# THE HOMOTOPY EXCISION THEOREM

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Dedicated to Professor R. L. Wilder on his seventieth birthday.

## 1. INTRODUCTION

This paper is devoted to a study of the homomorphism of homotopy groups induced by an inclusion map

$$e: (B, C) \subset (X, A)$$

such that $B - C = X - A$. Such an inclusion map is called an *excision map*, and we see that $C = A \cap B$ and $X = A \cup B$, so that an excision map is an inclusion map of the form $(B, A \cap B) \subset (A \cup B, A)$. In case A and B are open sets, it is well known [4, pp. 199-200], [6, p. 189] that the excision map induces isomorphisms of all the corresponding singular homology and cohomology groups; however, it need not induce isomorphisms of the corresponding homotopy groups.

We present an example to illustrate this. For a space Y, let SY be the join of Y with a pair of points p, p', and let $S: \pi_q(Y) \to \pi_{q+1}(SY)$ be the suspension homomorphism (thus, if $\alpha: S^q \to Y$ represents $[\alpha] \in \pi_q(Y)$, then $S[\alpha]$ is represented by the composite of a fixed homeomorphism $S^{q+1} \approx SS^q$ with $S\alpha: SS^q \to SY$). Then $SY - p$ and $SY - p'$ are contractible, $SY - (p \cup p')$ has the same homotopy type as Y, and there is a commutative diagram

$$\begin{array}{ccc}
\pi_{q+1}(SY - p', (SY - p) \cap (SY - p')) & \xrightarrow{e_\#} & \pi_{q+1}(SY, SY - p) \\
\approx \downarrow & & \uparrow \approx \\
\pi_q(SY - (p \cup p')) \approx \pi_q(Y) & \xrightarrow{S} & \pi_{q+1}(SY).
\end{array}$$

It follows that $e_\#: \pi_{q+1}(SY - p', (SY - p) \cap (SY - p')) \to \pi_{q+1}(SY, SY - p)$ is an isomorphism if and only if $S: \pi_q(Y) \to \pi_{q+1}(SY)$ is also an isomorphism. Since the suspension homomorphism is not generally an isomorphism, neither is the homomorphism induced on homotopy groups by an excision map.

On the other hand, with suitable hypotheses an excision map does induce an isomorphism in homotopy for a certain range of dimensions, and this result can be used to prove that the suspension homomorphism is an isomorphism for a corresponding range of dimensions.

A pair (X, A) is said to be n-*connected* for $n \geq 0$ if for $q \leq n$ every map $(E^q, S^{q-1}) \to (X, A)$ is homotopic relative to $S^{q-1}$ to a map sending all of $E^q$ into A. The following is the general result proved in this note.

HOMOTOPY EXCISION THEOREM. *Let* A, B *be subsets of a space* $X = A \cup B$ *such that* $(A, A \cap B)$ *is* n-*connected and* $(B, A \cap B)$ *is* m-*connected, where* $n, m \geq 0$. *If either*

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(a) $X = \text{int } A \cup \text{int } B$ *or*

(b) $A$ *and* $B$ *are closed and* $A \cap B$ *is a strong neighborhood deformation retract in* $A$ *(or in* $B$*),*

*then, for each* $x_0 \in A \cap B$*, the function*

$$e_{\#}: \pi_q(B, A \cap B, x_0) \to \pi_q(A \cup B, A, x_0)$$

*is injective for* $q \leq n + m - 1$ *and surjective for* $q \leq n + m$.

The triad homotopy groups of Blakers and Massey [2], [3] have been used to study homomorphisms of homotopy groups induced by excision maps. In particular, the homotopy excision theorem above is a consequence of their triad excision theorem [1], [3], [7]. Our proof uses neither triad homotopy groups nor spectral sequences. We use function spaces to derive the theorem from the Hurewicz isomorphism theorem [5, p. 166], [6, p. 397]. We present a self-contained proof based on the methods used to prove the homotopy excision theorem in [6, pp. 484-486] for $n, m \geq 2$.

If $Y$ is a space that is $(n - 1)$-connected for some $n \geq 1$, then

$$(SY - p, (SY - p) \cap (SY - p')) \qquad \text{and} \qquad (SY - p', (SY - p) \cap (SY - p'))$$

are $n$-connected, and the following is an easy consequence of the homotopy excision theorem.

SUSPENSION THEOREM. *If* $Y$ *is* $(n - 1)$*-connected for* $n \geq 1$*, then for* $y_0 \in Y$*, the suspension homomorphism*

$$S: \pi_q(Y, y_0) \to \pi_{q+1}(SY, y_0)$$

*is a monomorphism for* $q \leq 2n - 2$ *and an epimorphism for* $q \leq 2n - 1$.

For other applications of the homotopy excision theorem, see [6, pp. 487-490], [8].

We reduce the homotopy excision theorem to the case where (a) is satisfied. If $A \cap B$ is a strong deformation retract of some neighborhood $U$ in $A$, then $(B, A \cap B)$ is a strong deformation retract of $(U \cup B, U) = (U \cup B, A \cap (U \cup B))$, and there is a commutative triangle

$$\begin{array}{ccc} \pi_q(B, A \cap B, x_0) & \xrightarrow{\approx} & \pi_q(U \cup B, A \cap (U \cup B), x_0) \\ & {\scriptstyle e_{\#}} \searrow \quad \swarrow {\scriptstyle e'_{\#}} & \\ & \pi_q(A \cup B, A, x_0) \,. & \end{array}$$

Hence, we have reduced consideration of $e$ to consideration of $e'$ for sets $A$ and $U \cup B$ satisfying (a).

Similarly, if $A \cap B$ is a strong deformation retract of some neighborhood $V$ in $B$, then $A$ is a strong deformation retract of $A \cup V$, and there is a commutative square

$$\begin{array}{ccc} \pi_q(B, A \cap B, x_0) & \xrightarrow{e_\#} & \pi_q(A \cup B, A, x_0) \\ \approx\downarrow & & \downarrow\approx \\ \pi_q(B, (A \cup V) \cap B, x_0) & \xrightarrow{e''_\#} & \pi_q(A \cup B, A \cup V, x_0) . \end{array}$$

Hence, we have reduced consideration of $e$ to consideration of $e''$ for sets $A \cup V$ and $B$ satisfying (a). In either case we have reduced the problem to proving the homotopy excision theorem in case (a) is satisfied; in the sequel, we assume that (a) holds.

## 2. CONSEQUENCES OF CONNECTIVITY

This section derives some consequences of connectivity hypotheses. These will be used to prove the homotopy excision theorem in low dimensions (by direct means similar to techniques in [7]). We begin by stating without proof the following elementary fact [6, pp. 373, 402].

LEMMA 1. *For a pair* $(X, A)$, *the following properties are equivalent:*

(a) $(X, A)$ *is* $n$-*connected, where* $n \geq 0$.

(b) *Every path component of* $X$ *meets* $A$, *and*

$$\pi_q(X, A, x_0) = 0 \quad \textit{for } x_0 \in A \textit{ and } 1 \leq q \leq n.$$

(c) *Given a map* $f\colon (P, Q) \to (X, A)$, *where* $(P, Q)$ *is a polyhedral pair with* $\dim(P - Q) \leq n$, *then* $f$ *is homotopic relative to* $Q$ *to a map sending* $P$ *into* $A$.

LEMMA 2. *Let* $A$, $B$ *be subsets of a space* $X$ *such that* $X = \operatorname{int} A \cup \operatorname{int} B$ *and* $(A, A \cap B)$ *is* $n$-*connected. Given a polyhedron* $P$ *with* $\dim P \leq n$, *subpolyhedra* $P_A$, $P_B$ *of* $P$, *and a map* $f\colon P \to X$ *such that* $f(P_A) \subset A$ *and* $f(P_B) \subset B$, *there is a homotopy*

$$H\colon (P \times I, P_A \times I) \to (X, A)$$

*relative to* $P_B$ *from* $f$ *to some map sending* $P$ *into* $B$.

*Proof.* Let $K$ be a simplicial triangulation of $P$, with $P_A$ and $P_B$ triangulated by subcomplexes of $K$, and assume $K$ to be so fine that for every simplex $s \in K$ either $f(|s|) \subset A$ or $f(|s|) \subset B$ (such a $K$ exists, because

$$P \subset f^{-1}(\operatorname{int} A) \cup f^{-1}(\operatorname{int} B)) .$$

Let $K_A$ and $K_B$ be the subcomplexes of $K$ consisting, respectively, of the simplexes $s \in K$ such that $f(|s|) \subset A$ and $f(|s|) \subset B$. Define $Q_A = |K_A|$ and $Q_B = |K_B|$, and observe that $P = Q_A \cup Q_B$ and that $P_A \subset Q_A$ and $P_B \subset Q_B$.

Because $(A, A \cap B)$ is $n$-connected, it follows from Lemma 1 applied to the map

$$f|(Q_A, Q_A \cap Q_B)\colon (Q_A, Q_A \cap Q_B) \to (A, A \cap B)$$

that there exists a homotopy

$$H'\colon Q_A \times I \to A$$

relative to $Q_A \cap Q_B$ from $f \mid (Q_A, Q_A \cap Q_B)$ to a map sending $Q_A$ to $A \cap B$. Define $H: (P \times I, P_A \times I) \to (X, A)$ by

$$H(p, t) = \begin{cases} f(p) & p \in Q_B,\ t \in I, \\ H'(p, t) & p \in Q_A,\ t \in I. \end{cases}$$

Then H has all the desired properties.

COROLLARY 1. *If* $X = \operatorname{int} A \cup \operatorname{int} B$ *and* $(A, A \cap B)$ *is* n-*connected, then* $(X, B)$ *is also* n-*connected.*

*Proof.* Let $\alpha: (E^q, S^{q-1}) \to (X, B)$ be a map with $q \leq n$, and apply Lemma 2 with $P = E^q$, $P_A$ empty, and $P_B = S^{q-1}$ to obtain the result.

LEMMA 3. *If* $X = \operatorname{int} A \cup \operatorname{int} B$ *and* $(A, A \cap B)$ *is* n-*connected and* $(B, A \cap B)$ *is* 1-*connected, then given a map* $f: (E^{n+1}, S^n) \to (X, A)$ *and a proper polyhedral subset* $C \subset S^n$ *such that* $f(C) \subset B$, f *is homotopic relative to* C *to a map* $(E^{n+1}, S^n) \to (B, A \cap B)$.

*Proof.* The desired homotopy will be the result of a sequence of homotopies. Let K be a simplicial triangulation of $E^{n+1}$, with C triangulated by a subcomplex of K, and assume K so fine that $K = K_A \cup K_B$, where

$$K_A = \{s \in K \mid f(|s|) \subset A\} \quad \text{and} \quad K_B = \{s \in K \mid f(|s|) \subset B\}.$$

Because $(A, A \cap B)$ is n-connected, the map

$$f|(|K_A^n|, |K_A^n \cap K_B|): (|K_A^n|, |K_A^n \cap K_B|) \to (A, A \cap B)$$

is homotopic relative to $|K_A^n \cap K_B|$ to some map sending $|K_A^n|$ into $A \cap B$. This homotopy extends to a homotopy $H: E^{n+1} \times I \to X$ relative to $|K_B|$ such that $H(z, 0) = f(z)$ for $z \in E^{n+1}$ and $H(|K_A| \times I) \subset A$. Since $S^n \subset |K_A|$ and $C \subset |K_B|$, H is a homotopy relative to C from f to some map $f_1: (E^{n+1}, S^n) \to (X, A)$ such that $f_1$ maps all of $E^{n+1}$ except for a finite set of $(n+1)$-simplexes into B. By an additional homotopy relative to $S^n$, we can arrange it so that the $(n+1)$-simplexes not mapped into B by $f_1$ are pairwise disjoint and contained in $E^{n+1} - S^n$.

Because $E^{n+1}$ is path connected, we can enumerate these $(n+1)$-simplexes in a sequence $s_1, s_2, \cdots, s_k$ in such a way that there exist pairwise disjoint simple arcs $a_1, a_2, \cdots, a_k$ with the following properties: each $a_i$ meets $\bigcup |s_j|$ only in its endpoints $\dot{a}_i$; the arc $a_1$ is a path from some point of $S^n - C$ to some point of $|\dot{s}_1|$; and for $i > 1$, $a_i$ has one endpoint on $|\dot{s}_{i-1}|$ and one endpoint on $|\dot{s}_i|$. We "thicken" the arcs slightly to obtain "tubes" $T_i$ homeomorphic to $a_i \times I^n$ that are pairwise disjoint and have ends on $S^n - C$ or on $|\dot{s}_i|$. We describe a sequence of homotopies, all of which will be relative to the space Y equal to the closure of

$$E^{n+1} - \left(\bigcup |s_j| \cup \bigcup T_i\right).$$

Since $(B, A \cap B)$ is 1-connected, for $1 \leq i \leq k$ there exists a homotopy relative to $\dot{a}_i$ from $f_1 \mid a_i$ to some map sending $a_i$ into $A \cap B$. These homotopies extend to a homotopy relative to Y from $f_1$ to $f_2$, where $f_2\left(\bigcup a_i\right) \subset A \cap B$. By retracting a smaller tube $T_i'$ around $a_i$ inside $T_i$, we obtain a homotopy relative to Y from $f_2$ to $f_3$, where $f_3\left(\bigcup T_i'\right) \subset A \cap B$.

The set $\bigcup |s_j| \cup \bigcup T_i'$ is an $(n+1)$-cell $D$ with boundary $\dot{D} = D_1 \cup D_2$, where $D_1$ is the n-cell equal to $T_1' \cap (S^n - C)$ and $D_2$ is the closure of $\dot{D} - D_1$. Since $D_2$ is a strong deformation retract of D, there exists a homotopy relative to $E^{n+1} - D$ from $f_3$ to $f_4$, where $f_4(E^{n+1}) \subset B$. Since each of the homotopies

$$f \simeq f_1 \simeq f_2 \simeq f_3 \simeq f_4$$

is relative to C and keeps $S^n \times I$ mapped into A, the composite homotopy $f \simeq f_4$ is a homotopy with the desired properties.

## 3. LOW-DIMENSIONAL CASES

Throughout this section, we assume that $X = \text{int } A \cup \text{int } B$, where $(A, A \cap B)$ is n-connected and $(B, A \cap B)$ is m-connected, and we consider the excision maps $e: (B, A \cap B) \subset (X, A)$ and $e': (A, A \cap B) \subset (X, B)$.

LEMMA 4. *For* $q \geq 2$ *and any* $x_0 \in A \cap B$, *the following properties are equivalent:*

(a) $e_\#: \pi_k(B, A \cap B, x_0) \to \pi_k(X, A, x_0)$ *has trivial kernel for* $k = q - 1$ *and is surjective for* $k = q$.

(b) $e'_\#: \pi_k(A, A \cap B, x_0) \to \pi_k(X, B, x_0)$ *has trivial kernel for* $k = q - 1$ *and is surjective for* $k = q$.

*Proof.* Note that the kernel is defined even for $k = 1$, where the homotopy sets have distinguished elements but are not groups. We have exact sequences (all with base point $x_0$)

$$\cdots \to \pi_k(B, A \cap B) \xrightarrow{i_\#} \pi_k(X, A \cap B) \xrightarrow{j_\#} \pi_k(X, B) \xrightarrow{\partial} \pi_{k-1}(B, A \cap B) \to \cdots,$$

$$\cdots \to \pi_k(A, A \cap B) \xrightarrow{i'_\#} \pi_k(X, A \cap B) \xrightarrow{j'_\#} \pi_k(X, A) \xrightarrow{\partial'} \pi_{k-1}(A, A \cap B) \to \cdots,$$

and $e_\# = j'_\# i_\#$ and $e'_\# = j_\# i'_\#$. If $e_\#$ is surjective for $k = q$, so is $j'_\#$, and by exactness of the second sequence, $i'_\#$ has trivial kernel for $k = q - 1$. If $e_\#$ has trivial kernel for $k = q - 1$, so has $i_\#$, and by exactness of the first sequence, $j_\#$ is surjective for $k = q$. Therefore, if (a) holds, then $i'_\#$ has trivial kernel for $k = q - 1$ and $j_\#$ is surjective for $k = q$.

To prove the lemma, it suffices, in view of the symmetry, to prove that (a) implies (b). To prove $e'_\#$ has trivial kernel for $k = q - 1$, assume $e'_\#(\alpha) = 0$ for $\alpha \in \pi_{q-1}(A, A \cap B)$. Then $j_\# i'_\#(\alpha) = 0$, and by exactness of the first sequence, there is $\beta \in \pi_{q-1}(B, A \cap B)$ such that $i_\#(\beta) = i'_\#(\alpha)$. Clearly,

$$0 = j'_\# i'_\#(\alpha) = j'_\# i_\#(\beta) = e_\#(\beta),$$

and since $e_\#$ has trivial kernel for $k = q - 1$, it follows that $\beta = 0$. Therefore $0 = i_\#(\beta) = i'_\#(\alpha)$, and as we have already remarked, $i'_\#$ has trivial kernel for $k = q - 1$, so that $\alpha = 0$.

To prove $e'_\#$ is surjective for $k = q$, assume $\alpha \in \pi_q(X, B)$. As we noted in the first paragraph of this proof, $j_\#$ is surjective for $k = q$; hence, there exists a $\beta \in \pi_q(X, A \cap B)$ such that $j_\#(\beta) = \alpha$. Since $e_\#$ is surjective for $k = q$, there is $\gamma \in \pi_q(B, A \cap B)$ such that $e_\#(\gamma) = j'_\#(\beta)$. Therefore

$$j'_\#(\beta(i_\#(\gamma))^{-1}) = j'_\#(\beta)(e_\#(\gamma))^{-1} = 0,$$

and by exactness of the second sequence, there is a $\delta \in \pi_q(A, A \cap B)$ such that $i'_\#(\delta) = \beta(i_\#(\gamma))^{-1}$. Since $j_\# i_\# = 0$,

$$\alpha = j_\#(\beta) = j_\# i'_\#(\delta) = e'_\#(\delta).$$

*Proof of the homotopy excision theorem in case* $n \leq 1$ *or* $m \leq 1$. (a) We first consider the case where $n$ is arbitrary and $m = 0$, and we show that

$$e_\#: \pi_q(B, A \cap B, x_0) \to \pi_q(X, A, x_0)$$

is injective for $1 \leq q < n$ and surjective for $1 \leq q \leq n$. For the first part, let $\alpha_0, \alpha_1: (E^q, S^{q-1}, p_0) \to (B, A \cap B, x_0)$, for $q < n$, represent elements of $\pi_q(B, A \cap B, x_0)$ mapped by $e_\#$ to the same element of $\pi_q(X, A, x_0)$. Let $H: E^q \times I \to X$ be a homotopy relative to $p_0$ from $e\alpha_0$ to $e\alpha_1$. Apply Lemma 2 to $H$ with

$$P = E^q \times I, \qquad P_A = S^{q-1} \times I, \qquad P_B = E^q \times \dot{I} \cup p_0 \times I$$

to obtain a homotopy $H \simeq H'$ relative to $E^q \times \dot{I} \cup p_0 \times I$ such that $H'(E^q \times I) \subset B$ and $H'(S^{q-1} \times I) \subset A$. Then $H'$ is a homotopy from $\alpha_0$ to $\alpha_1$, and this shows that $e_\#$ is injective for $q < n$.

For the second part, let $\alpha: (E^q, S^{q-1}, p_0) \to (X, A, x_0)$ represent an element of $\pi_q(X, A, x_0)$, where $q \leq n$. Apply Lemma 2 with $P = E^q$, $P_A = S^{q-1}$, and $P_B = p_0$ to obtain a homotopy $\alpha \simeq \beta$ relative to $p_0$, where $\beta$ defines a map

$$\beta': (E^q, S^{q-1}, p_0) \to (B, A \cap B, x_0)$$

representing an element of $\pi_q(B, A \cap B, x_0)$ such that $e\beta' = \beta \simeq \alpha$; it follows that $e_\#$ is surjective for $q \leq n$.

(b) Next we consider the case $n = 0$ and $m \geq 1$. Observe that by Corollary 1, $(X, A)$ is $m$-connected, and therefore

$$e_\#: \pi_1(B, A \cap B, x_0) \to \pi_1(X, A, x_0)$$

is a one-to-one correspondence, because both homotopy sets are trivial. By (a), we know that

$$e'_\#: \pi_k(A, A \cap B, x_0) \to \pi_k(X, B, x_0)$$

is injective for $1 \leq q < m$ and surjective for $1 \leq q \leq m$. From Lemma 4 and the above observation it follows that $e_\#$ is also injective for $1 \leq q < m$ and surjective for $1 \leq q \leq m$.

(c) We now assume $n \geq 1$ and $m = 1$. To prove the result in this case, it is in view of (a) sufficient to show that $e_\#$ is injective for $q = n$ and surjective for $q = n + 1$. For the first part we apply Lemma 3 to a homotopy

$$H: (E^n \times I, E^n \times \dot{I} \cup S^{n-1} \times I) \to (X, A)$$

from $e\alpha_0$ to $e\alpha_1$ with $C = E^n \times \dot{I} \cup p_0 \times I$. We obtain a homotopy $H \simeq H'$, where

$$H': (E^n \times I, E^n \times \dot{I} \cup S^{n-1} \times I) \to (B, A \cap B)$$

is a homotopy from $\alpha_0$ to $\alpha_1$, and we see that $e_\#$ is injective for $q = n$.

For the second part, we apply Lemma 3 to a map $\alpha: (E^{n+1}, S^n) \to (X, A)$ with $C = p_0$ to obtain a homotopy $\alpha \simeq \beta$, where $\beta$ defines a map

$$\beta': (E^{n+1}, S^n) \to (B, A \cap B)$$

representing an element of $\pi_{n+1}(B, A \cap B, x_0)$ such that $e\beta' = \beta \simeq \alpha$; and we see that $e_\#$ is surjective for $q = n + 1$.

(d) In case $n = 1$ and $m \geq 1$, it follows from (c) that $e'_\#$ is injective for $1 \leq q \leq m$ and surjective for $1 \leq q \leq m + 1$. By Lemma 4, it follows that $e_\#$ is injective for $1 \leq q \leq m$ and surjective if $1 \leq q \leq m + 1$.

## 4. FIBRATIONS

In this section we recall some properties of fibrations (Hurewicz fiber spaces). By definition, a *fibration* $p: E \to B$ is a map satisfying the homotopy lifting property for arbitrary spaces (that is, if $f: Y \to E$ and $G: Y \times I \to B$ are maps such that $G(y, 0) = pf(y)$, then there exists a map $F: Y \times I \to E$ such that $F(y, 0) = f(y)$ and $pF = G$).

LEMMA 5. *If* $B$ *is a path-connected space, there exists a fibration* $p: E \to B$ *such that* $E$ *is simply connected, and such that for each* $x_0 \in E$,

$$p_\#: \pi_k(E, x_0) \approx \pi_k(B, p(x_0)) \qquad \textit{for } k > 1.$$

*Proof.* By successively attaching cells to $B$, beginning with dimension 3, we can imbed $B$ in a path-connected space $B'$ such that for any $b \in B$, $\pi_1(B, b) \approx \pi_1(B', b)$ and $\pi_k(B', b) = 0$ for $k > 1$. Let $p': E' \to B'$ be the path fibration over $B'$ (of paths in $E'$ beginning at some base point $b_0' \in B'$), and let $p: E \to B$ be the restriction of this fibration to $B$. Then $p: E \to B$ has the desired properties.

A fibration $p: E \to B$ having the properties stated in Lemma 5 will be called a *generalized universal covering space* of $B$. The next result will serve in place of the spectral sequence of a fibration in the later applications.

LEMMA 6. *Let* $p: E \to B$ *be a fibration with fiber* $F$ *over a point* $b_0 \in B' \subset B$. *Let* $E' = p^{-1}(B')$, *and assume that* $(B, B')$ *is* $n$-*connected and the reduced integral singular homology group* $\tilde{H}_q(F)$ *is* $0$ *for* $q < m$, *where* $m > 0$. *Then*

$$p_*: H_q(E, E') \to H_q(B, B')$$

*is an isomorphism for* $q \leq n + m$ *and an epimorphism for* $q = n + m + 1$.

*Proof.* Because $(B, B')$ is n-connected, we can successively attach cells to $B'$, beginning with dimension $n + 1$, to obtain a relative CW complex $(\overline{B}, B')$ such that $B'$ is the n-skeleton of $\overline{B}$ and such that there exists a weak homotopy equivalence $f: \overline{B} \to B$ with $f(b') = b'$ for all $b' \in B'$. Let $\overline{p}: \overline{E} \to \overline{B}$ be the fibration induced from $p$ by $f$, and using the exactness of the homotopy sequence of a fibration and the five-lemma, observe that $f$ induces a weak homotopy equivalence $f': \overline{E} \to E$. From the exactness of the homology sequence of a pair and the five-lemma it follows that there exists an isomorphism $H_*(\overline{E}, E') \approx H_*(E, E')$ induced by $f'$. Since $(\overline{B}, B')$ is also n-connected and there exists a commutative square

$$\begin{array}{ccc} H_*(\overline{E}, E') & \underset{\approx}{\rightarrow} & H_*(E, E') \\ \overline{p}_* \downarrow & & \downarrow p_* \\ H_*(\overline{B}, B') & \underset{\approx}{\rightarrow} & H_*(B, B'), \end{array}$$

we have reduced consideration to the case where the base pair is a relative CW complex $(\overline{B}, B')$ whose n-skeleton equals $B'$.

Let $B^k$ be the k-skeleton of this relative CW complex, and set $E^k = \overline{p}^{-1}(B^k)$. If $\{e_j\}_j$ is the set of k-cells of $(\overline{B}, B')$, there are isomorphisms

$$\begin{aligned} H_q(E_k, E_{k-1}) &\approx \bigoplus_j H_q(\overline{p}^{-1}(e_j), \overline{p}^{-1}(\dot{e}_j)) \\ &\approx [\bigoplus_j H_k(e_j, \dot{e}_j)] \otimes H_{q-k}(F) \\ &\approx H_k(B^k, B^{k-1}) \otimes H_{q-k}(F). \end{aligned}$$

Because $\tilde{H}_q(F) = 0$ for $q < m$, it follows that if $q - k < m$, then

$$H_q(E_k, E_{k-1}) \approx \begin{cases} 0 & q \neq k, \\ H_k(B^k, B^{k-1}) & q = k. \end{cases}$$

Hence, $\overline{p}_*: H_q(E_k, E_{k-1}) \approx H_q(B^k, B^{k-1})$ if $q - k < m$. Therefore, for each $k \geq n + 1$ the homomorphism

$$\overline{p}_*: H_q(E_k, E_{k-1}) \rightarrow H_q(B^k, B^{k-1})$$

is an isomorphism for $q \leq n + m$ and an epimorphism for $q = n + m + 1$ (it is an isomorphism for $q \leq n + m + 1$ except for $k = n + 1$, in which case $\overline{p}_*$ is an epimorphism because $H_{n+m+1}(B^{n+1}, B^n) = 0$, since $m > 0$). Using induction on k and the five-lemma, we find that for $k \geq n + 1$,

$$\overline{p}_*: H_q(E_k, E') \rightarrow H_q(B^k, B')$$

is an isomorphism for $q \leq n + m$ and an epimorphism for $q = n + m + 1$. The result follows on using the isomorphisms

$$\varinjlim_k \{H_q(E_k, E')\} \approx H_q(E, E'), \qquad \varinjlim_k \{H_q(B^k, B')\} \approx H_q(B, B').$$

## 5. THE GENERAL CASE

In this section we assume that $X = \operatorname{int} A \cup \operatorname{int} B$, where $(A, A \cap B)$ is n-connected and $(B, A \cap B)$ is m-connected with $n, m \geq 2$, and we consider the excision map

$$e: (B, A \cap B, x_0) \subset (X, A, x_0), \qquad x_0 \in A \cap B.$$

If we let $A'$ be the path component of $A$ containing $x_0$, and $B'$ the path component of $B$ containing $x_0$, then $A' \cap B$ and $A \cap B'$ are contained in the path component of $A \cap B$ containing $x_0$ (because $(A, A \cap B)$ and $(B, A \cap B)$ are 1-connected). Since the path component of $A \cap B$ containing $x_0$ is contained in $A' \cap B'$, we see that $A' \cap B'$ is the path component of $A \cap B$ containing $x_0$. Clearly $X' = A' \cup B'$ is the path component of $X$ containing $x_0$ (because $(X, A)$ and $(X, B)$ are 1-connected, by Corollary 1). There is a commutative square

$$\begin{array}{ccc} \pi_q(B', A' \cap B', x_0) & \xrightarrow{e'_\#} & \pi_q(X', A', x_0) \\ \approx\downarrow & & \downarrow\approx \\ \pi_q(B, A \cap B, x_0) & \xrightarrow{e_\#} & \pi_q(X, A, x_0). \end{array}$$

Hence, to study $e_\#$ it suffices to study $e'_\#$. Since $(X, A)$ is 1-connected, it follows that $X' \cap \operatorname{int} A \subset A'$, and since $(X, B)$ is 1-connected, $X' \cap \operatorname{int} B \subset B'$. Thus, in $X'$, $X' = \operatorname{int}_{X'} A' \cup \operatorname{int}_{X'} B'$, and so $A'$ and $B'$ satisfy all the original hypotheses, and in addition, $A' \cap B'$ is path connected.

Therefore, without loss of generality we may now assume that $A$ and $B$ satisfy the original hypotheses, and also that $A \cap B$ is path connected. Then $A$, $B$, and $X$ are also path connected. Let $p: E \to X$ be a generalized universal covering space of $X$ (which exists, by Lemma 5). Let $E_A$ and $E_B$ be the parts of $E$ over $A$ and $B$, respectively, and note that $E_A \cap E_B$ is the part of $E$ over $A \cap B$. Then $(E_A, E_A \cap E_B)$ is n-connected (because $(A, A \cap B)$ is n-connected) and $(E_B, E_A \cap E_B)$ is m-connected (because $(E_B, E_A \cap E_B)$ is m-connected). Since $n, m \geq 2$, the spaces $E_A$, $E_B$, and $E_A \cap E_B$ are all simply connected. Since $E = \operatorname{int} E_A \cup \operatorname{int} E_B$ and there is a commutative square

$$\begin{array}{ccc} \pi_q(E_B, E_A \cap E_B, z_0) & \xrightarrow{\bar{e}_\#} & \pi_q(E, E_A, z_0) \\ \downarrow\approx & & \downarrow\approx \\ \pi_q(B, A \cap B, x_0) & \xrightarrow{e_\#} & \pi_q(X, A, x_0) \end{array}$$

(where $z_0 \in p^{-1}(x_0)$), we have reduced the consideration of $e_\#$ to the consideration of $\bar{e}_\#$.

Thus, without loss of generality, we may assume that $A$ and $B$ satisfy all the hypotheses, and in addition, that $A \cap B$, and hence also $A$, $B$, and $X$, are simply connected. Let $P$ be the space of paths $\omega: (I, 0) \to (X, B)$ topologized by the compact-open topology, and define a fibration $p: P \to X$ by $p(\omega) = \omega(1)$. The fiber $F$ of $p$ over $x_0 \in A \cap B$ is the space of paths in $X$ beginning in $B$ and ending at $x_0$. Let $p': PX \to X$ be the path fibration of all paths in $X$ ending at $x_0$ and with $p'(\omega) = \omega(0)$. Then $F = p'^{-1}(B)$, and since $PX$ is contractible, there are isomorphisms

$$\pi_q(X, B, x_0) \underset{\approx}{\overset{p'_\#}{\leftarrow}} \pi_q(PX, F, \omega_0) \underset{\approx}{\overset{\partial}{\rightarrow}} \pi_{q-1}(F, \omega_0)$$

(where $\omega_0$ is the constant path at $x_0$). By Corollary 1, $(X, B)$ is n-connected, and so $F$ is $(n - 1)$-connected.

We let $E' = p^{-1}(A)$ and apply Lemma 6 (with $n$ and $m$ interchanged) to see that the homomorphism

$$p_*: H_q(E, E') \to H_q(X, A)$$

is an isomorphism for $q \leq n + m$ and an epimorphism for $q = n + m + 1$. Let $f: (B, A \cap B) \to (P, E')$ be the lifting of $e: (B, A \cap B) \subset (X, A)$ that assigns to each $b \in B$ the constant path at $b$. The map $f \mid B$ imbeds $B$ as a strong deformation retract of $P$ (with deformation retraction $P \to f(B)$ that contracts each path to the constant path at its initial point). Therefore, $f \mid B: B \to P$ is a homotopy equivalence.

From the commutative triangle

$$\begin{tikzcd}
H_q(B, A \cap B) \arrow[rr, "f_*"] \arrow[dr, "e_*"', "\approx"] & & H_q(P, E') \arrow[dl, "p_*"] \\
& H_q(X, A) &
\end{tikzcd}$$

it follows that $f_*: H_q(B, A \cap B) \to H_q(P, E')$ is an isomorphism for $q \leq n + m$. Since $(f \mid B)_*: H_*(B) \approx H_*(P)$, we deduce from the five-lemma that

$$(f \mid A \cap B)_*: H_q(A \cap B) \to H_q(E')$$

is an isomorphism for $q \leq n + m - 1$.

Since $A \cap B$ and $E'$ are both simply connected, it follows from the Whitehead theorem [5, p. 167], [6, p. 399] that

$$(f \mid A \cap B)_\#: \pi_q(A \cap B, x_0) \to \pi_q(E', \omega_0)$$

is an isomorphism for $q \leq n + m - 2$ and an epimorphism for $q = n + m - 1$. Since $(f \mid B)_*: \pi_*(B, x_0) \approx \pi_*(P, \omega_0)$, the five-lemma implies that

$$f_\#: \pi_q(B, A \cap B, x_0) \to \pi_q(P, E', \omega_0)$$

is an isomorphism for $q \leq n + m - 1$ and an epimorphism for $q = n + m$. The homotopy excision theorem now follows from commutativity of the triangle

$$\begin{tikzcd}
\pi_q(B, A \cap B, x_0) \arrow[rr, "f_\#"] \arrow[dr, "e_\#"'] & & \pi_q(P, E', \omega_0) \arrow[dl, "p_\#"] \\
& \pi_q(X, A, x_0) . &
\end{tikzcd}$$

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