AN HOMOLOGY ANALOGUE OF POSTNIKOV SYSTEMS

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INTRODUCTION

In [14], Postnikov presents a process for constructing semi-simplicial complexes by "adding" homotopy groups to more elementary semi-simplicial complexes. This process enables him to build a model complex equivalent to the singular complex of any given topological space. From the point of view of homotopy theory, the model is indistinguishable from the original space, but has the advantage that its structure is more conveniently displayed. The present paper studies an analogous process, in which homology groups are added to CW-complexes. This technique was used by J. C. Moore in [13], and by B. Eckmann and P. J. Hilton in their duality studies. The resulting model has the advantage that its elementary parts are complexes with relatively few cells (as compared with Postnikov complexes). On the other hand, much of the elegant algebraic structure associated with the Postnikov decomposition is lost. As an application of this homology decomposition, homotopy type classification theorems for spaces with only two nontrivial homology groups are presented. In connection with this last topic, a number of the groups $[X, Y]$ of homotopy classes of maps are described when X and Y are spaces with at most two nontrivial homology groups.

The process for adding homology groups is dual to the Postnikov construction. B. Eckmann and P. J. Hilton have made a systematic study of this duality [6], [7], [8] and [9].

1. PRELIMINARIES

Let G be an abelian group, and let $n > 1$ be an integer. According to J. C. Moore, a topological space L has homology type (G, n) if it is simply connected, if $H_q(L) = 0$ for $q \neq 0, n$, and if $H_n(L) \approx G$. (All homology groups will be taken with integer coefficients.) It is well known that such spaces exist and that any two CW-complexes of the same type are homotopically equivalent. L(G, n) will denote the class of CW-complexes of homology type (G, n). When no confusion is likely to result, we shall also denote a member of L(G, n) by L(G, n). The following lemma can be proved by standard CW-complex arguments.

LEMMA 1.1. *If* X *is an* (n - 1)*-connected space, there exists a map* $f\colon L = L(H_n(X), n) \to X$ *such that* $f_*\colon H_n(L) \approx H_n(X)$.

Let X and Y be two spaces, and let $f\colon X \to Y$ be a map. The cone $\hat{X}$ over X is formed from $X \times I$ by identifying $X \times \{0\}$ to a point. Y(f)X will denote $Y \cup \hat{X}$, with f(x) and (x, 1) identified for all $x \in X$. Let $i\colon Y \to Y(f)X$ be the inclusion map.

LEMMA 1.2. *There is a homomorphism* α *such that the sequence*

$$\to H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{i_*} H_n(Y(f)X) \xrightarrow{\alpha} H_{n-1}(X) \to$$

is exact.

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Proof. Let C_f be the mapping cylinder of f, that is, let $C_f = Y \cup (X \times I)$ with f(x) and (x, 1) identified. Y(f)X may be obtained from C_f by identifying $X \times \{0\}$ to a point. Let g: $C_f \to Y(f)X$ be the identification map, and let p be the vertex of $\hat{X}$. A simple excision argument shows that g: $(C_f, X) \to (Y(f)X, p)$ induces isomorphisms in homology. Lemma 1.2 then follows from the exact sequence

$$\to H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{i_*} H_n(C_f, X) \xrightarrow{\partial} H_{n-1}(X)$$

if we take $\alpha = \partial g_*^{-1}$.

2. STRATIFIED COMPLEXES

Suppose A is a CW-complex such that $H_q(A) = 0$ for $q > n$. Let G be an abelian group, and let g: $L(G, n) \to A$ be a map such that g_*: $H_n(L(G, n)) \to H_n(A)$ is trivial. Let $A' = A(g)\, L(G, n)$, and let i: $A \to A'$ be the inclusion map. Then A' is a CW-complex, and Lemma 1.2 yields

$$i_*\colon H_q(A) \approx H_q(A') \quad (q \neq n+1), \qquad H_{n+1}(A') \approx G.$$

Given a sequence of abelian groups $\{H_n\}$ $(n > 1)$, we may construct a simply connected CW-complex A such that $H_n(A) \approx H_n$, as follows: Let A_1 be a point. Suppose we have constructed a CW-complex A_{n-1} such that

$$H_q(A_{n-1}) \approx H_q \qquad (q < n),$$

$$H_q(A_{n-1}) = 0 \qquad (q \geq n).$$

Choose a map g_n: $L(H_n, n-1) \to A_{n-1}$ which induces the trivial homomorphism in homology. Let $A_n = A_{n-1}(g_n)\, L(H_n, n-1)$. Then A_n has the same homology groups as A_{n-1} except that H_n has been added as its nth homology group. Finally we take $A = \bigcup A_n$. We shall call A a *stratified complex* and A_n its *nth stratum.*

THEOREM 2.1. *If* X *is a simply connected space, there exists a stratified complex* A *and a map* f: $A \to X$ *such that* f_*: $H_q(A) \approx H_q(X)$ *for all* q.

This theorem is a special case of Proposition 5 in [13, p. 22-13] and of the Eckmann-Hilton decomposition of maps (unpublished). It might also be deduced from the following theorem.

THEOREM 2.2. *Suppose* X *and* A *are simply connected spaces, and* f: $A \to X$ *is a map such that* f_*: $H_q(A) \approx H_q(X)$ *for* $q < n$, *and* $H_q(A) = 0$ *for* $q \geq n$. *Then there exist maps*

$$g\colon L = L(H_n(X), n-1) \to A \qquad \textit{and} \qquad f'\colon A' = A(g)L \to X$$

such that

(i) g_*: $H_{n-1}(L) \to H_{n-1}(A)$ *is trivial;*

(ii) $f'|A = f$;

(iii) f'_*: $H_q(A') \approx H_q(X)$ *for* $q \leq n$.

Proof. We prove 2.1 for the case where f is an inclusion map. The general case then follows by the usual mapping cylinder arguments. Let E_1 and E_2 be the spaces of paths in X starting at a fixed point $x_0 \in X$ and ending in A and in X, respectively. Let $p_1: E_1 \to A$ and $p_2: E_2 \to X$ be the maps which assign to each path its endpoint. From the homology sequence of the pair (X, A) and the hypotheses of Theorem 2.2, we deduce that

$$H_q(X, A) = 0 \quad (q < n) \qquad \text{and} \qquad H_q(X) \approx H_q(X, A) \qquad (q \geq n).$$

But $\pi_q(X, A) \approx \pi_{q-1}(E_1)$. Hence, by the Hurewicz theorem,

$$\pi_q(E_1) \begin{cases} = 0 & (q < n - 1), \\ \approx H_n(X) & (q = n - 1). \end{cases}$$

From Lemma 1.1 it follows that there is a map $g': L \to E_1$ such that

$$g'_*: H_{n-1}(L) \approx H_{n-1}(E_1).$$

Let $g = p_1 g'$. Note that fp_1 is inessential and hence $f_* g_*: H_{n-1}(L) \to H_{n-1}(X)$ is trivial. But $f_*: H_{n-1}(A) \approx H_{n-1}(X)$. Therefore $g_*: H_{n-1}(L) \to H_{n-1}(A)$ is trivial.

The map $f': A' \to X$ is defined as follows:

$$f'(a) = a \qquad (a \in A),$$

$$f'(s, t) = g'(s)(t) \qquad (s \in L, \ t \in I).$$

Note that $f'(s, 0) = g'(s)(0) = x_0$ and $f'(s, 1) = g'(s)(1) = g(s) = f'(g(s))$ when $s \in L$. Hence f' is consistent with the identifications made in A'. Clearly $f' | A = f$.

Finally we show that $f'_*: H_q(A') \approx H_q(X)$ for $q \leq n$. For $q < n$ this follows from the fact that g_* is trivial and hence that $i_*: H_q(A) \approx H_q(A')$. Consider the diagram:

$$\text{(2.3)} \qquad \begin{array}{ccccccc} H_n(A') & \xrightarrow{k_{1*}} & H_n(A', A) & \xleftarrow{j_*} & H_n(\hat{L}, L) & \xrightarrow{\partial} & H_{n-1}(L) \\ \downarrow f'_* & & \downarrow f'_* & & \downarrow g''_* & & \downarrow g'_* \\ H_n(X) & \xrightarrow{k_{2*}} & H_n(X, A) & \xleftarrow{p_{2*}} & H_n(E_2, E_1) & \xrightarrow{\partial} & H_{n-1}(E_1) \end{array}$$

where g'' is given as follows:

$$g''(s, t)(u) = g'(s)(tu) \qquad (t, u \in I, \ s \in L).$$

We claim that the maps in the horizontal rows are isomorphisms. The two ∂'s are isomorphisms because E_2 and $\hat{L}$ are contractible. As previously observed, $H_q(X, A) = 0$ for $q < n$. Hence

$$H_q(X, A) \approx \pi_q(X, A) \approx \pi_q(E_2, E_1) \approx H_q(E_2, E_1)$$

for $q \leq n$. We have also seen above that k_{2*} is an isomorphism. Finally, the fact that k_{1*} is an isomorphism may be deduced from the homology sequence of the pair (A', A). It is easily verified that the diagram is commutative. The homomorphism g'_* is an isomorphism by construction, and hence f'_* is an isomorphism.

3. THE PROBLEM OF THE UNIQUENESS OF A_n

Given a space X, one might hope that the homotopy type of a CW-complex A_n is determined by the conditions

$$H_q(A_n) = 0 \qquad (q > n), \tag{3.1}$$

(3.2) there exists a map $f\colon A_n \to X$ such that $f_*\colon H_q(A_n) \approx H_q(X)$ for $q \leq n$.

Unfortunately this is not in general true. Below we give a counter-example. We then give a condition on X which insures the uniqueness of the homotopy type of A_n.

In the following paragraphs it is convenient to form the cone $\hat{Z}$ over a space Z by identifying $Z \times \{0\} \cup \{z_0\} \times I \subset Z \times I$ to a point ($z_0 \in Z$). The definition of Y(f)X is correspondingly modified. Note that this alteration does not change the homotopy type of $\hat{Z}$ or Y(f)X. The *suspension* S(Z) is $\hat{Z}$ with $Z \times \{1\}$ identified to a point. The points in $\hat{Z}$ and S(Z) which are images of $\{x_0\} \times I$ will be called the *base points*.

Choose integers n, m and p such that $m < n - 1$ and $\pi_n(S^m)$ contains an element α which is of order p, but which is not divisible by p. Construct $L(Z_p, n)$ by attaching an $(n+1)$-cell E^{n+1} to S^n by a map of degree p. Let $h\colon S^n \to S^m$ represent α, and let $h'\colon L(Z_p, n) \to S^m$ be an extension of h. Let $j\colon S^n \to L(Z_p, n)$ be the inclusion map, and let $k\colon L(Z_p, n-1) \to S^n$ be the map which collapses S^{n-1} to a point. Let

$$X = (S^m(h')\, L(Z_p, n)) \vee S(L(Z_p, n-1)),$$

where the wedge is formed by identifying the base points $d \in S^m(h')\, L(Z_p, n)$ and $d' \in S(L(Z_p, n-1))$. Let $A_n = S^m \vee S(L(Z_p, n-1))$ be formed by identifying $h'(d)$ and d'. Let $A_n' = S^m(hk)\, L(Z_p, n-1)$. Let $f\colon A_n \to X$ be the inclusion map, and let $f'\colon A_n' \to X$ be defined as follows:

$$f'(s) = s \qquad (s \in S^m),$$

$$f'(e, t) = \begin{cases} (e, 2t) & (0 < t \leq 1/2) \\ (jke, 2t-1) & (1/2 \leq t \leq 1) \end{cases} \qquad (e \in L(Z_p, n-1)).$$

One easily verifies that f' is consistent with the identifications made in A_n' and X. The spaces X, A_n and A_n' are stratified complexes whose homology groups are

$$H_q(X) = \begin{cases} Z & (q = 0, m), \\ Z_p & (q = n, n+1), \\ 0 & \text{otherwise}, \end{cases}$$

$$H_q(A_n) = H_q(A_n') = \begin{cases} Z & (q = 0, m), \\ Z_p & (q = n), \\ 0 & \text{otherwise}. \end{cases}$$

One easily verifies that f and f' satisfy condition (3.2). But A_n and A_n' do not have the same homotopy type, for hk is essential (the kernel of

$$k^*: \pi_n(S^m) \to [L(Z_p, n-1); S^m]$$

is the subgroup $p\pi_n(S^m)$). Thus the conclusion follows from Theorem 4.1.

We next show that the above phenomenon does not occur when $H_{n+1}(X) = 0$.

THEOREM 3.3. *Suppose that* X, A_n *and* A_n' *are simply connected, that* $H_{n+1}(X) = 0$, *and that* $f: A_n \to X$ *and* $f': A_n' \to X$ *are maps satisfying the conditions* (3.1) *and* (3.2). *Then there is a homotopy equivalence* $h: A_n \to A_n'$ *such that* $f'h$ *and* f *are homotopic.*

Proof. By Theorem 2.1 we may assume that X is a stratified complex. Let B_q be the strata of X. By Theorem 2.2 we can construct CW-complexes

$$A_n \subset A_{n+1} \subset A_{n+3} \subset \cdots$$

and a map $u: A = \bigcup A_q \to X$ which extends f and is a homotopy equivalence. In constructing B_q and A_q, we may use $L(G, q)$'s which have cells only in dimensions 0, q and $q + 1$. Since $H_{n+1}(X) = 0$, we may take $B_{n+1} = B_n$. Then $X^{n+1} = B_n$ (X^k is the k-skeleton of X) and $A^{n+1} \subset A_n$. The map u has a homotopy inverse $v: X \to A$ which we may assume to be cellular. Then $v(B_n) \subset A_n$. Let $k = v \mid B_n: B_n \to A_n$. Then $fk = uv \mid B_n$. Since uv is homotopic to the identity, fk is homotopic to the inclusion map $i: B_n \to X$.

$$i_*: H_q(B_n) \approx H_q(X) \quad \text{and} \quad f_*: H_q(A_n) \approx H_q(X) \quad \text{for } q \leq n.$$

Therefore $k_* = f_*^{-1} i_*: H_q(B_n) \approx H_q(A_n)$ for all q, and hence k is a homotopy equivalence. We can do the same thing for A_n' and obtain a homotopy equivalence $k': B_n \to A_n'$ such that $f'k'$ is homotopic to i. Let $\bar{k}$ be the homotopy inverse of k, and let $h = k'\bar{k}$. Then $f'h = f'k'\bar{k} \sim i\bar{k} \sim fk\bar{k} \sim f$, where "$\sim$" denotes homotopy equivalence of maps. This completes the proof.

Suppose X, A_{n-1}, and $f: A_{n-1} \to X$ are spaces and a map such that

$$f_*: H_q(A_{n-1}) \approx H_q(X) \quad (q < n) \quad \text{and} \quad H_q(A_{n-1}) = 0 \quad (q \geq n).$$

Let L_0 be a member of $L(H_n(X), n-1)$. By Theorem 2.2 there are maps $g_0: L_0 \to A_{n-1}$ and $F_0: A_n^0 = A_{n-1}(g_0)L_0 \to X$ such that $F_{0*}: H_q(A_n^0) \approx H_q(X)$ for $q \leq n$. If $H_{n+1}(X) = 0$, the homotopy type of A_n^0 is determined by X. We next investigate the extent to which the map g_0 is determined by $f: A_{n-1} \to X$. Let L_1 be a member of $L(H_n(X), n-1)$, let $g_1: L_1 \to A_{n-1}$ and let $A_n^1 = A_{n-1}(g_1)L_1$.

THEOREM 3.4. *If* X *is* 2-*connected and* $H_{n+1}(X) = 0$, *then* f *can be extended to a map* $f_1: A_n^1 \to X$ *such that* $f_{1*}: H_q(A_n^1) \approx H_q(X)$ *for* $q \leq n$ *if and only if there is an homotopy equivalence* $h: L_1 \to L_0$ *such that* g_1 *and* g_0h *are homotopic.*

Proof. If an homotopy equivalence h exists such that g_1 and g_0h are homotopic, the identity map on A_{n-1} can be extended to a homotopy equivalence $u: A_n^1 \to A_n^0$ in the obvious way. Then $f_1 = f_0u$ is the desired map.

Suppose f_1 exists. By the usual mapping cylinder arguments, we may assume that f is an inclusion map. Let E_1, E_2, p_1 and p_2 be the path spaces and maps described in the proof of Theorem 2.2. Recall that E_1 is $(n-2)$-connected. Let $g_0': L_0 \to E_1$ be defined as follows:

$$g_0'(s)(t) = f_0(s, t) \qquad ((s, t) \in \hat{L}_0).$$

Then $g_0 = p_1 g_0'$, and we are in exactly the same situation as described in the proof of Theorem 2.2. In particular, the diagram (2.3) is applicable. By diagram (2.3), $f_{0*}\colon H_n(A_0) \approx H_n(X)$ implies that $g_{0*}'\colon H_{n-1}(L_0) \approx H_{n-1}(E_1)$. By the same argument, there is a map $g_1'\colon L_1 \to E_1$ such that

$$g_1 = p_1 g_1' \qquad \text{and} \qquad g_{1*}'\colon H_{n-1}(L_1) \approx H_{n-1}(E_1).$$

We show below that $H_n(E_1) = 0$. If we assume this for the moment, it follows from Theorem 3.3 that there is a homotopy equivalence $h\colon L_1 \to L_0$ such that g_1' and $g_0'h$ are homotopic. Therefore $g_1 = p_1 g_1'$ and $g_0 h = p_1 g_0' h$ are homotopic.

It remains to show that $H_n(E_1) = 0$. Let E be the paths in X which start in X and end in A_{n-1}. Let $p\colon E \to X$ be the map which assigns to each path its initial point. Then p is a fiber map with fiber E_1, and E and A are homotopically equivalent. E_1 is $(n - 2)$-connected and X is 2-connected. The Serre-Wang sequence for a fiber space [15] then yields the exact sequence,

$$H_{n+2}(X) \to H_n(E_1) \to H_n(E) \to \cdots.$$

But $H_{n+1}(X) = H_n(E) = 0$, and therefore $H_n(E_1) = 0$.

4. SPACES WITH ONLY TWO NONTRIVIAL HOMOLOGY GROUPS

Let H_1 and H_2 be abelian groups, and let n and m be integers such that $1 < n < m$. Let $L_1 = L(H_1, n)$, and let $L_2 = L(H_2, m - 1)$. Let $g_i\colon L_2 \to L_1$ $(i = 0, 1)$, and let $A_i = L_1(g_i)L_2$. Note that if $n < m - 1$, then

$$H_q(A_i) \begin{cases} = Z & (q = 0), \\ \approx H_1 & (q = n), \\ \approx H_2 & (q = m), \\ = 0 & \text{otherwise}. \end{cases}$$

THEOREM 4.1. *If* $2 < n < m - 1$, *then* A_0 *and* A_1 *have the same homotopy type if and only if there are homotopy equivalences* $h\colon L_1 \to L_1$ *and* $k\colon L_2 \to L_2$ *such that* $g_2 k$ *and* hg_1 *are homotopic.*

Proof. If k and h exist, a homotopy equivalence of A_0 into A_1 can be constructed in the obvious way.

Suppose $G\colon A_0 \to A_1$ is a homotopy equivalence. Because $n < m - 1$, we may assume $G(L_1) \subset L_1$. Let $h = G \mid L_1\colon L_1 \to L_1$. Clearly $h_*\colon H_n(L_1) \approx H_n(L_1)$, and hence h is a homotopy equivalence. Let h' be the homotopy inverse to h. Let $F\colon L_1(hg_0)L_2 \to L_1(h\,hg_0)L_2$ be defined as follows:

$$F(s) = h'(s) \qquad (s \in L_1),$$

$$F(r, t) = (r, t) \qquad ((r, t) \in \hat{L}_2).$$

F induces isomorphisms in homology and is therefore a homotopy equivalence.

Since g_0 and $h'hg_0$ are homotopic, we can extend the identity map on L_1 to a homotopy equivalence $H: L_1(n'ng_0)L_2 \to L_1(g_0)L_2 = A_0$. Then $GHF: L_1(g_1h)L_2 \to A_1$ is a homotopy equivalence, and $GHF \mid L_1 = ihh'$, where $i: L_1 \to A_1$ is the inclusion map. The map hh' is homotopic to the identity map, and hence, by the homotopy extension theorem, GHF is homotopic to a map $K: L_1(gh)L_2 \to A_1$ such that $K \mid L_1 = i$. The identity map of A_1 is also an extension of i. In addition, $H_{n+1}(A_1) = 0$. Therefore, by Theorem 3.4, there exists a homotopy equivalence $k: L_2 \to L_2$ such that g_1h and kg_2 are homotopic. This completes the proof.

We now wish to classify, up to homotopy type, the complexes X with only two nontrivial homology groups of positive dimension. Let us take these groups to be $H_n(X) = H_1$ and $H_m(X) = H_2$ $(n < m)$. Any such complex is of the homotopy type of a complex of the form $L_1(g)L_2$. If $m = n + 1$, then the complex has the homotopy type of an A_n^2-polyhedron. Such spaces have been classified by Chang [4]. Thus it suffices to describe the classification when $m > n + 1$.

First note that $R(H_1, n) = [L_1; L_1]$ and $R(H_2, m - 1) = [L_2; L_2]$ are rings with identities, and that $[L_2; L_1]$ is an $(R(H_1, n), R(H_2, m - 1))$-bi-module. That is, it is a left $R(H_1, n)$-module and a right $R(H_2, m - 1)$-module such that $(\alpha\gamma)\beta = \alpha(\gamma\beta)$ for all $\alpha \in R(H_1, n)$, $\gamma \in [L_2; L_1]$ and $\beta \in R(H_2, m - 1)$. The operations are induced by composition of maps. Let $U(H_1, n)$ and $U(H_2, m - 1)$ denote the groups of units (= homotopy equivalences) of $R(H_1, n)$ and of $R(H_2, m - 1)$. Let $V(H_1, n, H_2, m - 1)$ denote the set of equivalence classes in $[L_2; L_1]$ under the relation: $\gamma \equiv \gamma'$ if and only if there exist $\alpha \in U(H_1, n)$ and $\beta \in U(H_2, m - 1)$ such that $\alpha\gamma = \gamma'\beta$.

THEOREM 4.2. *If* $L_1 = L(H_1, n)$ *and* $L_2 = L(H_2, m - 1)$ *and if* $m > n + 1$, *then* $V(H_1, n, H_2, m - 1)$ *is in one-to-one correspondence with the set of homotopy equivalence classes of spaces of the form* $L_1(g)L_2$.

This follows at once from Theorem 4.1.

A necessary condition for two spaces to have the same homotopy type is that their homology groups be isomorphic. On the other hand, given a pair of abelian groups H_1, H_2 and a pair of integers n, m $(n < m)$, the complex

$$X = L(H_1, n) \vee L(H_2, m)$$

has $H_n(X) \approx H_1$ and $H_m(X) \approx H_2$. These observations, together with Theorem 4.2, reduce the problem of classifying CW-complexes with two nontrivial homology groups to the problems of classifying abelian groups and of computing the sets

$$V(H_1, n, H_2, m - 1).$$

The following theorem is a partial solution of the second problem.

THEOREM 4.3. *Let integers* n *and* m $(n + 1 < m)$ *and finitely generated abelian groups* H_1 *and* H_2 *be given. If* $m \neq k(n - 1) + 2$ $(k = 1, 2, \cdots)$, *or if* H_1 *or* H_2 *is finite, then* $V(H_1, n, H_2, m - 1)$ *is finite, and there is an effective procedure for computing* $V(H_1, n, H_2, m - 1)$ *in terms of* $R(H_1, n)$, $R(H_2, m - 1)$ *and the bi-module* $[L(H_2, m - 1); L(H_1, n)]$.

Proof. We shall first show that $[L_2; L_1]$, where

$$L_2 = L(H_2, m - 1) \quad \text{and} \quad L_1 = L(H_1, n),$$

is finite under these hypotheses. The finiteness of $V(H_1, n, H_2, m - 1)$ will then follow at once. If H_1 is finite, then $\pi_p = \pi_p(L(H_1, n))$ is finite [14]. Since $[L_2; L_1]$ is an abelian extension of $H^m(L_2; \pi_m)$ by $H^{m-1}(L_2; \pi_{m-1})$ [2], the group $[L_2; L_1]$ is finite if either H_1 or H_2 is finite. Using only the hypothesis that H_2 is finitely generated, we may write H_2 as the direct sum of a free abelian group H_2' and a finite group H_2''. If $A_2 = L(H_2', m - 1)$ and if $B_2 = L(H_2'', m - 1)$, then L_2 has the homotopy type of $A_2 \vee B_2$. Thus $[L_2; L_1] \approx [A_2; L_1] + [B_2; L_1]$. The previous argument shows that $[B_2; L_1]$ is finite. Since A_2 is a union of $(m - 1)$-spheres joined at a point, $[A_2; L_1]$ is a direct sum of copies of $\pi_{m-1}(L_1)$. Let $L_1 = A_1 \vee B_1$ be a decomposition similar to that just given for L_2. A result of J. Milnor [12] shows that $\pi_{m-1}(L_1)$ is isomorphic to the direct sum $\pi_{m-1}(A_1) + \pi_{m-1}(B_1) + \Sigma\, \pi_{m-1}(X_\alpha)$, where each of the groups $\pi_{m-1}(X_\alpha)$ is finite, and all but finitely many are zero. From our previous reasoning it follows that $\pi_{m-1}(B_1)$ is finite. But A_1 is a union of n-spheres joined at a point. P. J. Hilton [11] has shown that

$$\pi_{m-1}(A_1) \approx \sum_{k,\alpha} \pi_{m-1}(S^{k(n-1)+1}),$$

where each $S_\alpha^{k(n-1)+1}$ is a $(k(n - 1) + 1)$-sphere, and there are only finitely many such spheres for each k. Thus all but a finite number of the groups in the sum are zero, and each of the groups $\pi_{m-1}(S_\alpha^{k(n-1)+1})$ is finite unless $m - 1 = k'(n - 1) + 1$ $(k' = k$ or $2k)$. There remains only the demonstration of the effective procedure.

The underlying group of the ring $R_1 = R(H_1, n)$ is isomorphic to the direct sum

$$[A_1; A_1] + [A_1; B_1] + [B_1; A_1] + [B_1; B_1].$$

Let S_1 be the image in R_1 of $[A_1; A_1]$, and let T_1 be the image of

$$[A_1; B_1] + [B_1; A_1] + [B_1; B_1].$$

It is immediate that T_1 is finite. Note that S_1 is a subring of R_1. Since A_1 may be taken to be the union of k n-spheres (k is the rank of H_1'), it follows that S_1 is ring-isomorphic to the ring of k-by-k matrices with integer entries. The group V_1 of units of S_1 is carried, by the isomorphism, onto the (multiplicative) group of matrices of determinant ± 1. This matrix group is finitely generated by three matrices with entries from $\{0, 1\}$ [5]. A similar discussion may be made of the ring $R_2 = R(H_2, m - 1)$; we shall denote the corresponding quantities by the same letters, with subscript 2.

Let N be the order of $[L_2; L_1]$. Then the quotient rings $R_{1N} = R_1/NR_1$ and $R_{2N} = R_2/NR_2$ have the same action on $[L_2; L_1]$ as R_1 and R_2. That is, if $\phi_i\colon R_i \to R_{iN}$ $(i = 1, 2)$ is the natural projection, then $[L_2; L_1]$ is an (R_{1N}, R_{2N})-bimodule under the operations $(\phi_1 r_1)\alpha(\phi_2 r_2) = r_1 \alpha r_2$ for $\alpha \in [L_2; L_1]$ and $r_i \in R_i$ $(i = 1, 2)$. It follows that $V(H_1, n, H_2, m - 1)$ is the decomposition of $[L_2; L_1]$ under the following relation: α and β in $[L_2, L_1]$ are equivalent if and only if there exist elements ρ_i in the image groups $\phi_i[U_i]$ $(U_1 = U(H_1, n)$ and $U_2 = U(H_2, m - 1))$ such that $\rho_1\alpha = \beta\rho_2$. Thus it suffices to construct the groups $\phi_i[U_i]$.

Each element of R_1 may be written uniquely in the form $s + t$ with $s \in S_1$ and $t \in T_1$. In particular, the identity element of R_1 may be written $1 = e + \overline{e}$ $(e \in S_1$ and $\overline{e} \in T_1)$, and e is the identity of the subring S_1. Note that the product of an element of finite (additive) order with an arbitrary element of R_1 is in T_1. The element $s + t$ is a unit of R_1 if and only if there are elements $s' \in S_1$ and $t' \in T_1$ such that $(s + t)(s' + t') = 1$. This holds if and only if $ss' = e$ (whence $s \in V_1$ and $s' = s^{-1}$) and $st' + ts^{-1} + tt' = \overline{e}$. Thus

$$\sigma + \tau \in \phi_1[U_1] \qquad (\sigma \in \phi_1[S_1],\ \tau \in \phi_1[T_1])$$

if and only if $\sigma \in \phi_1[V_1]$ and there exists an element $\tau' \in \phi_1[T_1]$ such that

$$\sigma\tau' + \tau\sigma^{-1} + \tau\tau' = \phi_1\bar{e}.$$

Let v_1, v_2, v_3 be the generators of V_1. Then $\phi_1 v_1, \phi_1 v_2, \phi_1 v_3$ generate $\phi_1[V_1]$. The order of $\phi_1[V_1]$ is at most $c = N^{k2} - 1$, whence $\phi_1[V_1]$ may be determined by at most 3^c calculations. Let d be the order of T_1. The group $\phi_1[U_1]$ may now be determined by testing at most cd^2 triples (σ, τ, τ') in the equation $\sigma\tau' + \tau\sigma^{-1} + \tau\tau' = \phi_1\bar{e}$. The determination of $\phi_2[U_2]$ is similar.

5. THE GROUPS $[L(H_1, m); L(H_2, n)]$, FOR $m = n+1, n+2$.

M. G. Barratt [1] and [2] has given a procedure for computing the group $[X; Y]$ when $X = L(H_1, m)$ as an extension of $H^{m+1}(X; \pi_{m+1}(Y))$ by $H^m(X; \pi_m(Y))$. The extension is determined by a Steenrod square. We present below an alternative procedure, together with explicit calculations of some of these groups.

We define the following CW-complexes.

$M(0, n) = S^n(\eta_n)S^{n+1}$.

$M(s, n) = L(Z_{2^s}, n) \cup M(0, n)$ with $s > 0$ and with $L(Z_{2^s}, n) \cap M(0, n) = S^n$.

$N(0, s', n) = (S^n \vee S^{n+1})(\eta_n + 2^{s'-1}\iota_{n+1})S^{n+1}$ with $s' > 1$ and $\eta_n: S^{n+1} \to S^n$ essential,

$N(s, s', n) = N(0, s', n) \cup L(Z_{2^s}, n)$ with $s > 0$ and $N(0, s', n) \cap L(Z_{2^s}, n) = S^n$.

These are all A_n^2-polyhedra, and they are classified by P. J. Hilton as types 6, 7, 10 and 11, respectively [10, p. 129]. S. C. Chang [4] has shown that these complexes, together with the complexes $L(G, m)$ (for suitable choices of G and m) form a basis for the A_n^2-polyhedra. That is, any A_n^2 has the homotopy type of a $\vee$-product of these complexes.

The elements $\iota_n \in \pi_n(S^n)$, $\eta_n \in \pi_{n+1}(S^n)$ and $\nu_n \in \pi_{n+3}(S^n)$ are, as usual, the generators of these groups. We need to know about them only that $\eta_n\eta_{n+1}$ generates $\pi_{n+2}(S^n)$ and that $12\nu_n = \eta_n\eta_{n+1}\eta_{n+2}$. A complete description of ν_n may be found in [17]. The notation for the other elements is according to the following scheme. The element $\begin{pmatrix} n & k \\ s & r \end{pmatrix}$ is in $[L(Z_{p^r}, n+k); L(Z_{p^s}, n)]$ (p a prime) when $r, s > 1$. If $r = 0$ (or $s = 0$), then $L(Z_{p^r}, n)$ (or $L(Z_{p^s}, n+k)$) is replaced by S^n (or S^{n+k}). We have not yet told which element of $[L(Z_{p^r}, n+k); L(Z_{p^s}, n)], \cdots$ is intended; this is specified in the text, following the tables. Nor does our notation distinguish between an element of $[L(Z_{p^r}, n+k); L(Z_{p^s}, n)]$ and an element of $[L(Z_{g^r}, n+k); L(Z_{g^s}, n)]$ where p and g are different primes; these must be distinguished by the context. A similar scheme is used in case one of the spaces involved is $M(s, m)$ or $N(s, s', m)$.

If $a \in [X; Y]$ and $b \in [Y; Z]$, then $ba \in [X; Z]$ is the class of the composite of the representative maps of a and b. If $a \in [X; Y]$, $b \in [X; Y]$ and $r_i: X_1 \vee X_2 \to X_i$ $(i = 1, 2)$ are the natural retractions, then $a + b = ar_1 + br_2 \in [X_1 \vee X_2; Y]$. We permit ourselves the lattitude of using the same symbol for a map as for its homotopy class, whenever we believe that this will work no hardship on the reader.

TABLE 1 OF GROUPS [X; Y] AND THEIR GENERATORS

	$Y = S^n$	$L(Z_2, n)$	$L(Z_{2^s}, n)$, $s > 1$	$L(Z_{p^s}, n)$, $s > 0$	$M(0, n)$	$M(s, n)$, $s > 0$	$N(0, s', n)$, $s' > 1$	$N(S, s', n)$, $s > 0$, $s' > 1$
$X = L(Z_{2^r}, n-1)$	Z_{2^r}	Z_2	Z_{2^t}	0	Z_{2^r}	Z_{2^t}	Z_{2^r}	Z_{2^t}
$r > 0$	$\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$		$\begin{pmatrix} n & 0 \\ 0,* & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s,* & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 0,s' & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s,s' & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$
$L(Z_{p^r}, n-1)$	Z_{p^r}	0	0	Z_{p^t}	Z_{2^r}	0	Z_{p^r}	0
$r > 0$	$\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$			$\begin{pmatrix} n & -1 \\ s & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 0,* & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$		$\begin{pmatrix} n & 0 \\ 0,s' & 0 \end{pmatrix}\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$	
S^n	Z	Z_2	Z_{2^s}	Z_{p^s}	Z	Z_{2^s}	Z	Z_{2^s}
	ι_n	$\begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 0,* & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s,* & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 0,s' & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s,s' & 1 \end{pmatrix}\begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix}$
$L(Z_2, n)$	Z_2	Z_4	$Z_2 + Z_2$	0	0	Z_2	Z_2	$Z_2 + Z_2$
	$\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s & 1 \end{pmatrix}, \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & 1 \end{pmatrix}$			$\begin{pmatrix} n & 0 \\ s,* & 1 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 0,s' & 0 \end{pmatrix}\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s,s' & 1 \end{pmatrix}, \begin{pmatrix} n & 1 \\ s,s' & 0 \end{pmatrix}\begin{pmatrix} n+1 & -1 \\ 0 & 1 \end{pmatrix}$
$L(Z_{2^r}, n)$	Z_2	$Z_2 + Z_2$	$Z_{2^t} + Z_2$	0	0	Z_{2^t}	$Z_{2^{t'}}$	$Z_{2^t} + Z_{2^{t'}}$
$r > 1$	$\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 1 & r \end{pmatrix}$, $\begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix}\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s & r \end{pmatrix}$ $\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}$			$\begin{pmatrix} n & 0 \\ s,* & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ 0,s' & 0 \end{pmatrix}\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s,s' & r \end{pmatrix}$, $\begin{pmatrix} n & 1 \\ s,s' & 0 \end{pmatrix}\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}$
$L(Z_{p^r}, n)$	0	0	0	Z_{p^t}	0	0	0	0
$r > 0$				$\begin{pmatrix} n & 0 \\ s & r \end{pmatrix}$				

$t = \min(r, s)$, $t' = \min(r, s')$, p is an odd prime. Note that $[L(Z_{q^s}, m)] = 0$ when $p \neq q$ are primes and m, n, r, s are positive integers.

The element $\begin{pmatrix} n & -1 \\ 0 & r \end{pmatrix}$ is the class of the map which collapses the (n - 1)-skeleton of $L(Z_{p^r}, n-1)$, thereby forming the sphere S^n. The element $\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}$ is the class of the inclusion of S^n in $L(Z_{p^s}, n)$. The element $\begin{pmatrix} n & 0 \\ s & r \end{pmatrix}$ is an extension over $L(Z_{p^r}, n)$ of $\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}$ when $r \geq s$, and it is an extension over $L(Z_{p^r}, n)$ of $2^{s-r}\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}$ when $r < s$. The element $\begin{pmatrix} n & 0 \\ s,* & s \end{pmatrix}$ is the inclusion of S^n in $M(0, n)$ when $s = 0$, and it is the inclusion of $L(Z_{2^s}, n)$ in $M(s, n)$ when $s > 0$. The element $\begin{pmatrix} n & 0 \\ s, s' & s \end{pmatrix}$ is the inclusion of S^n in $N(0, s', n)$ when $s = 0$, and it is the inclusion of $L(Z_{2^s}, n)$ in $N(s, s', n)$ when $s > 0$. The inclusion of S^{n+1} in $N(s, s', n)$ is denoted by $\begin{pmatrix} n & 1 \\ s, s' & 0 \end{pmatrix}$. We abbreviate

$$\begin{pmatrix} n & 0 \\ s, s' & r \end{pmatrix} = \begin{pmatrix} n & 0 \\ s, s' & s \end{pmatrix}\begin{pmatrix} n & 0 \\ s & r \end{pmatrix}.$$

The entries in the first four columns of Table 1, as well as the entries in the third row, all are due to M. G. Barratt [3]. The entries in the fifth, sixth and seventh columns, in the first two rows, and in the last row are obtained from obstruction theory. There remain the two entries corresponding to $X = L(Z_{2^r}, n)$ with $r > 0$ and $Y = N(s, s', n)$ with $s > 0$.

Let $A = L(Z_{2^s}, n) \vee S^{n+1}$, and let

$$d = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}\eta_n + 2^{s+-1}\iota_{n+1} \in \pi_{n+1}(A).$$

Then $Y = N(s, s', n) = A(d)S^{n+1}$. Let $j = \begin{pmatrix} n & 0 \\ s, s' & s \end{pmatrix} + \begin{pmatrix} n & 1 \\ s, s' & 0 \end{pmatrix}$ be the inclusion map of A in Y. If $X = L(Z_{2^r}, n)$, then the sequence

$$\cdots \to [X; S^{n+1}] \xrightarrow{d} [X; A] \to [X; Y] \to 0$$

is exact, since $[X; Y, A] = [X; S^{n+2}] = 0$. The subgroup $d[X; S^{n+1}] \subset [X; A]$ is generated by

$$d\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}\eta_n\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix} + 2^{s'-1}\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}.$$

Thus $[X; Y] \approx Z_{2^t} + Z_{2^{t'}}$ is generated by $\begin{pmatrix} n & 0 \\ s, s' & r \end{pmatrix}$ and $\begin{pmatrix} n & 1 \\ s, s' & 0 \end{pmatrix}\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}$.

The products ab listed below are easily obtained.

b / a	$\begin{pmatrix} n & 0 \\ r & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ r & u \end{pmatrix}$ $r \geq u \geq 0$	$\begin{pmatrix} n & 0 \\ r & u \end{pmatrix}$ $0 < r < u$
$\begin{pmatrix} n+1 & -1 \\ 0 & r \end{pmatrix}$	0	$p^{u-r}\begin{pmatrix} n+1 & -1 \\ 0 & u \end{pmatrix}$	$\begin{pmatrix} n+1 & -1 \\ 0 & u \end{pmatrix}$
$\begin{pmatrix} n & 0 \\ s & r \end{pmatrix}$ $0 < s \leq r$	$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}$	$\begin{pmatrix} n & 0 \\ s & u \end{pmatrix}$	$p^{r-u}\begin{pmatrix} n & 0 \\ s & u \end{pmatrix}$
$\begin{pmatrix} n & 0 \\ s & r \end{pmatrix}$ $s > r > 0$	$p^{s-r}\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix}$	$p^{s-r}\begin{pmatrix} n & 0 \\ s & u \end{pmatrix}$	$p^{s-u}\begin{pmatrix} n & 0 \\ s & u \end{pmatrix}$

Also note that $2\begin{pmatrix} n & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix} \eta_n \begin{pmatrix} n+1 & -1 \\ 0 & 1 \end{pmatrix}$.

The entries in Table 2 are groups $[X; Y]$ with $X = L(G, m)$ and $Y = L(G', n)$ (G and G' are cyclic groups, $m = n + 1, n + 2$). The calculation of these entries uses the fact that $[X; P] \approx [X; Y]$ if P is, roughly speaking, the $(m + 3)$-skeleton of the $(m - 1)$-connective fibre space over Y. More precisely:

LEMMA 5.1. *Let* $X = L(Z_{p^r}, m) = S^m(p^r \iota_m)S^m$, *and let* $p\colon P \to Y$ *be a map which induces isomorphisms* $\pi_m(P) \approx \pi_m(Y)$ *and* $\pi_{m+1}(P) \approx \pi_{m+1}(Y)$. *Then* $p_{\#}\colon [X; P] \approx [X; Y]$.

Proof. The diagram below is commutative, the functions $p'_{\#}$ and $p''_{\#}$ are isomorphisms, and the horizontal sequences are exact.

$$\begin{tikzcd}
\pi_m(P) \arrow[d, "p'_{\#}"] & \pi_m(P) \arrow[l, "p^r \iota_m"'] \arrow[d, "p'_{\#}"] & {[X; P]} \arrow[l] \arrow[d, "p_{\#}"] & \pi_{m+1}(P) \arrow[l] \arrow[d, "p''_{\#}"] & \pi_{m+1}(P) \arrow[l, "p^r \iota_{m+1}"'] \arrow[d, "p'_{\#}"] \\
\pi_m(Y) & \pi_m(Y) \arrow[l, "p^r \iota_m"] & {[X; Y]} \arrow[l] & \pi_{m+1}(Y) \arrow[l] & \pi_{m+1}(Y) \arrow[l, "p^r \iota_{m+1}"]
\end{tikzcd}$$

The proof of the lemma is completed by applying the "five-lemma."

Let $Y = S^n$ and $P = L(Z_2, n + 1)$. Let $\begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix}\colon P \to Y$ be an extension over P of $\eta_n\colon S^{n+1} \to Y$. Then

$$\begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix}\colon \pi_j(P) \approx \pi_j(Y) \qquad (j = n + 1, n + 2).$$

From the lemma it follows that $[L(Z_2, n + 1); Y] \approx Z_4$ is generated by $\begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix}$, and $[L(Z_{2^r}, n + 1)\colon Y] \approx Z_2 + Z_2$ $(r > 1)$ is generated by

TABLE 2 OF GROUPS [X; Y]

X \ Y	S^n	$L(Z_2, n)$	$L(Z_4, n)$	$L(Z_{2^s}, n)$ $s > 2$	$L(Z_{3^s}, n)$ $s > 0$
S^{n+1}	Z_2	Z_2	Z_2	Z_2	0
$L(Z_2, n+1)$	Z_4	$Z_2 + Z_2$	$Z_4 + Z_2$	$Z_4 + Z_2$	0
$L(Z_{2^r}, n+1)$ $r > 1$	$Z_2 + Z_2$	$Z_2 + Z_2$	$Z_2 + Z_2 + Z_2$	$Z_2 + Z_2 + Z_2$	0
S^{n+2}	Z_2	Z_4	$Z_2 + Z_2$	$Z_2 + Z_2$	0
$L(Z_2, n+2)$	Z_4	$Z_2 + Z_2 + Z_2$	$Z_4 + Z_2 + Z_2$	$Z_4 + Z_2 + Z_2$	0
$L(Z_4, n+2)$	$Z_2 + Z_2$	$Z_4 + Z_2 + Z_2$	$Z_4 + Z_2 + Z_2 + Z_2$	$Z_4 + Z_2 + Z_2 + Z_2$	0
$L(Z_{2^r}, n+2)$ $r > 2$	$Z_2 + Z_2$	$Z_4 + Z_2 + Z_2$	$Z_4 + Z_2 + Z_2 + Z_2$	$Z_8 + Z_2 + Z_2 + Z_2$	0
$L(Z_{3^r}, n+2)$ $r > 0$	0	0	0	0	Z_3

$$\begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1 & r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1 & 0 \end{pmatrix} \eta_{n+1} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix} = \eta_n \eta_{n+1} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix}.$$

Let $Y = L(Z_2, n)$, let $P = N(1, 2, n+1)$, and let $\begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} \in \pi_{n+2}(Y) \approx Z_4$ be a generator. Note that $2\begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix} \eta_m \eta_{n+1}$. Let $\begin{pmatrix} n & 1 \\ 1 & 1,2 \end{pmatrix} : P \to Y$ be an extension over P of

$$\begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} : L(Z_2, n+1) \vee S^{n+2} \to Y.$$

Then

$$\begin{pmatrix} n & 1 \\ 1 & 1,2 \end{pmatrix} : \pi_j(P) \approx \pi_j(Y) \qquad (j = n+1, n+2).$$

Thus $[L(Z_2, n+1); Y] \approx Z_2 + Z_2$ is generated by

$$\begin{pmatrix} n & 1 \\ 1 & 1,2 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1,2 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} n & 1 \\ 1 & 1,2 \end{pmatrix} \begin{pmatrix} n+1 & 1 \\ 1,2 & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & 1 \end{pmatrix},$$

while for $r > 1$ $[L(Z_{2^r}, n+1); Y] \approx Z_2 + Z_4$ is generated by

$$\begin{pmatrix} n & 1 \\ 1 & 1,2 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1,2 & r \end{pmatrix} = \begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1 & r \end{pmatrix} \qquad \text{(of order 2)},$$

and by

$$\begin{pmatrix} n & 1 \\ 1 & 1,2 \end{pmatrix} \begin{pmatrix} n+1 & 1 \\ 1,2 & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix} \quad \text{(of order 4)}.$$

Let $Y = L(Z_{2^s}, n)$ with $s > 1$. The group $\pi_{n+2}(Y) \approx Z_2 + Z_2$ has

$$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \eta_n \eta_{n+1} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1 & 0 \end{pmatrix} \eta_{n+1} \quad \text{and} \quad \begin{pmatrix} n & 2 \\ s & 0 \end{pmatrix}$$

for generators. The latter generator satisfies the equation $\begin{pmatrix} n+1 & -1 \\ 0 & s \end{pmatrix} \begin{pmatrix} n & 2 \\ s & 0 \end{pmatrix} = \eta_{n+1}$. Let $\begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix}$ be an extension over $L(Z_2, n+2)$ of $\begin{pmatrix} n & 2 \\ s & 0 \end{pmatrix}$. If

$$P = L(Z_2, n+1) \vee L(Z_2, n+2),$$

then

$$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} : \pi_j(P) \approx \pi_j(Y) \qquad (j = n+1, n+2).$$

Thus $[L(Z_2, n+1); Y] \approx Z_4 + Z_2$ is generated by $\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix}$, which is of order 4, and

$$\begin{pmatrix} n & 2 \\ s & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & 1 \end{pmatrix}.$$

If $r > 1$, then $[L(Z_{2^r}, n+1); Y] \approx Z_2 + Z_2 + Z_2$ is generated by

$$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1 & r \end{pmatrix},$$

$$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n+1 & 0 \\ 1 & 0 \end{pmatrix} \eta_{n+1} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \eta_n \eta_{n+1} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix}$$

and

$$\begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} n & 2 \\ s & 0 \end{pmatrix} \begin{pmatrix} n+2 & -1 \\ 0 & r \end{pmatrix}.$$

Let $Y = S^n$, and let $P = N(1, 2, n+2) \vee L(Z_3, n+3)$. Let

$$\begin{pmatrix} n & 2 \\ 0 & 1,3 \end{pmatrix} : N(1, 3, n+2) \to Y$$

be an extension of

$$\eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 1 \end{pmatrix} + 3\nu_n : L(Z_2, n+2) \vee S^{n+3} \to Y,$$

and let

$$\begin{pmatrix} n & 3 \\ 0 & 1 \end{pmatrix} : L(Z_3, n+3) \to Y$$

be an extension of $8\nu_n$: $S^{n+3} \to Y$. Then

$$\begin{pmatrix} n & 2 \\ 0 & 1,3 \end{pmatrix} + \begin{pmatrix} n & 3 \\ 0 & 1 \end{pmatrix} : \pi_j(P) \approx \pi_j(Y) \qquad (j = n+2,\ n+3).$$

We obtain:

$$[L(Z_{2^r}, n+2); Y] \approx Z_2 + Z_k \qquad (k = \min(2^r, 8))$$

is generated by

$$\begin{pmatrix} n & 2 \\ 0 & 1,3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1,3 & r \end{pmatrix} = \eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & r \end{pmatrix} \quad \text{(of order 2)},$$

and

$$\begin{pmatrix} n & 2 \\ 0 & 1,3 \end{pmatrix} \begin{pmatrix} n+2 & 1 \\ 1,3 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix} = 3\nu_n \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix}.$$

Let $Y = L(Z_2, n)$, and let $P = L(Z_4, n+2) \vee L(Z_2, n+3)$. Let

$$\begin{pmatrix} n & 2 \\ 1 & 2 \end{pmatrix} : L(Z_4, n+2) \to Y$$

be an extension of

$$\begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} : S^{n+2} \to Y,$$

and let

$$\begin{pmatrix} n & 3 \\ 1 & 1 \end{pmatrix} : L(Z_2, n+3) \to Y$$

be an extension of $12\nu_n$: $S^{n+3} \to Y$. Then

$$\begin{pmatrix} n & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} n & 3 \\ 1 & 1 \end{pmatrix} : \pi_j(P) \approx \pi_j(Y) \qquad (j = n+2,\ n+3).$$

Thus $[L(Z_2, n+2); Y] \approx Z_2 + Z_2 + Z_2$ is generated by

$$\begin{pmatrix} n & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 1 & 0 \end{pmatrix} \eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} n & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 2 & 0 \end{pmatrix} \eta_{n+2} \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} \eta_{n+2} \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} n & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n+3 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix} = 12\nu_n \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix}.$$

If $r > 1$, then $[L(Z_{2^r}, n+2); Y] \approx Z_4 + Z_2 + Z_2$ is generated by

$$\begin{pmatrix} n & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 2 & r \end{pmatrix} \quad \text{(of order 4)},$$

$$\begin{pmatrix} n & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 2 & 0 \end{pmatrix} \eta_{n+2} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix} \eta_{n+2} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix}$$

and

$$\begin{pmatrix} n & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n+3 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix} = 12\nu_n \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $Y = L(Z_4, n)$, and let $P = N(1, 2, n+2) \vee L(Z_2, n+2)$. Let

$$\begin{pmatrix} n & 2 \\ 2 & 1,2 \end{pmatrix} : N(1, 2, n+2) \to Y$$

be an extension of

$$\begin{pmatrix} n & 0 \\ 2 & 0 \end{pmatrix} \left(\eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 1 \end{pmatrix} + 6\nu_n \right) : L(Z_2, n+2) \vee S^{n+3} \to Y.$$

Then

$$\begin{pmatrix} n & 2 \\ 2 & 1,2 \end{pmatrix} + \begin{pmatrix} n & 2 \\ 2 & 1 \end{pmatrix} : \pi_j(P) \approx \pi_j(Y) \qquad (j = n+2,\ n+3).$$

We now deduce that $[L(Z_2, n+2); Y] \approx Z_2 + Z_2 + Z_4$ is generated by

$$\begin{pmatrix} n & 2 \\ 2 & 1,2 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1,2 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 2 & 0 \end{pmatrix} \eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} n & 2 \\ 2 & 1,2 \end{pmatrix} \begin{pmatrix} n+2 & 1 \\ 1,2 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 2 & 0 \end{pmatrix} (6\nu_n) \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix}$$

and $\begin{pmatrix} n & 2 \\ 2 & 1 \end{pmatrix}$, this last being an element of order 4. If $r > 1$, then

$$[L(Z_{2r}, n+2); Y] \approx Z_4 + Z_2 + Z_2 + Z_2$$

is generated by

$$\begin{pmatrix} n & 2 \\ 2 & 1,2 \end{pmatrix} \begin{pmatrix} n+2 & 1 \\ 1,2 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} n & 0 \\ 2 & 0 \end{pmatrix} (6\nu_n) \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix} \quad \text{(of order 4)},$$

$$\begin{pmatrix} n & 2 \\ 2 & 1,2 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1,2 & r \end{pmatrix} = \begin{pmatrix} n & 0 \\ 2 & 0 \end{pmatrix} \eta_n \begin{pmatrix} n+1 & 1 \\ 0 & r \end{pmatrix}, \quad \begin{pmatrix} n & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & r \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} n & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & 0 \end{pmatrix} \eta_{n+2} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix}.$$

Let $Y = L(Z_{2s}, n)$ with $s > 2$, and let $P = N(1, 3, n+2) \vee L(Z_2, n+2)$. Let $\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} : N(1, 3, n+2) \to Y$ be an extension of

$$\begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \left(\eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 1 \end{pmatrix} + 3\nu_n \right) : L(Z_2, n+2) \vee S^{n+3} \to Y.$$

Then

$$\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} + \begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} : \pi_j(P) \approx \pi_j(Y) \qquad (j = n+2,\ n+3).$$

Thus $[L(Z_2, n+2); Y] \approx Z_2 + Z_2 + Z_4$ is generated by

$$\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1,3 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} \begin{pmatrix} n+2 & 1 \\ 1,3 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} (3\nu_n) \begin{pmatrix} n+3 & -1 \\ 0 & 1 \end{pmatrix}$$

and $\begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix}$, this last being an element of order 4. The group

$$[L(Z_4, n+2); Y] \approx Z_4 + Z_2 + Z_2 + Z_2$$

is generated by

$$\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} \begin{pmatrix} n+2 & 1 \\ 1,3 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} (3\nu_n) \begin{pmatrix} n+3 & -1 \\ 0 & 2 \end{pmatrix} \qquad \text{(of order 4)},$$

$$\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1,3 & 2 \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \eta_n \begin{pmatrix} n+1 & 1 \\ 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & 0 \end{pmatrix} \eta_{n+2} \begin{pmatrix} n+3 & -1 \\ 0 & 2 \end{pmatrix}.$$

If $r > 2$, then $[L(Z_{2^r}, n+2); Y] \approx Z_8 + Z_2 + Z_2 + Z_2$ is generated by

$$\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} \begin{pmatrix} n+2 & 1 \\ 1,3 & 0 \end{pmatrix} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} (3\nu_n) \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix} \qquad \text{(of order 8)},$$

$$\begin{pmatrix} n & 2 \\ s & 1,3 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1,3 & 2 \end{pmatrix} = \begin{pmatrix} n & 0 \\ s & 0 \end{pmatrix} \eta_n \begin{pmatrix} n+1 & 1 \\ 0 & r \end{pmatrix},$$

$$\begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n & 2 \\ s & 1 \end{pmatrix} \begin{pmatrix} n+2 & 0 \\ 1 & 0 \end{pmatrix} \eta_{n+2} \begin{pmatrix} n+3 & -1 \\ 0 & r \end{pmatrix}.$$

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