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A NEW CONDITION FOR A MODULAR LATTICE

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A lattice $\mathbf{L}$ is said to be modular if it satisfies the following axiom:

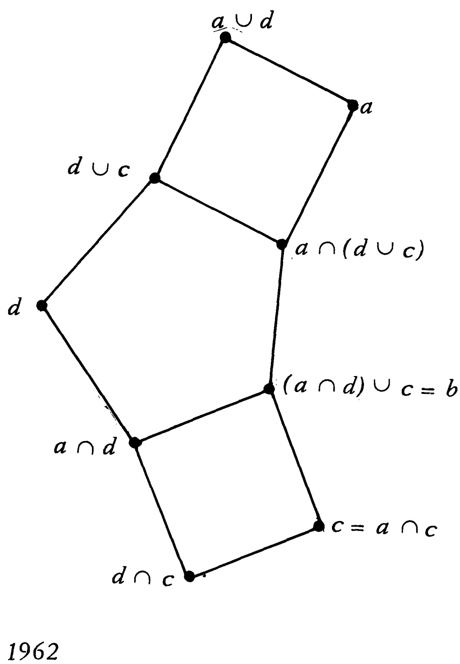
M. $[a, b, c]$: *If* $a, b, c \in \mathbf{L}$ *and* $a \geqslant c$, *then* $a \cap (b \cup c) = (a \cap b) \cup c$.

Several conditions equivalent to *M* are known. This paper introduces another characterization of a modular lattice which as far as I know has not been noted.

M'. $[a, b, c, d]$: *If* $a, b, c, d \in \mathbf{L}$, $a \cap c \leqslant b$, $a \cap d \leqslant b$, *and c is comparable to a, or c is comparable to d, then* $a \cap (c \cup d) \leqslant b$.

The expression "*a is comparable to b*" means: $a \leqslant b$ or $a > b$.

In the finite lattice shown below the elements are represented by dots and $x < y$ if x appears below y and is connected to y by a line segment. This lattice is known to be non-modular and we note that M' does not hold.



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Theorem. In any lattice condition M is satisfied, if and only if, M' is satisfied.

Proof: Assume that M' holds and that $a \geqslant c$. From the definition of l.u.b. we have

$$a \cap b \leqslant (a \cap b) \cup c. \tag{1}$$

Similarly, $c \leqslant (a \cap b) \cup c$; and from the definition of g.l.b, $a \cap c \leqslant c$. Therefore,

$$a \cap c \leqslant (a \cap b) \cup c. \tag{2}$$

Since $(a \cap b) \cup c \in \mathbf{L}$, and $a \geqslant c$, we may apply M' to (1) and (2) which gives

$$a \cap (b \cup c) \leqslant (a \cap b) \cup c. \tag{3}$$

In any lattice there is a one-sided modular law

$$a \cap (b \cup c) \geqslant (a \cap b) \cup c. \tag{4}$$

Then (3) and (4) give M.

Conversely, assume M, $a \cap c \leqslant b$, and $a \cap d \leqslant b$, and that either c is comparable to d, or c is comparable to a. Then, if:

(i) c is comparable to d, we have $c \cup d = d$ or $c \cup d = c$, and in either case $a \cap (c \cup d) \leqslant b$ is true.

And, if:

(ii) c is comparable to a, then if

(a) $a \leqslant b$, we note that $a \cap (c \cup d) \leqslant a$, so that $a \cap (c \cup d) \leqslant b$.

And if:

(b) $a \nleqslant b$, then $a \leqslant c$ implies that $a \cap c = a$. But $a \cap c \leqslant b$, so that this case cannot arise. Hence

$$a > c \tag{5}$$

holds. Then (5) implies

$$a \geqslant c \tag{6}$$

and

$$a \cap c = c. \tag{7}$$

Then, by (7) and our assumption, $a \cap c \leqslant b$, we have

$$c \leqslant b \tag{8}$$

and by M and (6)

$$a \cap (d \cup c) = (a \cap d) \cup c. \tag{9}$$

But $a \cap d \lessgtr b$ (assumption) and (8) imply

$$(a \cap d) \cup c \lessgtr b \tag{10}$$

and, therefore, by (9) and (10) we have

$$a \cap (d \cup c) \lessgtr b$$

i.e.

$$a \cap (c \cup d) \lessgtr b.$$

Hence, both subcases (a) and (b) of (ii) give the conclusion M'. Therefore, since this conclusion follows from (i) and from (ii) we have proved that condition M implies M'. Thus the proof of the theorem is complete.

It should be noted that M' is a disjunction of six theorems, instead of "*c is comparable to a, or c is comparable to d*" we could have taken separately each of the conditions: $c < a$, $c = a$, $c > a$, $c < d$, $c = d$, $c > d$. No one of these conditions, however, is strong enough to imply M, and no two of these conditions, except $c \lessgtr a$, imply M.

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