FACTORING OF SECOND-ORDER DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS

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We consider the following difference equation:

$$f(x+i) - p(x)f(x) - f(x-i) = 0 \qquad (i = \sqrt{-1}) \tag{1}$$

under the hypothesis that there exists an integer n for which

$$p(x+ni) = p(x).$$

Equation (1) is completely general, for if

$$a(x)g(x+i) + b(x)g(x) + c(x)g(x-i) = 0,$$

then, setting

$$g(x) = r(x)f(x),$$

where

$$a(x)r(x+i) = -c(x)r(x-i),$$

we obtain (1) with

$$p(x) = \frac{b(x)r(x)}{a(x)r(x+i)}.$$

A continued fraction solution of (1) is given in [1], but no discussion of convergence is given there, and in practice to prove convergence may be quite difficult. The periodic coefficient case is discussed by Fort [2], under the restriction that the values of f are required only at integral multiples of i. He shows the existence of a second order equation with constant coefficients, some of whose solutions comprise all the solutions of (1). His result is useful in considering the asymptotic behavior of solutions.

We shall show that the problem of solving (1) can be reduced to that of solving certain first order linear difference equations whose coefficients we give explicitly in terms of p and n. There are two cases. In the first case, we give two homogeneous first order linear difference equations with the property that any pair of solutions of these two equations are solutions of (1) and are linearly independent. (Throughout this paper, linear independence is with respect to the ring of functions that have period i.) In the second case, these two equations turn out to be identical and a second, independent, solution is obtained as a solution of a nonhomogeneous, first order, linear difference equation with periodic coefficients whose right-hand side is the solution of a homogeneous equation.

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An important example, which arises from scattering by a wedge with impedance boundary conditions, is the equation

(2) $$f(x+i) - \eta(\coth \alpha x) f(x) - f(x-i) = 0,$$

where $\alpha = m\pi/n$ and m, n are integers.

We introduce, for convenience, the n functions

$$p_0(x),\ p_1(x),\ \cdots,\ p_{n-1}(x)$$

defined by the relations

$$p_\nu(x) = p(x+\nu i) \qquad (\nu = 0, \cdots, n-1).$$

Similarly, we define

$$f_{-1}(x),\ f_0(x),\ \cdots,\ f_{n-1}(x)$$

by the conditions

$$f_\nu(x) = f(x+\nu i) \qquad (\nu = -1, \cdots, n-1).$$

These notations permit us to omit the symbol x when writing these functions, and we shall do so when the meaning is clear. For each pair of integers s, t in the range $0, \cdots, n-1$ and such that

$$s + t \leq n,$$

we define a function $P_{s,t}$ of x, by

(3) $$P_{s,t} = \begin{vmatrix} -p_{n-s} & -1 & 0 & \cdots & 0 \\ 1 & -p_{n-s-1} & -1 & & \vdots \\ 0 & 1 & & & 0 \\ \vdots & & & & -1 \\ 0 & \cdots & 0 & 1 & -p_t \end{vmatrix},$$

where the right-hand side is an $n - s - t + 1$ dimensional determinant. More precisely, the determinant consists of the main diagonal with entries $-p_{n-s}, \cdots, -p_t$, the diagonal directly below the main diagonal with entries all equal to 1, the diagonal directly above the main diagonal with entries all equal to -1, and all the remaining entries equal to 0.

We now show that $P_{s,t}$ satisfies two recursion relations which are crucial for our results. If we expand (3) by the minors of the first column, we obtain only two terms, the first of which is $-p_{n-s} P_{s+1,t}$ and the second of which is a determinant whose first column consists of a 1 in the first row and zeros elsewhere. This second term is thus immediately reducible again to $P_{s+2,t}$. Hence we have shown that

(4) $$P_{s,t} = -p_{n-s} P_{s+1,t} + P_{s+2,t}.$$

In a similar manner, if we expand (3) by the minors of the last column, we obtain the relation

$$P_{s,t} = -p_t P_{s,t+1} + P_{s,t+2}. \tag{5}$$

We define two functions g_1, g_2 to be the two solutions of the quadratic equation

$$g^2 - (p_0 P_{1,1} - P_{2,1} - P_{1,2})g + (-1)^n = 0. \tag{6}$$

We shall use the symbol g to refer to either g_1 or g_2. We now prove that

$$g(x + i) = g(x).$$

This is equivalent to showing that

$$p_0(x + i)P_{1,1}(x + i) - P_{2,1}(x + i) - P_{1,2}(x + i) = p_0(x)P_{1,1}(x) - P_{2,1}(x) - P_{1,2}(x)$$

or, by the definitions of p_ν and $P_{s,t}$, that

$$p_1 P_{0,2} - P_{1,2} - P_{0,3} = p_0 P_{1,1} - P_{2,1} - P_{1,2}.$$

By (4), with $s = 0$, $t = 1$, we obtain the equation

$$p_0 P_{1,2} - P_{2,1} - P_{1,2} = -P_{0,1} - P_{1,2};$$

and by (5) with $s = 0$, $t = 1$, we obtain the equation

$$p_1 P_{0,2} - P_{1,2} - P_{0,3} = -P_{0,1} - P_{1,2}.$$

Thus the periodicity of g has been proved.

We now prove that any solution of the first order difference equation

$$P_{2,0} f_0 - (g + P_{2,1})f_{-1} = 0 \tag{7}$$

is a solution of (1). From the periodicity of g, it follows that

$$f_1 = \frac{g + P_{1,2}}{P_{1,1}} f_0,$$

$$f_{-1} = \frac{P_{2,0}}{g + P_{2,1}} f_0.$$

Applying (1), we see that

$$\begin{aligned} &\left(\frac{g + P_{1,2}}{P_{1,1}} - p_0 - \frac{P_{2,0}}{g + P_{2,1}} \right) f_0 \\ &= \frac{g^2 + (P_{1,2} + P_{2,1})g + P_{1,2} P_{2,1} - p_0 P_{1,1} g - p_0 P_{1,1} P_{2,1} - P_{2,0} P_{1,1}}{P_{1,1}(g + P_{2,1})} f_0 \\ &= \frac{g^2 - (p_0 P_{1,1} - P_{1,2} - P_{2,1})g + P_{1,2} P_{2,1} - p_0 P_{1,1} P_{2,1} - P_{2,0} P_{1,1}}{P_{1,1}(g + P_{2,1})} f_0. \end{aligned} \tag{8}$$

And from (5) with $s = 2$, $t = 0$, we obtain the relations

$$
\begin{aligned}
P_{1,2}P_{2,1} - p_0P_{1,1}P_{2,1} - P_{2,0}P_{1,1} &= P_{1,2}P_{2,1} - P_{1,1}(p_0P_{2,1} + P_{2,0}) \\
&= P_{1,2}P_{2,1} - P_{1,1}P_{2,2}.
\end{aligned} \tag{9}
$$

On the other hand, it follows from (5) (with $s = 1$) that

$$p_tP_{1,t+1} = -P_{1,t} + P_{1,t+2}$$

and (with $s = 2$) that

$$-p_tP_{2,t+1} = P_{2,t} - P_{2,t+2}.$$

Cross multiplying the left and right-hand members of these two equations yields the result

$$p_tP_{1,t+1}P_{2,t} - p_tP_{1,t+1}P_{2,t+2} = p_tP_{1,t}P_{2,t+1} - p_tP_{1,t+2}P_{2,t+1},$$

or

$$P_{1,t+1}P_{2,t} - P_{1,t}P_{2,t+1} = -[P_{1,t+2}P_{2,t+1} - P_{1,t+1}P_{2,t+2}].$$

Note that the right-hand side of the last equation may be obtained from the left-hand side by everywhere increasing the second subscript by 1. Thus by induction we derive the equation

$$P_{1,t+1}P_{2,t} - P_{1,t}P_{2,t+1} = (-1)^{n-3-t}[P_{1,n-2}P_{2,n-3} - P_{1,n-3}P_{2,n-2}].$$

But the right-hand side can now be computed; namely

$$
\begin{aligned}
&P_{1,n-2}P_{2,n-3} - P_{1,n-3}P_{2,n-2} \\
&= \begin{vmatrix} -p_{n-1} & -1 \\ 1 & -p_{n-2} \end{vmatrix} \begin{vmatrix} -p_{n-2} & -1 \\ 1 & -p_{n-3} \end{vmatrix} - p_{n-2} \begin{vmatrix} -p_{n-1} & -1 & 0 \\ 1 & -p_{n-2} & -1 \\ 0 & 1 & -p_{n-3} \end{vmatrix} = 1.
\end{aligned}
$$

Therefore we have shown that

$$P_{1,t+1}P_{2,t} - P_{1,t}P_{2,t+1} = (-1)^{n-3-t}.$$

If $t = 1$,

$$P_{1,2}P_{2,1} - P_{1,1}P_{2,2} = (-1)^{n-4} = (-1)^n. \tag{10}$$

Thus from (8), (9), and (6) we see that

$$\left(\frac{g + P_{1,2}}{P_{1,1}} - p_0 - \frac{P_{2,0}}{g + P_{2,1}}\right) f_0 = \frac{g^2 - (p_0P_{1,1} - P_{1,2} - P_{2,1})g + (-1)^n}{P_{1,1}(g + P_{2,1})} f_0 = 0.$$

This shows that the solutions of (7) are solutions of (1). Let us denote by $u(x)$ and $v(x)$ the solutions of (1) which are obtained when $g_1(x)$ and $g_2(x)$, respectively, are used in (7). We want to find the conditions under which u, v are linearly independent. If we let

$$A(x) = \frac{g_1(x) + P_{2,1}(x)}{P_{2,0}(x)} \tag{11}$$

and

$$B(x) = \frac{g_2(x) + P_{2,1}(x)}{P_{2,0}(x)}, \tag{12}$$

then u, v satisfy the equations

$$u(x) = A(x)u(x - i), \tag{13}$$

$$v(x) = B(x)v(x - i). \tag{14}$$

Let us suppose that

$$\alpha(x)u(x) + \beta(x)v(x) = 0, \tag{15}$$

where

$$\alpha(x + i) = \alpha(x), \quad \beta(x + i) = \beta(x).$$

Then

$$\alpha(x)u(x - i) + \beta(x)v(x - i) = 0;$$

and from (13), (14), and (15), it follows that

$$\alpha(x)A(x)u(x - i) + \beta(x)B(x)v(x - i) = 0.$$

Therefore,

$$\alpha(x)A(x)u(x - i) - \alpha(x)B(x)u(x - i) = 0,$$

from which it follows that

$$A(x) = B(x).$$

But together with (11), (12), this implies that

$$g_1(x) = g_2(x).$$

From (6), it then follows that

$$(p_0 P_{1,1} - P_{2,1} - P_{1,2})^2 = (-1)^n 4. \tag{16}$$

It is obvious that if (16) is satisfied, then (7) will not give two independent solutions. Thus, (7) gives two independent solutions if and only if (16) is not satisfied.

Let us now suppose that (16) is satisfied. As a matter of fact, this assumption will not be used. The method we are about to use works even if (16) is not satisfied. However, it is a bit more complicated in this case, so its use should be restricted to the case where equation (16) holds.

We define the operator E by

$$(Ef)(x) = f(x+i).$$

We shall also interpret an ordinary function to be an operator in the sense of pointwise multiplication by that function. By the product of two operators, we mean composition. We shall use the well-known arithmetic rules of operator calculus such as associativity, distributivity, etc.

We may now write (1) in the form

$$(E^2 - p_0 E - 1)f_{-1} = 0, \tag{17}$$

and the problem is to "factor" the operator $E^2 - p_0 E - 1$. But we already know that one factor is

$$E - \frac{g + P_{2,1}}{P_{2,0}}.$$

Keeping in mind that we do not have commutativity, we learn that

$$E^2 = p_0 E - 1 = \left(E + \frac{P_{2,0}}{g + P_{2,1}}\right)\left(E - \frac{g + P_{2,1}}{P_{2,0}}\right). \tag{18}$$

In fact, from (6), (10), and (5), it follows that

$$\begin{aligned}
\left(E + \frac{P_{2,0}}{g + P_{2,1}}\right)\left(E - \frac{g + P_{2,1}}{P_{2,0}}\right) &= E^2 + \frac{P_{2,0}}{g + P_{2,1}} E - E\,\frac{g + P_{2,1}}{P_{2,0}} - 1 \\
&= E^2 + \left[\frac{P_{2,0}}{g + P_{2,1}} - \frac{g + P_{1,2}}{P_{1,1}}\right] E - 1 \\
&= E^2 - \left[\frac{g^2 + (P_{2,1} + P_{1,2})g - P_{1,1}P_{2,0} + P_{2,1}P_{1,2}}{(g + P_{2,1})P_{1,1}}\right] E - 1, \\
&= E^2 - \left[\frac{p_0 P_{1,1} g - (-1)^n - P_{1,1}P_{2,0} + P_{2,1}P_{1,2}}{P_{1,1}(g + P_{2,1})}\right] E - 1, \\
&= E^2 - \left[\frac{p_0 g - P_{2,0} + P_{2,2}}{g + P_{2,1}}\right] E - 1 = E^2 - \frac{p_0 g + p_0 P_{2,1}}{g + P_{2,1}} E - 1, \\
&= E^2 - p_0 E - 1.
\end{aligned}$$

Let us assume now that f is a function satisfying the equation

$$\left(E - \frac{g + P_{2,1}}{P_{2,0}}\right) f_{-1} = 0. \tag{19}$$

From (17), (18), and (19), it follows that f is a solution of (1). Suppose $\tilde{f}$ is not zero and satisfies the equation

$$\left(E + \frac{P_{2,0}}{g + P_{2,1}}\right) \tilde{f}_{-1} = 0,$$

and suppose f^* satisfies the equation,

$$\left(E - \frac{g + P_{2,1}}{P_{2,0}}\right) f^*_{-1} = \tilde{f}_{-1}. \tag{20}$$

From (53) we see that f^* is a solution of (1). Suppose now that $af = f^*$, where $a(x + i) = a(x)$. Then,

$$\left(E - \frac{g + P_{2,1}}{P_{2,0}}\right) af_{-1} = \left(E - \frac{g + P_{2,1}}{P_{2,0}}\right) f^*_{-1} = \tilde{f}_{-1}.$$

But,

$$\left(E - \frac{g + P_{2,1}}{P_{2,0}}\right) af_{-1} = a\left(E - \frac{g + P_{2,1}}{P_{2,0}}\right) f_{-1} = 0.$$

Therefore $\tilde{f}_{-1} = 0$, which is a contradiction. Hence f and f^* are linearly independent solutions of (1).

At this point it may be of interest to give an example in which equation (16) is satisfied. In equation (2) let us take $n = 4$ and $m = 2$ so that

$$p_0(x) = \eta \coth\frac{\pi}{2}x,$$

$$p_1(x) = \eta \tanh\frac{\pi}{2}x,$$

$$p_2(x) = \eta \coth\frac{\pi}{2}x,$$

$$p_3(x) = \eta \tanh\frac{\pi}{2}x,$$

and

$$p_0 P_{1,1} - P_{2,1} - P_{1,2} = 2 + p_0 p_1 + p_1 p_2 + p_2 p_3 + p_3 p_0 + p_0 p_1 p_2 p_3 = 2 + 3\eta^2 + \eta^4. \tag{21}$$

Thus, there are seven values of η satisfying the relations

$$\eta^4 + 3\eta^2 + 2 = \pm 2.$$

If we take one of them, say $\eta = i\sqrt{3}$, then the quadratic equation may be written in the form

$$g^2 - 2g + 1 = 0.$$

Therefore $g = 1$ and the first order equation, (7) is

$$(p_0 + p_2 + p_0 p_1 p_2)f_0 - (g - p_1 p_2 - 1)f_{-1} = 0$$

or

(22) $$(2 + \eta^2)\left(\coth\frac{\pi}{2}x\right) f_0 - \eta f_{-1} = 0,$$

and (20) becomes

(23) $$(2 + \eta^2)\left(\coth\frac{\pi}{2}x\right) f_0^* - \eta f_{-1}^* = \tilde{f}_{-1},$$

where

(24) $$\eta\tilde{f}_0 + (2 + \eta^2)\left(\coth\frac{\pi}{2}x\right) \tilde{f}_{-1} = 0.$$

Equation (22) has the solution

$$f(x) = \exp\left\{-ix \log\frac{i\eta}{2 + \eta^2}\right\} \sinh\frac{\pi}{2}x$$

and (24) has the solution

$$\tilde{f}(x) = \exp\left\{-ix \log\frac{\eta}{i(2 + \eta)^2}\right\} \sinh\frac{\pi}{2}x.$$

To proceed one has to solve the first order equation (23) with the given expression for $\tilde{f}$.

For any other values of η (in particular for η sufficiently large), we may use the simpler method. From (21) and (6) we obtain two values g_1, g_2 for g, both of which are constants. Let $j = 1$ or 2. Then the first order equation is

$$(2\eta + \eta^3)\left(\coth\frac{\pi}{2}x\right) f_0 - (g_j - 1 - \eta^2) f_{-1} = 0,$$

and the two solutions of (1) are given by

$$f(x) = \exp\left\{-ix \log\frac{i(g_1 - 1 - \eta^2)}{2\eta + \eta^3}\right\} \sinh\frac{\pi}{2}x,$$

$$f(x) = \exp\left\{-ix \log\frac{i(g_2 - 1 - \eta^2)}{2\eta + \eta^3}\right\} \sinh\frac{\pi}{2}x.$$

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