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A NOTE ON THE GENERALIZED CONTINUUM HYPOTHESIS. I

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It is well-known[1] that the following set-theoretical formulas:

A *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m} = \mathfrak{n}$, *then* $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$

B *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $2^{\mathfrak{m}} < 2^{\mathfrak{n}}$, *then* $\mathfrak{m} < \mathfrak{n}$

C *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m} < \mathfrak{n}$, *then* $2^{\mathfrak{m}} < 2^{\mathfrak{n}}$

D *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$, *then* $\mathfrak{m} = \mathfrak{n}$

are such that a) A is provable without any difficulty in the general set theory,[2] b) we do not know whether it is possible to prove B without the help of the axiom of choice, and that c) we are unable to prove C and D without the aid of the generalized continuum hypothesis, i.e. the formula:

$\mathfrak{A}$ *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there exists no cardinal* $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$

which, as we know,[3] is equivalent to

$\mathfrak{B}$ *The axiom of choice*

taken in conjunction with the formula

$\mathfrak{C}$ *for any ordinal number* α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

i.e. Cantor's hypothesis on alephs.

In this note I present several sets of assumptions such that

α) each of these sets is equivalent either to $\mathfrak{A}$ or to $\mathfrak{C}$.

β) almost each of these sets contains either formula C or formula D or a certain instance of one of these formulas.

In the considerations given below I am using constantly several of my results published in [6], and, especially, that formula $\mathfrak{C}$, i.e. Cantor's hypothesis on alephs, is equivalent to the conjunction of the following two formulas:

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$\mathbf{E}_1$ *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are alephs and* $\mathfrak{b} < 2^{\mathfrak{a}}$, *then* $\mathfrak{b} \leq \mathfrak{a}$

and

$\mathbf{E}_2$ *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are alephs and* $\mathfrak{b} < 2^{2^{\mathfrak{a}}}$, *the* $\mathfrak{b} \leq 2^{\mathfrak{a}}$

Concerning these formulas it should be noted that

γ) the deductions presented in [6] show clearly that $\{\mathfrak{B}:\mathbf{E}_1\}$ is equivalent to $\{\mathfrak{A}\}$.

δ) the problems whether $\{\mathfrak{B};\mathbf{E}_2\}$ implies $\{\mathfrak{A}\}$ and whether the formulas $\mathbf{E}_1$ and $\mathbf{E}_2$ are mutually independent in the field of the general set theory remain open.

§1

(i) *The set of formulas* $\mathfrak{B}$, $\mathbf{E}_2$ *and* D *is equivalent to* $\mathfrak{A}$. Obviously, it suffices to prove that $\{\mathfrak{B};\mathbf{E}_2;D\}$ implies $\{\mathfrak{A}\}$. In order to prove it, first of all we have to note that formula B follows from $\mathfrak{B}$. Then, assume that

(1) $\mathfrak{m}$ *is an arbitrary cardinal number which is not finite*

and suppose that the negation of the consequence of $\mathfrak{A}$ is true, i.e. that

(2) *there is a cardinal number* $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$

Hence, in virtue of *(1)*, *(2)* and the general set theory, we know that

(3) *either* $2^{\mathfrak{n}} = 2^{2^{\mathfrak{m}}}$ *or* $2^{\mathfrak{n}} < 2^{2^{\mathfrak{m}}}$

But, since, by *(1)* and *(2)*, $\mathfrak{n}$ and $2^{\mathfrak{m}}$ are cardinal numbers, the first case of *(3)*, viz. $2^{\mathfrak{n}} = 2^{2^{\mathfrak{m}}}$, together with D implies that $\mathfrak{n} = 2^{\mathfrak{m}}$ which contradicts *(2)*. Hence, the second case of *(3)*, viz.

(4) $2^{\mathfrak{n}} < 2^{2^{\mathfrak{m}}}$

holds. Now, due to *(1)*, *(2)* and $\mathfrak{B}$ we know that cardinals $\mathfrak{m}$, $\mathfrak{n}$ and $2^{\mathfrak{n}}$ are alephs. Hence, in virtue of $\mathbf{E}_2$, *(4)* yields that

(5) $2^{\mathfrak{n}} \leq 2^{\mathfrak{m}}$ *where* $\mathfrak{n}$ *and* $\mathfrak{m}$ *are alephs*

which, since formula B is at our disposal, by it and D, gives at once

(6) $\mathfrak{n} \leq \mathfrak{m}$

which is incompatible with our assumption *(2)*. Hence, the negation of (2), i.e.

(7) *there exists no cardinal* $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$

holds, and, therefore, we proved that formula $\mathfrak{A}$ follows from $\{\mathfrak{B};\mathbf{E}_2;D\}$. Hence, $\{\mathfrak{B};\mathbf{E}_2;D\}\rightleftarrows\{\mathfrak{A}\}$.

(ii) *The set of the formulas* $\mathfrak{B}$, $\mathbf{E}_2$ *and C is equivalent to* $\mathfrak{A}$. Obviously, due to deductions given above it suffices to prove that formula D follows from $\{\mathfrak{B};\mathbf{E}_2;C\}$. Since in the case where at least one of the cardinals $\mathfrak{m}$ and $\mathfrak{n}$ is finite, D holds banally, it remains to prove this formula in the case where both cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ are not finite. Hence, let us assume that

(8) $\mathfrak{m}$ *and* $\mathfrak{n}$ *are arbitrary cardinal numbers which are not finite*

and that

(9) $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$

Then, due to *(8)* and $\mathfrak{B}$ we know that $\mathfrak{m}$ and $\mathfrak{n}$ are alephs, and, therefore, we have

(10) *either* $\mathfrak{m} = \mathfrak{n}$ *or* $\mathfrak{m} < \mathfrak{n}$ *or* $\mathfrak{n} < \mathfrak{m}$

Since the second and the third cases of *(10)*, viz. $\mathfrak{m} < \mathfrak{n}$ and $\mathfrak{n} < \mathfrak{m}$, together with *(8)* and formula C imply $2^{\mathfrak{m}} < 2^{\mathfrak{n}}$ and $2^{\mathfrak{n}} < 2^{\mathfrak{m}}$ respectively, these cases contradict *(9)*. Hence, the first case of *(10)*, i.e.

(11) $\mathfrak{m} = \mathfrak{n}$

holds. Thus, formula D follows from $\mathfrak{B}$ and C alone, and, therefore, we proved that $\{\mathfrak{B};\mathbf{E}_2;C\}\rightleftarrows\{\mathfrak{A}\}$.

(iii) It is obvious that the following formula

F *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there exists no cardinal* $\mathfrak{n}$ *such that* $\mathfrak{m} < 2^{\mathfrak{n}} < 2^{\mathfrak{m}}$

is a particular instance of $\mathfrak{A}$. I shall show here that each of the sets of the formulas $\{F;D\}$ and $\{F;C\}$ is equivalent to $\mathfrak{A}$. Naturally, it suffices to show only that $\mathfrak{A}$ follows from each of these sets of assumptions.

(α) *The formulas F and D imply* $\mathfrak{A}$. Let us assume that

(12) $\mathfrak{m}$ is an arbitrary cardinal number which is not finite

and suppose that the negation of the consequence of $\mathfrak{A}$ is true, i.e. that

(13) *these exists a cardinal number* $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$

Hence, in virtue of *(12)*, *(13)* and the general set theory, the formula

(14) *either* $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$ *or* $2^{\mathfrak{m}} < 2^{\mathfrak{n}}$

holds. Since the first case of *(14)*, viz. $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$, together with *(12)*, *(13)* and D implies that $\mathfrak{m} = \mathfrak{n}$ which contradicts *(13)*, the second case of *(14)*, viz.

(15) $2^{\mathfrak{m}} < 2^{\mathfrak{n}}$

must be accepted which, by *(13)*, gives at once

(16) $\mathfrak{n} < 2^{\mathfrak{m}} < 2^{\mathfrak{n}}$

Moreover, since, by *(13)*, $\mathfrak{m} < \mathfrak{n}$, and since, by *(12)*, $\mathfrak{m}$ is not finite, we can also establish that

(17) $\mathfrak{n}$ *is a cardinal number which is not finite*

Since formula F is incompatible with the conjunction of *(16)* and *(17)*, the negation of *(13)* must be true which means that $\mathfrak{A}$ follows from F and D. Thus, we proved that $\{F;D\} \rightleftarrows \{\mathfrak{A}\}$.

($\mathfrak{b}$) *The formulas F and C imply* $\mathfrak{A}$. Let us assume the same conditions which we adopted in ($\mathfrak{a}$), i.e. the points *(12)* and *(13)*. Then, using the same reasonings as in (*a*) we obtain *(17)* from *(12)* and *(13)*. And, in virtue of C and *(12)*, *(13)* implies directly *(15)* and, therefore, also *(16)*. Since we have F, *(17)* and *(16)* show that our assumption *(13)* is false. Hence, F and C imply $\mathfrak{A}$, and, therefore, we know that $\{F;D\} \rightleftarrows \{\mathfrak{A}\}$.

It should be noted that

ϵ) In each set of assumptions $\{\mathfrak{B};\mathbf{E}_2;D\}$ and $\{\mathfrak{B};\mathbf{E}_2;C\}$ the formulas D and C can by substituted by their respective instances, namely, D by

D1 *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are alephs and* $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$, *then* $\mathfrak{a} = \mathfrak{b}$

and C by

C1 *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are alephs and* $\mathfrak{a} < \mathfrak{b}$, *then* $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$

and that

ζ) it is not known whether the formulas belonging to each of the set of assumptions discussed in this paragraph, viz. $\{\mathfrak{B};\mathbf{E}_2;D1\}$ (or $\{\mathfrak{B};\mathbf{E}_2;D\}$), $\{\mathfrak{B};\mathbf{E}_2;C1\}$ (or $\{\mathfrak{B};\mathbf{E}_2;C\}$), $\{F;D\}$ and $\{F;C\}$, are mutually independent.

NOTES

1. Concerning the connection of formula B with the axiom of choice, *cf.*, e.g., [3], p. 414 (where there appears an obvious misprint, viz. instead of "without" must be "with") and [4], p. 168, and concerning the proof of C and D, *cf.*, [3], p. 439 and [4], p. 167.

2. In this note general set theory is understood as the set theory from which the axiom of choice and all its consequences otherwise unprovable have been removed. It is well-known that if we base a so defined set

theory on an axiomatic system in which the notions of cardinal and ordinal numbers cannot be defined, we have to introduce these concepts into the system by means of special axioms.

The symbols $\rightleftarrows$ and $\rightarrow$ used below mean "is inferentially equivalent to" and "inferentially implies" respectively.

3. This was announced without proof by Lindenbaum and Tarski, *cf.* [2], pp. 313-314, theorem **95**. Sierpiński published a proof in [3], pp. 434-437. *Cf.* also his [5] and [4], pp. 166-167 and pp. 193-197. *Cf.* also [1], pp. 245-247. In [2], [4] and [5] the condition of $\mathfrak{A}$ is stronger than given here, viz. $\mathfrak{m}$ is assumed to be a transitive cardinal, i.e. $\mathfrak{m} \geq \aleph_0$. Such a strong assumption is superfluous, *cf.* [3], pp. 434-437.

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To be continued

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