Notre Dame Journal of Formal Logic
Volume VII, Number 1, January 1966

A CALCULUS OF ANTINOMIES

F. G. ASENJO

1. *Truth tables for antinomies.* Let us assume that atomic propositions have either one or two truth values. Propositions (atomic or compound) will then be either true, false, or true and false. Let us call those propositions which are both true and false "antinomies." Our immediate purpose is to extend the classical propositional calculus to include operations with antinomies. Therefore, we take the truth tables for the five classical propositional connectives and add the following rule for operations that involve antinomies. The truth value or values of $A\tau B$ (where τ stands for any binary connective) are the value or values that result from giving A and B all possible combinations of the truth values. If A and B have only one value, then the classical pattern follows. But if A or B, or both, are antinomies, then the value or values of $A\tau B$ depend on the values that are obtained when A and B assume all their different truth values in succession. For example, the following case arises for $A \supset B$: (a) A antinomic, B true, then $A \supset B$ shall be considered true, since $A \supset B$ is true whatever the truth value of A; (b) A false, B antinomic, then $A \supset B$ will be true for a similar reason. By indicating true, false, and antinomic with the symbols *0, 1, 2,* respectively, the entire situation can be described with these tables.

$A \supset B$

$A \backslash B$	0	1	2
0	0	1	2
1	0	0	0
2	0	2	2

$A \& B$

$A \backslash B$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$A \vee B$

$A \backslash B$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$A \sim B$

$A \backslash B$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

$\neg A$

A	$\neg A$
0	1
1	0
2	2

If one applies the tables in the order in which a compound proposition is generated, it is possible to assign one of the three truth values to any well-formed proposition. Three different cases arise, depending on the domain of atomic propositions to which the five operations are to be applied. The atomic propositions can be (1) all single-valued, (2) all antinomic, or (3) some single-valued and some antinomic. Case 1 is the

Received June 20, 1964

classical one with proper devices to avoid antinomies whenever necessary. Case 2 is the classical one without those devices and with at least one antinomy—the existence of a single antinomy converts any other formula in the domain into an antinomy. Case 3 requires restricted axiom systems, like the one mentioned in the next paragraph, for example. Assuming the last case, the following compound proposition has the indicated truth values. $A \supset (B \supset A)$ has the value 0 for $A = 2$ and $B = 1$, or $A = 1$ and $B = 2$, or $A = 0$ and $B = 2$. It has the value 2 for $A = 2$ and $B = 0$, or $A = 2$ and $B = 2$. The other postulates for the classical propositional calculus also remain true for some combinations of values that include antinomies. But for the other combinations of the values, the postulates themselves become antinomies. (Note: The five connectives can still be reduced to combinations of the appropriate extension of Sheffer's stroke.)

2. *An axiom system for antinomies.* We move now from semantics to syntax. What we require is a system of axioms for a nontrivially inconsistent propositional calculus. It should not be possible to prove any proposition from A and $\neg A$, which means that the rule of reduction to the absurd should not be deducible. Of course, the principle of contradiction should not be deducible either. Both objectives are obtained by eliminating Axiom 7, $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$, from Kleene's presentation of the classical propositional calculus. This is indicated by Da Costa in his note [4]. Here is Da Costa's axiom system (after Jaśkowski and Kleene) in its simplest form. (1) $A \supset (B \supset A)$. (2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C)$. (3) A, $A \supset B \vdash B$. (4) $A \,\&\, B \supset A$. (5) $A \,\&\, B \supset B$. (6) $A \supset (B \supset A \,\&\, B)$. (7) $A \supset A \vee B$. (8) $B \supset A \vee B$. (9) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$. (10) $A \vee \neg A$. (11) $\neg\neg A \supset A$. (Compare with Kleene [6].)

If we apply this axiom system to a domain of antinomic atomic propositions (cases 2 and 3 in paragraph 1), then every provable formula is at least true and at most antinomic (from rule of inference 3 above, only true or antinomic propositions can be obtained from antinomies). Therefore, we have a basis for a calculus of antinomies. This calculus is inconsistent without the reduction-to-the-absurd deduction rule, and it is incomplete as well. For completeness, of course, every true formula, antinomies included, must be provable. Theorems of the classical propositional calculus that cannot be proved without Kleene's Axiom 7 form the class of propositions in the calculus of antinomies which are true (or antinomic) and unprovable.

Finally, Russell's paradox can be produced by conveniently extending Da Costa's propositional calculus into a predicate calculus with axioms of membership. However, without reduction to the absurd, $T \varepsilon T \sim \neg(T \varepsilon T)$ does not yield $T \varepsilon T$ and $\neg(T \varepsilon T)$. But if $T \varepsilon T \sim \neg(T \varepsilon T)$ holds, then $T \varepsilon T$ has to be true and false, although not provable (the same is true for $\neg(T \varepsilon T)$).

REFERENCES

[1] Asenjo, F. G., "La idea de un cálculo de antinomies," *Seminario Matemático*, Universidad de La Plata, 1954.

[2] Carnap, R., *Introduction to Semantics*, Harvard University Press, 1942, p. 163.

[3] Curry, H. B., *Leçons de logique algébrique*, Gauthier-Villars, Paris, 1952, p. 98.

[4] Da Costa, N. C. A., "Calculs propositionnels pour les systems formels inconsistants," *Comptes Rendus des Séances de L'Académie des Sciences*, No. 25, Dec. 1963, pp. 3790-3792.

[5] Jaśkowski, S., "Rachunek zdań dla systemów dedukcyjnych sprzecznych" (Un calcul des propositions pour les systemes déductifs contradictoires). *Studia Societatis Torunensis*, Lectio A, Vol. 1, 1948, pp. 57-74. (Reviewed by A. Mostowski in *The Journal of Symbolic Logic*, Vol. 14, p. 66).

[6] Kleene, S. C., *Introduction to Metamathematics*, Van Nostrand, Princeton, 1952.

University of Pittsburgh
Pittsburgh, Pennsylvania