SOME OBSERVATIONS ON QUASICOHESIVE SETS

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1. In this note we shall derive a few results concerning *quasicohesive* sets of natural numbers. One or two of these results would be of considerable interest if they could be obtained for sets with recursively enumerable (r. e.) complements. Even in their present broad form, they lead to some moderately interesting facts, as we shall see in Section 5.

The results contained in Section 4 answer questions that were raised partly by J. S. Ullian and partly by myself.

After the original version of this paper was submitted for publication, I learned that Prof. Hartley Rogers has proved, in a draft of Chapter 12 of his forthcoming book on recursive function theory, a result closely related to Theorem 1. In fact, Rogers' proof establishes just exactly the first of the two assertions comprising the statement of Theorem 1. (His construction does not furnish the additional information that K_2 may be any cohesive subset of a cosimple set.)

D. A. Martin, in a letter of Sept. 18, 1963, has communicated to me the following theorem: there are nonhyperhyperimmune, r-cohesive infinite number sets. This result implies both Theorem 2 and Theorem 3 of the present paper. Martin's construction is substantially different from the one I have used for Theorems 2 and 3.

2. As usual, 'W_j' denotes the j-th term in some fixed uniform enumeration of the class of all r. e. sets of natural numbers. We use 'N' to denote the set of all natural numbers. A subset α of N is called *cohesive* (the term is due to Ullian) provided α is infinite and, for each j, either $\alpha \cap W_j$ or $\alpha \cap \overline{W_j}$ is finite. (In general, we use '$\overline{\alpha}$' as notation for N - α, α any subset of N.) By a *quasicohesive* set of natural numbers we mean one that is a finite union of cohesive sets. We say that a subset β of N *splits* a set $\alpha \subset$ N (the symbol '$\subset$' indicates inclusion in the wide sense) if $\alpha \cap \beta$ and $\alpha \cap \bar{\beta}$ are both infinite. Finally, we call a subset α of N r-*cohesive* provided α is infinite and is not split by any recursive set. Sometimes, r-cohesive sets are referred to as being "recursively indecomposable." [An infinite set α of natural numbers is said to be *decomposable* if and only if there exist disjoint sets W_i, W_j such that (1) both $\alpha \cap W_i$ and $\alpha \cap W_j$ are infinite, and (2) $\alpha = (\alpha \cap W_i) \cup (\alpha \cap W_j)$.]

3. The lemmas listed in this section form the basis of the proofs to be presented in Section 4.

LEMMA 1. *If* S *is a nonempty, countable collection of infinite subsets of* N *with the property that if* s_i, $s_j \in$ S *then also* $s_i \cap s_j \in$ S, *then there exists an infinite set* β *such that* $\beta \subset \bigcup$S *and* β - s_i *is finite for all* $s_i \in$ S.

Lemma 1 was proved by Dekker in [1].

LEMMA 2. *There exists an r. e. set with a cohesive complement.*

Lemma 2 was proved by Friedberg in [3].

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LEMMA 3. *If* β *is a nonrecursive r. e. set, then there exists a recursive sequence* $\beta_{\phi(0)}, \beta_{\phi(1)}, \cdots$ *of pairwise disjoint, nonrecursive r. e. subsets of* β, *such that* $\beta - \beta_{\phi(i)}$ *is r. e. for every* i. (By *recursiveness* of the sequence, we mean of course that the indexing function ϕ is recursive.)

Lemma 3 is a clear consequence of the proof of the first theorem in [3].

LEMMA 4. *Any infinite set of natural numbers has a cohesive subset.*

Lemma 4 follows at once from [2, p. 102], where it is noted that every *immune* infinite set of numbers has a cohesive subset.

4. In this section we investigate some of the splitting properties of the class of quasicohesive sets.

THEOREM 1. *There exist disjoint cohesive sets* K_1, K_2 *such that*

$$(\exists j)(K_1 \subset W_j \text{ and } K_2 \subset \overline{W}_j) \quad \text{and} \quad \sim (\exists j)(K_2 \subset W_j \text{ and } K_1 \subset \overline{W}_j).$$

Indeed, for K_2 *we may take any cohesive subset of the complement of a simple set.*

Proof. Let ζ be a simple set, and (applying Lemma 4) let K_2 be a cohesive subset of $\bar{\zeta}$. Let $S = \{W_j \mid K_2 \subset W_j\}$, and let S' be the class $\{\zeta \cap W_j \mid W_j \in S\}$. We can easily verify (using the simplicity of ζ) that S' is a nonempty, countable collection of infinite sets, closed with respect to intersection. Hence, by Lemma 1, there exists an infinite set β, $\beta \subset \zeta$, such that $\beta - \lambda$ is finite for all $\lambda \in S'$. By Lemma 4, β has a cohesive subset, K_1. A fortiori, $K_1 - \lambda$ is finite for all $\lambda \in S'$. Clearly, then, $K_2 \subset W_j \Rightarrow K_1 \not\subset \overline{W}_j$ for every j. On the other hand, $K_1 \subset \zeta$, $K_2 \subset \bar{\zeta}$, with ζ r. e. This finishes the proof.

COROLLARY 1. r-*cohesiveness does not imply cohesiveness.*

Proof. Let K_1, K_2 be as in Theorem 1; then $K_1 \cup K_2$ is clearly not cohesive. But $K_1 \cup K_2$ is r-cohesive. For suppose δ splits $K_1 \cup K_2$, δ recursive. Since both K_1 and K_2 are cohesive, we see that one of them must lie in δ but for finitely many elements, and the other must lie in $\bar{\delta}$ but for finitely many elements. But this yields recursive separation of K_1 and K_2, and so a contradiction to Theorem 1.

The fact that Theorem 1 entails the above corollary was pointed out to the writer by J. S. Ullian. We shall shortly demonstrate a much stronger version of this corollary; in particular, it will follow from our discussion that r-cohesiveness does not imply indecomposability.

Regarding the wording of Theorem 1, it may be of interest to point out that we have indeed limited K_2 by asking that it be in the complement of some simple set. For it is easily proved that there are cohesive sets K that do not lie in the complement of any simple set: just observe that the class of all simple sets satisfies the hypotheses of Lemma 1, and apply Lemmas 1 and 4.

THEOREM 2. *There exist infinite, nonquasicohesive sets of numbers that cannot be split by any recursive set.*

Proof. Applying Lemma 2, let M be an r. e. set with a cohesive complement. Bringing Lemma 3 to bear on M, let $M_{\phi(0)}, M_{\phi(1)}, \cdots$ be an infinite recursive sequence of pairwise disjoint, nonrecursive r. e. subsets of M such that $M - M_{\phi(i)}$ is r.e. for each i. For arbitrary i, let S_i be the set $\{s_{i,j} = W_j \cap M_{\phi(i)} \mid \overline{M} \subset W_j\}$. We claim that each S_i is a nonempty, countable collection of infinite sets, closed with respect to intersection. The only one of these assertions that may require comment is that the $s_{i,j}$ are infinite. Notice, then, that if $s_{i,j}$ were finite, for some i and j such that $\overline{M} \subset W_j$, we should have

$$\overline{M}_{\phi(i)} = (M - M_{\phi(i)}) \cup (W_j - s_{i,j}) = \text{an r. e. set},$$

which contradicts the nonrecursivity of $\overline{M}_{\phi(i)}$. Consider, then, the sets $M \cap s_i$, for each element of the class $S = \{s_i = W_j \mid \overline{M} \subset W_j\}$. For each $s_i \in S$, let $\{d_{i,0}, d_{i,1}, \cdots\}$ be an enumeration of $M \cap \bigcap_{k \leqq i} s_k$. We define a set K as follows:

k_0 = the first term of $\{d_{0,0}, d_{0,1}, \cdots\}$ that belongs to $M_{\phi(0)}$;

k_1 = the first term of $\{d_{1,0}, d_{1,1}, \cdots\}$ that belongs to $M_{\phi(0)}$ and is $>k_0$;

k_2 = the first term of $\{d_{2,0}, d_{2,1}, \cdots\}$ that belongs to $M_{\phi(1)}$ and is $>k_1$;

k_3 = the first term of $\{d_{3,0}, d_{3,1}, \cdots\}$ that belongs to $M_{\phi(0)}$ and is $>k_2$;

k_4 = the first term of $\{d_{4,0}, d_{4,1}, \cdots\}$ that belongs to $M_{\phi(1)}$ and is $>k_3$;

k_5 = the first term of $\{d_{5,0}, d_{5,1}, \cdots\}$ that belongs to $M_{\phi(2)}$ and is $>k_4$;

$\cdots$;

$K = \{k_0, k_1, k_2, \cdots\}$. We claim that K is a set of the required sort. In the first place, K is not quasicohesive, since each of the pairwise disjoint r. e. sets $M_{\phi(i)}$ contains infinitely much of K. Furthermore, K cannot be split by any recursive set. For let δ be a recursive set; since $\overline{M}$ is cohesive, we may suppose, without loss of generality, that one of two situations confronts us with regard to δ: (A) $\delta \subset M$, or (B) $\overline{M} \subset \delta$. In case (A), δ cannot split K, since otherwise $\overline{\delta}$ would also split K, whereas $\overline{M} \subset \overline{\delta}$, and therefore $\overline{\delta}$ contains all but finitely much of K. For the same sort of reason, δ cannot split K in case $\overline{M} \subset \delta$. The proof is complete.

THEOREM 3. *There exist a recursive sequence* $\{W_{\phi(i)}\}$ *of pairwise disjoint r. e. sets and a sequence* $\{K_i\}$ *of cohesive sets such that*

(1) $\bigcup_i K_i$ *is* r-*cohesive and*

(2) $K_i \subset W_{\phi(i)}$ *for every* i.

Proof. Let M, $M_{\phi(i)}$, and K be as in the proof of Theorem 2. Applying Lemma 4, let K_i be a cohesive subset of $K \cap M_{\phi(i)}$, for each i; it is then clear that $\{M_{\phi(i)}\}$, $\{K_i\}$ are sequences with the required properties.

Since the sets K_i are pairwise recursively inseparable, K_i can pick up only finitely much of any recursive subset of $M_{\phi(i)}$. A more general proposition:

THEOREM 4. *If* α *is an infinite nonrecursive set of numbers, then* α *has a cohesive subset* K *such that* $K \cap R$ *is finite for every recursive subset* R *of* α.

Proof. Let $S = \{\alpha - R \mid R \text{ is recursive, } R \subset \alpha\}$. Since α is not recursive, every member of S is infinite. Since the union of two recursive sets is recursive, we see that S is closed with respect to intersection. Hence Lemma 1 applies to S, and α has an infinite subset β that picks up only finitely much of any recursive subset of α. The proof is completed by an appeal to Lemma 4.

Notice that it is an immediate consequence of Theorem 3 that there exist decomposable sets which are r-cohesive.

5. In this section, we point out some applications of the results in Section 4. Proofs will be sketched, details being left to the reader.

For the notion "α is recursively equivalent to β" (symbolized by '$\alpha \simeq \beta$'), see [2]; for the definition of *retraceability*, see [2] or [5]. Let '$\alpha] \beta$' mean that $\alpha \subset W_j \subset \bar{\beta}$ for some j. Let '$\alpha | \beta$' mean (as is customary) that there exist i and j for which $\alpha \subset W_i$, $\beta \subset W_j$, and $W_i \cap W_j$ is empty. Finally, let '$\alpha \| \beta$' mean that α and β are recursively separable. Myhill posed, in [4], the following three problems:

(1) Suppose $\alpha \simeq \gamma \subset \beta$ and $\gamma] \beta - \gamma$. Must there exist $\delta \subset \beta$ such that $\alpha \simeq \delta$ and $\delta | \beta - \delta$?

(2) Suppose $\alpha \simeq \gamma \subset \beta$ and $\gamma | \beta - \gamma$. Must there exist $\delta \subset \beta$ such that $\alpha \simeq \delta$ and $\delta \| \beta - \delta$?

(3) If the answer to (2) is no, then is the answer to (2) yes, provided we require also that $\bar{\alpha}$, $\bar{\beta}$ be r. e.?

We shall provide answers for two of these three questions.

COROLLARY 2. *The answer to question* (1) *is no.*

Proof. Let K_1, K_2 be as in Theorem 1. Let $\alpha = \gamma = K_1$, $\beta = K_1 \cup K_2$. Then $\alpha \simeq \gamma$ and $\gamma] \beta - \gamma$. But it is easily verified that the existence of a subset δ of β such that $\alpha \simeq \delta$ and $\delta | \beta - \delta$ would contradict the properties that Theorem 1 ascribes to K_1 and K_2.

COROLLARY 3. *The answer to question* (2) *is no.*

Proof. Applying Theorem 3, let K_1 and K_2 be disjoint cohesive sets such that $K_1 | K_2$ but not $K_1 \| K_2$. Define α, β, γ as in the proof of Corollary 2, and check that the existence of a δ such that $\delta \subset \beta$, $\alpha \simeq \delta$, and $\delta \| \beta - \delta$ would violate the hypotheses on K_1 and K_2.

Next, we apply our results to answer a rather natural question about retraceable sets.

LEMMA 5. *Let α be an infinite set of numbers. If there exists a recursive sequence $\{W_{\phi(i)}\}$ of pairwise disjoint r. e. sets each of which has a nonempty intersection with α, then α has an infinite retraceable subset.*

A proof of Lemma 5 can be found in [5].

LEMMA 6. *Let α be an infinite retraceable set. If α can be retraced by a general recursive function, then α is not* r-*cohesive.*

Proof. If g is a recursive function which retraces α, the set

$$\{x \mid [(\exists n \geq 1)(g^n(x) = g^{n-1}(x) \text{ and } (\forall m < n)(g^m(x) \geq g^{m+1}(x))]$$

$$\text{and } [2 \text{ divides the smallest } n \text{ for which } n \geq 1 \text{ and } g^n(x) = g^{n-1}(x)]\}$$

is easily seen to be a recursive set that splits α.

Remark. It can be shown that recursive decomposability is not a *sufficient* condition for a retraceable set to admit a recursive retracing function.

COROLLARY 4. *There exist retraceable sets not retraced by any general recursive function.*

Proof. The proposition is an evident consequence of Theorem 3, Lemma 5, and Lemma 6.

6. We offer some concluding remarks relative to Theorems 1 and 2. In the first place, suppose Theorem 1 could be proved with the additional requirement that $K_1 \cup K_2$ have an r. e. complement. We would then have the affirmative answer to a rather interesting open question: does there exist a quasimaximal set of rank 2 (that is, an r. e. set whose complement is the union of two, but not fewer than two, cohesive sets) whose complement is indecomposable? All quasimaximal sets of rank 2 so far constructed have decomposable complements, either because of evident features of their construction or else by virtue of a theorem of P. R. Young (in [6]). Again, suppose Theorem 2 could be strengthened to assert the existence of infinite, nonquasicohesive sets, not splittable by any recursive set, and having r. e. complement. It would then follow that at least one of the following two questions (both of which, so far as the writer knows, are open) would have an affirmative answer: (1) does there exist a hyperhypersimple set whose complement is not quasicohesive? (2) does there exist a hypersimple but not hyperhypersimple set with an indecomposable complement?

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