

CORRELATIVE REMARKS CONCERNING
ELEMENTARY NUMBER THEORY, GROUPS AND MUTANT SETS

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0. Introduction. The philosophical works coming from the Pythagoreans and from the logical investigations of Aristotle, to cite some ancient philosophers, emphasize hosts of dichotomies used by man's reasoning faculties, e.g., odd and even, finite and infinite and male and female. Kant, Hegel and C. S. Peirce, among others, sought to develop parts of their philosophies around triads compounded of dichotomies carried forward to a synthesis of sorts.

With a similar spirit in mind this brief note gives a general theorem dealing with closed and anti-closed (see references [1], [2]) subsets of general algebraic systems. In particular the theorem shows a logical relation between naive (i.e., non-axiomatic) algebraic number theory and such mutant (i.e., anti-closed) sets of general algebraic systems.

1. Proposition. A (λ, T) -mutant of a general algebraic system $(A, *)$ is a subset M of A which satisfies the condition that $\{a_1 * a_2 * \dots * a_\lambda : a_1 \in M, a_2 \in M, \dots, a_\lambda \in M\} \subseteq T \cap \bar{M}$, where λ is an integer greater than or equal to 2, T together with $*$ forms a general algebraic subsystem of $(A, *)$ and \bar{M} is the set-theoretic complement of M . For a fixed λ and T , a (λ, T) -mutant of $(A, \bar{*})$ is said to be a maximal (λ, T) -mutant of $(A, *)$ provided there is no (λ, T) -mutant of $(A, *)$ properly containing it.

Theorem: Put $\mathbf{G}(\lambda) = \{2^\lambda m : m = 0, \pm 1, \pm 2, \dots\}$, where λ is a non-negative integer. Consider the additive groups $(\mathbf{G}(\lambda), +)$ of all 2^λ multiples of the set of all integers. Then for every non-negative integer λ there exist a maximal $(2, \mathbf{G}(\lambda + 1))$ -mutant M_λ of $(\mathbf{G}(\lambda), +)$ and a maximal $(2, \mathbf{G}(\lambda + 2))$ -mutant M'_λ of $(\mathbf{G}(\lambda), +)$.

Proof: Every general algebraic system has a mutant and a maximal mutant [2]. It is trivial to establish the existence of M_λ . Thus, recall that (i) the property of being a maximal mutant is preserved under isomorphism [3], that (ii) the set of all odd integers is a maximal $(2, \mathbf{G}(1))$ -mutant of $(\mathbf{G}(0), +)$ and that (iii) the $(\mathbf{G}(\lambda), +)$ are isomorphic. It is somewhat less trivial to establish the existence of M'_λ . Once again make use of (i) and

(iii) given earlier in the proof. In place of (ii) substitute the observation that $\{2(1 + 2n) : n = 0, \pm 1, \pm 2, \dots\}$ is a maximal $(2, \mathbf{G}(2))$ -mutant of $(\mathbf{G}(0), +)$, i.e., $\{2(1 + 2n) : n = 0, \pm 1, \pm 2, \dots\}$ is a maximal mutant of the additive group of all integers; that result is so just as the set of all odd integers is a maximal mutant of the additive group of all integers, too.

2. Observations. It can be shown that all of the maximal mutants of any infinite cyclic group have the same cardinality, \aleph_0 , [3]. Note that if a maximal $(2, \mathbf{G}(\lambda))$ -mutant is composed with itself it yields a closed system, e.g., odds + odds = evens. In addition, it can be shown [4], in a constructive sense, that the set of all natural numbers, i.e., $\{1, 2, 3, \dots\}$ can be represented as the union of a countable collection of disjoint maximal mutants.

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