} U_i$, then it is clear that $\|\nabla v_i - \nabla u^*\|_p \to 0$ and that $v_i \to u^*$ at ψ_p-almost all points of $E_n - V_j$, where $\psi_p(V_j) < 2^{1-j}$. Since j is arbitrary, $v_i \to u^*$ at ψ_p-almost all points, and therefore u^* is admissible and p-precise.

Corollary 4.8 and Lemma 4.9 imply that Θ_p is a true capacity on each subset of B. Since Θ_p and ψ_p agree on compact sets, this leads to our last theorem.

4.10. THEOREM. *If* E *is a bounded Suslin set, then*

$$\Theta_p(E) = \psi_p(E) \qquad (1 < p < \infty).$$

By employing an argument similar to that in Lemma 4.9, one can easily show that there exists an admissible function u such that $\|\nabla u\|_p^p = \Theta_p(E)$.

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