AN ARCHIMEDEAN PROPERTY OF CARDINAL ALGEBRAS

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A partially ordered group is called *Archimedean* if $na \leq b$ for all integers n implies $a = 0$, and *integrally closed* if $na \leq b$ for all $n \geq 1$ implies $a \leq 0$ [1, pp. 225, 229]. The latter property, expressed in terms of positive elements, reads as follows:

$$na \leq nb + c \text{ for all } n \geq 1 \quad \text{implies} \quad a \leq b.$$

Because of the presence of infinite elements, this property fails in every nontrivial cardinal algebra. It is the purpose of this note to show that every cardinal algebra satisfies the following closely related condition:

$$na \leq nb + c \text{ for all } n \geq 1 \quad \text{implies} \quad a + c \leq b + c.$$

The study of such properties was undertaken in the hope (as yet unfulfilled) of shedding some light on the simple cardinal algebras.

A *cardinal algebra*, as defined by Tarski [2], is an algebraic system consisting of a set A, a binary operation $+$ on A, and an operation Σ of countably infinite rank on A satisfying axioms which assert closure under the operations, the existence of a zero element 0, unrestricted commutativity and associativity of the operations, that $+$ is the restriction of Σ to two nonzero summands, and the validity of the following two principles:

Refinement. If $a + b = \Sigma c_i$, then there exist $a_i, b_i \in A$ such that $c_i = a_i + b_i$ for all $i < \infty$, $a = \Sigma a_i$, and $b = \Sigma b_i$.

Remainder. If $a_n = b_n + a_{n+1}$ for all $n < \infty$, then there exists $c \in A$ such that $a_n = c + \Sigma_i b_{n+i}$ for all $n < \infty$. (Summation indices are to run over the natural numbers, and the phrase "all $n < \infty$" refers to the set of natural numbers.)

A cardinal algebra can be partially ordered by defining $a \leq b$ to mean $a + x = b$ for some $x \in A$; this is the order referred to above. We begin by listing the results from [2] that we shall need, the numbering being that of [2]:

1.29. $a + b = b$ if and only if $\infty a \leq b$.

2.10. If $a + nc \leq b + (n+1)c$ for some $n < \infty$, then $a \leq b + c$.

2.21. If $a_0 + a_1 + \cdots + a_n \leq b$ for all $n < \infty$, then $\Sigma a_i \leq b$.

2.24. Every increasing sequence of elements of A has a least upper bound in A.

2.28. If $a_m \leq b_n$ for all $m, n < \infty$, then there exists $c \in A$ with $a_m \leq c \leq b_n$ for all $m, n < \infty$.

2.29. If $a \leq b_n + c$ for all $n < \infty$, then there exists $d \in A$ with $a \leq d + c$ and $d \leq b_n$ for all $n < \infty$.

2.33. If $na \leq nb$ for some $n < \infty$, then $a \leq b$.

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LEMMA 1. *If* $x \leq 2a, 2b$, *then there exist* $x_1, x_2 \in A$ *with* $x = x_1 + x_2$ *and* $x_1, x_2 \leq a, b$.

Proof. Since $2x \leq 2a + 2b$, we get $x \leq a + b$ by 2.33. From $x \leq 2a$ and $x \leq a + b$ follows by 2.29 the existence of $c \in A$ with $c \leq a, b$ and $x \leq a + c$. Similarly, there exists $d \in A$ with $d \leq a, b$ and $x \leq b + d$. By 2.28 there is $e \in A$ such that $c, d \leq e \leq a, b$, and so $x \leq a + e, b + e$, from which one deduces by 2.29 again the existence of $f \leq a, b$ with $x \leq e + f$. By the refinement property, $x = x_1 + x_2$ with $x_1 \leq e$ and $x_2 \leq f$, so that $x_1, x_2 \leq a, b$.

LEMMA 2. *If* $x \leq 2^n a, 2^n b$ *for some* $n < \infty$, *there exists* $y \in A$ *with* $x \leq 2^n y$ *and* $y \leq a, b$.

Proof. We use induction. For $n = 1$, we can by Lemma 1 write $x = x_1 + x_2$ with $x_1, x_2 \leq a, b$. By 2.28 there exists $y \in A$ with $x_1, x_2 \leq y \leq a, b$, and so $x \leq 2y$.

If $x \leq 2^{n+1} a, 2^{n+1} b$, there is $z \in A$ such that $x \leq 2z$ and $z \leq 2^n a, 2^n b$. By the induction hypothesis there exists $y \leq a, b$ with $z \leq 2^n y$. But then $x \leq 2^{n+1} y$, as required.

THEOREM. *If* $na \leq (n + 1)b$ *for all* $n < \infty$, *then* $a \leq b$.

Proof. We construct an increasing sequence $\{y_n\}$ with $(2^n - 1)a \leq 2^n y_n$ and $y_n \leq a, b$. For $n = 1$ we see that $a \leq 2a, 2b$, so that by Lemma 2 there exists $y_1 \in A$ with $a \leq 2y_1$ and $y_1 \leq a, b$. Assume the elements $y_1 \leq y_2 \leq \cdots \leq y_n$ have been determined. Since

$$x = (2^{n+1} - 1)a \leq 2^{n+1} a, \; 2^{n+1} b,$$

Lemma 2 implies that there exists $y'_{n+1} \leq a, b$ with $x \leq 2^{n+1} y'_{n+1}$. An application of 2.28 guarantees the existence of y_{n+1} with $y_n, y'_{n+1} \leq y_{n+1} \leq a, b$. Plainly then $x \leq 2^{n+1} y_{n+1}$ and $y_n \leq y_{n+1} \leq a, b$.

Now let y be the least upper bound of the y_n, which exists by 2.24. Then $y \leq a, b$, and we let $a = y + z$. For all $n < \infty$,

$$(2^n - 1)y + (2^n - 1)z = (2^n - 1)a \leq 2^n y_n \leq 2^n y,$$

and therefore $(2^n - 1)z \leq y$ for all $n < \infty$, by 2.10. Hence $a = y + z = y \leq b$, by 2.21 and 1.29.

COROLLARY. *If* $na \leq nb + c$ *for all* $n < \infty$, *then* $a + c \leq b + c$.

Proof. $n(a + c) = na + nc \leq nb + (n + 1)c \leq (n + 1)(b + c)$ for all $n < \infty$.

Remarks. 1) It is known [1, p. 229] that every σ-complete lattice-ordered group is Archimedean. The above reasoning may be used to obtain the following generalization: let G be a partially ordered directed abelian group with the properties (i) if $a_1 \leq a_2 \leq \cdots \leq b$, then the a_n have a least upper bound in G, (ii) if $a, b \leq c, d$, then there exists $x \in G$ with $a, b \leq x \leq c, d$. Then G is Archimedean.

2) According to [1, p. 229], the completion by nonvoid cuts of an integrally closed partially ordered group is a complete lattice-ordered group. Is the completion by cuts of a (simple) cardinal algebra a (simple) cardinal algebra? This question is of interest inasmuch as a simple cardinal algebra that is a lattice in its natural order must be either the set of nonnegative real numbers with ∞ adjoined, or the set of nonnegative integers with ∞ adjoined.

REFERENCES

1. G. Birkhoff, *Lattice theory*, American Mathematical Society Colloquium Publications, vol. 25, revised edition, 1948.

2. A. Tarski, *Cardinal algebras*, Oxford University Press, 1949.

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