INNER AUTOMORPHISMS OF GROUPS IN TOPOLOGICAL ALGEBRAS

Bertram Yood

1. INTRODUCTION. Let B be a complex Banach algebra with an identity e, and let G be the multiplicative group of all regular elements of B. This group plays an important role in the theory of Banach algebras; for expositions and further references see [3] and [5]. Here we investigate the group $\mathfrak{I}$ of all inner automorphisms of G. The group $\mathfrak{I}$ can equally well be considered as the group of all inner automorphisms of the algebra B inasmuch as, given $u, v \in G$, the equality $uxu^{-1} = vxv^{-1}$ holds for all $x \in G$ if and only if it holds for all $x \in B$. The group $\mathfrak{I}$ is, of course, isomorphic to the group G/Z, where Z is the center of G, and is trivial if B is commutative; we are concerned only with algebras which are not commutative.

The quotient group formed by a group modulo its center can, as is well known, readily have a nontrivial center. However, we show in Theorem 2.3 that, for all semi-simple Banach algebras B (or, more generally, for any normed Q-algebra B whose center is semi-simple), the group G/Z has only the identity in its center. (In the special case where B is the algebra of all matrices of degree n over the complex field K, G is the general linear group GL(n, K), Z is the set of nonzero scalar multiples of the identity, and G/Z is isomorphic to the projective group in $n - 1$ dimensions over K; see, for example, [1, p. 297]. In this case Theorem 2.3 states that the projective group has a trivial center.) That G/Z has only the identity in its center is true in spite of the fact that, for all such B which are not commutative, the power of G/Z is at least that of the continuum. The latter property holds in a more general setting (Theorem 2.6), but there are incomplete real normed algebras that are not commutative and for which G/Z is trivial.

The author is greatly indebted to Dr. John A. Lindberg, Jr. for his suggestions and assistance. In particular he supplied Lemma 2.2 and pointed out that it could be used in conjunction with the author's arguments to show that G/Z has a trivial center (Theorem 2.3). This provided a substantial improvement over the original version which required the additional hypothesis that every ideal not equal to (0) of the center should contain a minimal ideal of the center.

2. ON THE GROUP G/Z. As in [4], we call a topological ring A a *Q-ring* if the set of quasi-regular elements of A is open; A is a Q-ring if and only if there is a neighborhood of zero consisting entirely of quasi-regular elements [4, p. 154]. By [3, p. 695], any modular maximal right (left) ideal of A is closed. Any Banach algebra is a Q-algebra [4, p. 155]. Suppose that A is a Q-algebra over the reals with identity e. Then to each $x \in A$ there corresponds a real number $b \neq 0$ such that $e + bx \in G$. It follows that $Z = C \cap G$, where C is the center of A.

For an element x in an algebra A over the real or complex numbers, we denote its spectrum by Sp(x). If x lies in a subalgebra A_1 and we wish to consider its spectrum when x is considered as an element of A_1, we denote this set by $Sp(x) \mid A_1)$.

Normed Q-algebras have been investigated in [6].

Received April 2, 1962.

This research was supported by the National Science Foundation, Grant NSF-G-14111.

LEMMA 2.1. *Let* B *be a normed* Q-*algebra with center* C. *Then the radical of* C *is the intersection of* C *with the radical of* B.

In view of [5, Theorem 1.6.14] C is also a normed Q-algebra. As shown in [6, Lemma 2.1], the spectral radius of an element x in a normed Q-algebra is given by the expression $\nu(x) = \lim \|x^n\|^{1/n}$. Now $\nu(xy) \leq \nu(x)\nu(y)$ if $xy = yx$ [5, p. 10]. Let x be in the radical of C. Then $\nu(x) = 0$, and xB consists entirely of quasi-regular elements so that x lies in the radical of B. Conversely if $x \in C$ and x is in the radical of B, then xC is made up entirely of elements quasi-regular in C, and x is in the radical of C.

LEMMA 2.2 (Lindberg). *Let* B *be a complex Banach algebra with* $u, y \in G$. *Suppose that* $a = uyu^{-1}y^{-1} \in Z$. *Then* Sp(a) *is contained in the set of complex numbers of modulus one.*

Note that $ayu = uy = y^{-1}(yu)y$. Thus the relation $a^n yu = y^{-n}(yu)y^n$ is valid for n = 0, 1. We establish the relation by induction for all positive integers. Suppose it is valid for n. Then

$$a^{n+1}yu = a(y^{-n}yuy^n) = y^{-n}(ayu)y^n = y^{-n}(uy)y^n = y^{-(n+1)}(yu)y^{n+1}.$$

Therefore $Sp(yu) = Sp(a^n yu)$ $(n = 1, 2, \cdots)$. Since $a^{-1} \in Z$, we also see that $yu = y^{-n}(a^{-n}yu)y^n$ for each positive integer n so that $Sp(yu) = Sp(a^n yu)$ for negative integers also.

Now let B_1 be a maximal commutative subalgebra containing yu. Clearly B_1 contains C so that $a \in B_1$. Observe that, by [5, p. 35], $Sp(x) = Sp(x|B_1)$ for all $x \in B_1$ and that $x^{-1} \in B_1$ if x is regular in B.

Now let $\mathfrak{M}$ be the space of maximal ideals of B_1, and let $M \in \mathfrak{M}$. Clearly $(yu)(M) \neq 0$, and $a(M) \neq 0$. Suppose that $|a(M)| > 1$. Then

$$|a^n(yu)(M)| \to \infty \quad \text{as } n \to \infty.$$

Since $Sp(yu|B_1) = Sp(a^n yu|B_1)$, this is impossible. Likewise it is impossible that $|a(M)| < 1$. Thus $|a(M)| = 1$.

If the center of a group is the identity, we say that the group has a *trivial center.*

THEOREM 2.3. *Let* B *be a complex normed* Q-*algebra with identity* e *whose center* C *is semi-simple. Then* G/Z *has a trivial center.*

By Lemma 2.1, the hypothesis on C is fulfilled if B is semi-simple. Suppose first that B is complete (a Banach algebra).

Let $y \in G$, and suppose that $uyu^{-1}y^{-1} \in Z$ for all $u \in G$. Take a fixed u, and set $a = uyu^{-1}y^{-1}$. Now $uy = ayu = yau$. An easy induction shows that $u^n y = ya^n u^n$ $(n = 1, 2, \cdots)$. Therefore

$$\left(e + \frac{u}{1!} + \cdots + \frac{u^n}{n!}\right) y = y\left(e + \frac{au}{1!} + \cdots + \frac{(au)^n}{n!}\right). \tag{2.1}$$

Passing to the limit as $n \to \infty$, we obtain from (2.1), in terms of the exponential function exp(x) [3, p. 172], the relation

$$\exp(u)\,y = y\exp(au). \tag{2.2}$$

Now $\exp(u) \in G$. Setting $b = \exp(u)y\exp(-u)y^{-1}$, we see that

$$\exp(u)\,y = by\exp(u)\,. \tag{2.3}$$

From (2.2) and (2.3) and since $b \in Z$, we obtain the conclusion $b\exp(u) = \exp(au)$ or

$$b = \exp((a - e)u)\,. \tag{2.4}$$

If in the above development we replace u by λu for any scalar $\lambda \neq 0$ and we let

$$b(\lambda) = \exp(\lambda u)\,y\exp(-\lambda u)\,y^{-1},$$

we obtain the equality

$$b(\lambda) = \exp((a - e)\lambda u) \tag{2.5}$$

with the same a as in (2.4).

Next let B_1 be a maximal commutative subalgebra containing u. Then $B_1 \supset C$ so that $a \in B_1$. Let $\mathfrak{M}$ be the space of maximal ideals of B_1, and let $M \in \mathfrak{M}$. It follows from (2.5) that $b(\lambda) \in B_1$ and from Lemma 2.2 that $|b(\lambda)(M)| = 1$. Then (2.5) shows that

$$1 = |\exp[\lambda u(M)(a(M) - 1)]|\,. \tag{2.6}$$

Since $u(M) \neq 0$ and λ is arbitrary but not zero, the equation (2.6) requires that $a(M) = 1$. Therefore $Sp(a \mid B_1)$ is the one element set $\{1\}$ so that [5, p. 35] $Sp(a) = \{1\}$.

Next we recall that $a \in C$ and observe from [5, Theorem 1.6.12] that $Sp(a \mid C) = \{1\}$. Then $Sp(e - a \mid C) = \{0\}$ so that $\nu(e - a) = 0$.

Now we turn to the case where B is a (possibly incomplete) Q-algebra. Let B_0 be its completion, let G_0 the set of regular elements of B_0, and let Z_0 be the center of G_0. We consider B to be embedded in B_0. Let $x \in B$. We show first that x is right quasi-regular in B if and only if it is right quasi-regular in B_0. For suppose x is not right quasi-regular in B. Then the set $K = \{xw - w \mid w \in B\}$ is a proper modular right ideal of B and is contained in a modular maximal right ideal M of B. As M is closed, x is at a distance $d > 0$ from M. Set $K_0 = \{xv - v \mid v \in B_0\}$. The modular right ideal K_0 of B_0 lies in the closure of K so that $x \notin K_0$. From this we see that x is not right quasi-regular in B_0.

In the same way, $x \in B$ is left quasi-regular in B if and only if it is left quasi-regular in B_0 and, consequently, $x \in B$ is quasi-regular in B if and only if it is quasi-regular in B_0. In particular, this implies that $G_0 \cap B = G$.

Next let $z \in G_0$, and suppose $z = \lim x_n$, where each $x_n \in B$. Since G_0 is open in B_0, we may suppose that each $x_n \in G_0$ and therefore that each $x_n \in G$. Let $v \in Z$. Since $vx = xv$ for all $x \in G$, $vx = xv$ for all $x \in G_0$. This shows that $Z_0 \cap B = Z$.

Suppose that $y \in G$ and that $uyu^{-1}y^{-1} \in Z$ for all $u \in G$. Our task is to show that $y \in Z$. Take $z \in G_0$, and suppose $z = \lim x_n$ as above. Since $x_n^{-1} \to z^{-1}$ and each $x_n y x_n^{-1} y^{-1} \in Z_0$,

$$zyz^{-1}y^{-1} \in Z_0 \quad \text{for } \textit{each } z \in G_0\,. \tag{2.7}$$

The first part of the proof shows that $\nu(e - zyz^{-1}y^{-1}) = 0$ for each $z \in G_0$. In particular, $\nu(e - uyu^{-1}y^{-1}) = 0$ for $u \in G$ where $e - uyu^{-1}y^{-1} \in C$. The hypothesis on C

shows that $e - uyu^{-1}y^{-1} = 0$ or $uy = yu$ for each $u \in G$. Therefore G/Z has a trivial center.

As in [3] or [5], we denote by G_1 the component of G that contains e. In the complete case we can obtain a bit more than in Theorem 2.3.

COROLLARY 2.4. *Let* B *be a complex Banach algebra with identity* e *whose center is semi-simple. Let* Z_1 *be the center of* G_1. *Then* G_1/Z_1 *has a trivial center.*

To each $x \in B$ there corresponds a number $b > 0$ such that $e + \alpha x \in G_1$ for all α, $|\alpha| \leq b$. Hence each $v \in Z_1$ lies in C and therefore in Z. Thus the conclusion of Lemma 2.2 holds for the commutator a of two elements in G_1, if $a \in Z_1$. The arguments of [5, Theorem 1.4.10] show that $\exp(x) \in G_1$ for all $x \in B$. Let $y \in G_1$, and suppose that $uyu^{-1}y^{-1} \in Z_1$ for all $u \in G_1$. For a given $u \in G_1$ and any scalar $\lambda \neq 0$, each $b(\lambda)$, in the notation of the proof of Theorem 2.3, must lie in Z_1. The arguments of that proof now show that $\nu(e - uyu^{-1}y^{-1}) = 0$ and that $uy = yu$. Therefore $y \in Z_1$.

Let A be a real topological algebra with an identity e. We assume A to be a Hausdorff space and that the set of regular elements G is a topological group under multiplication. The latter is always the case if A is a normed algebra [5, Theorem 1.4.8], but there are real topological algebras for which the inverse operation is not continuous on G. We call A *almost radially* Q if there exists a dense set S in A such that to each $x \in S$ there corresponds a number $b > 0$ such that αx is quasi-regular for all real numbers $0 < \alpha \leq b$. Trivially, any Q-algebra (over the reals) is almost radially Q. The converse is false even for normed algebras. As an example take for A the algebra of all complex-valued functions defined and continuous on the closed disc $|z| \leq 1$ of the complex plane and holomorphic in $|z| < 1$, where A is made into an incomplete normed algebra by setting

$$\|f\| = \sup_{|z| \leq 1/2} |f(z)|. \tag{2.8}$$

Since A is a Banach algebra in its usual norm, it is clearly almost radially Q. But, in terms of the norm (2.8), z^n is not quasi-regular in A and $z^n \to 0$ so that A is not a Q-algebra.

In the sequel we denote the power of the continuum by $\mathfrak{c}$.

LEMMA 2.5. *Suppose that the group* G *in* A *is not commutative and the closed linear manifold generated by* G_1 *contains* G. *Then the power of* G/Z *is at least* $\mathfrak{c}$.

Inasmuch as Z is a closed subgroup of G, the quotient group G/Z is a Hausdorff space and therefore, by the theory of topological groups, is completely regular. We show first that it is enough to establish that G/Z contains a connected set E which is not a single point. For then, by complete regularity, there is a continuous function $f(t)$ from G/Z to $[0, 1]$ taking on the values 0 and 1 on E. Consequently $f(E) = [0, 1]$, and the cardinality of E is at least $\mathfrak{c}$.

Consider the natural homomorphism π of G onto G/Z. Since $\pi(G_1)$ is connected, the above shows that we are through unless $\pi(G_1)$ has one element. But then $G_1 \subset Z$ so that $xy = yx$ for all $x \in G_1$, $y \in G$. The hypothesis on G_1 shows that G is commutative which is impossible.

THEOREM 2.6. *Suppose that the real topological algebra* A *is almost radially* Q *and is not commutative. Then the power of* G/Z *is at least* $\mathfrak{c}$.

Let S be the set of elements $x \in A$ for which there exists a number $b > 0$ such that $(e - \alpha x)^{-1}$ exists for $0 < \alpha \leq b$. By hypothesis, S is dense in A. If G is commutative, then the elements of S permute pairwise, and A is commutative. Therefore G is not abelian. It is readily seen that the linear manifold generated by G_1 contains S. Lemma 2.5 can now be applied to obtain the desired conclusion.

The hypothesis that A is almost radially Q cannot be dropped. We exhibit a real normed algebra that is not commutative but for which G consists only of the nonzero scalar multiples of e so that G/Z is trivial. Let t be a real variable with range [0, 1]. Consider the set A of all polynomials f(t) on [0, 1] where, for

$$f(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

a_0 is real and each a_k $(k \geq 1)$ is a quaternion. Under the usual operations for polynomials, A is a real algebra; it is a normed algebra under the definition

$$\|f\| = \sup_{0 \leq t \leq 1} |f(t)|, \tag{2.9}$$

where $|f(t)|$ is the modulus of the quaternion f(t). It is readily verified that A has the desired properties.

3. ON INVOLUTIONS. By an *involution* of a complex algebra is meant a conjugate-linear anti-automorphism of period two. We apply the ideas of Section 2 to the theory of involutions.

THEOREM 3.1. *Let* A *be a complex topological algebra that is not commutative and is almost radially* Q. *Suppose that* A *has a continuous involution* $x \to x^*$. *Then the set of involutions on* A *has power at least* $\mathfrak{c}$.

This is an improvement of [2, Theorem 2.20], which demands the additional hypothesis that A is a Banach algebra and uses a category argument to show that the set of involutions is not denumerable.

We adopt the notation established in Section 2. Let $u \in Z$, and consider the set S of Theorem 2.6. For each $x \in S$, there exists a number $b > 0$ such that

$$u(e - bx) = (e - bx)u \quad \text{for all } u \in Z.$$

Hence $ux = xu$ for all $u \in Z$, $x \in A$, and $Z = C \cap G$.

By algebra, $G^* = G$. Since $x \to x^*$ is a homeomorphism, $G_1^* = G_1$. Let $G_2 = \{yy^* \mid y \in G_1\}$. Since G_1 is a subgroup of G, G_2 is a connected subset of G_1. We show that $\pi(G_2)$ cannot be a single element. For suppose otherwise. Then G_2 lies in C which is, of course, a closed linear manifold in A. For each $x \in S$ there exists a number $b > 0$ such that

$$(e - \alpha x)(e - \alpha x)^* \in C \quad \text{for all } \alpha \ (0 < \alpha \leq b).$$

Thus, $\alpha^2 xx^* - \alpha(x + x^*) \in C$. Dividing by α and letting $\alpha \to 0$, we see that $x + x^* \in C$ for each $x \in S$. Since S is dense and the involution is bicontinuous, $w + w^* \in C$ for all $w \in A$. If we replace w by iw, we see that $w - w^* \in C$ as well so that A is commutative. It follows (as in the proof of Lemma 2.5) that the set $\pi(G_2)$ in G/Z has cardinal at least $\mathfrak{c}$.

For each $y \in G$ let $A_y(x) = yxy^{-1}$ be the corresponding automorphism of the algebra A. We then see that the set of A_u $(u \in G_2)$ has cardinal at least $\mathfrak{c}$.

For each $u \in G_2$, u is self-adjoint. It is thus easy to verify that the mapping J_u $(u \in G_2)$, defined by $J_u(x) = ux^*u^{-1}$ $(x \in A)$, is an involution on A. If also $w \in G_2$, then $J_u = J_w$ if and only if $A_u = A_w$. Therefore the set of all J_u $(u \in G_2)$ is a set of involutions with cardinal at least $\mathfrak{c}$.

REFERENCES

1. G. Birkhoff and S. MacLane, *A survey of modern algebra*, rev. ed., Macmillan Co., New York, 1953.
2. P. Civin and B. Yood, *Involutions on Banach algebras*, Pacific J. Math. 9 (1959), 415-436.
3. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Publ., vol. 31, rev. ed. Amer. Math. Soc., Providence, 1957.
4. I. Kaplansky, *Topological rings*, Amer. J. Math. 69 (1947), 153-183.
5. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand Co., Princeton, 1960.
6. B. Yood, *Homomorphisms on normed algebras*, Pacific J. Math. 8 (1958), 373-381.

University of Oregon
and
Institute for Advanced Study