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AN EQUATIONAL AXIOMATIZATION OF ASSOCIATIVE NEWMAN ALGEBRAS

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An associative Newman algebra is a Newman algebra[1] in which the binary multiplicative operation $\times$ is associative for all elements belonging to the carrier set of the considered system. In [2], p. 265 and p. 271, Theorem 5 and Example E10, Newman has established that such an algebraic system is a proper extension of his complemented mixed algebra,[2] and that it is a direct join of an associative Boolean ring with unity element and a Boolean lattice (i.e. a Boolean algebra). Moreover, he has shown there that this system can be constructed by an addition of a rather weak formula, viz. *K1* given in section 1 below, as a new postulate, to the axiom-system formulated in [2] of Newman algebra. On the other hand, it is almost self-evident that an associative Newman algebra is not necessarily a Boolean algebra.

In this note it will be shown that the addition of formula *K1* mentioned above, as a new postulate, to the set of axioms of system $\mathfrak{B}$ discussed in [3] allows us to construct a very simple and compact equational axiom-system for associative Newman algebra.

1 We define a system under consideration as follows:

Any algebraic system

$$\mathfrak{D} = \langle B, =, +, \times, - \rangle$$

with *one binary relation* $=$, *two binary operations* $+$ *and* $\times$, *and one unary operation* $-$, *is an associative Newman algebra, if it satisfies the postulates*

1. An acquaintance with the the papers [2] and [3] is presupposed. An enumeration of the formulas used in this note is a continuation of the enumeration which is given in [3]. As in that paper, the properties of "even" and "odd" elements will be not discussed in this note, and the axioms *A1–A11* given below will be used mostly tacitly in the deductions.

2. I.e., of Newman algebra, *cf.* [3].

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A1-A11, C1, C2, F1, F2 and F3 of System $\mathfrak{B}$ (defined in [3], section **1**) *of Newman algebra, and, additionally, an axiom*

$K1$ $[ab] . a, b \epsilon A . a + a = a \times \bar{a} . b + b = b \times \bar{b} . \supset . a \times (b \times b) = (a \times b) \times b$

Concerning the form of *K1*, *cf.* [2], p. 285, Theorem 5, and *D2* given in section **2.2** of [3]. The following algebraic table

$+$	0	η
0	0	η
η	η	0

$\times$	0	η
0	0	0
η	0	η

x	$\bar{x}$
0	η
η	0

which is constructed by Stone, *cf.* [4], p. 730, example $*\mathrm{P}6_1$, and [2], p. 268, and which is adjusted here to the primitive unary operation of complementation of system $\mathfrak{D}$ shows that this system is not necessarily a Boolean algebra. Namely this example satisfies all postulates of $\mathfrak{D}$, but falsifies

$$[a] : a \epsilon B . \supset . a = a + a$$

for a/η: (i) $\eta = \eta$, and (ii) $\eta + \eta = 0$.

2 Let us assume the axioms of $\mathfrak{D}$. Since, clearly, system $\mathfrak{B}$ is a subsystem of $\mathfrak{D}$, we have at our disposal all formulas which are proved in sections **2.2** and **3.1** of [3]. Moreover, since it has been established, *cf.* [3], section **2.3**, that system $\mathfrak{B}$ is inferentially equivalent or inferentially equivalent up to isomorphism to the original formalization of Newman algebra, we know that any formula which is proved in [2] is also provable analogously in the field of $\mathfrak{B}$. Hence, we can add the following formulas

$F34$ $[abc] : a, b, c \epsilon B . \supset . a + (b + c) = (a + b) + c$ [*Cf.* **P18** in [2], p. 260]

$F35$ $[ab] : a, b \epsilon B . a + a = a . \supset . (a \times b) + (a \times b) = a \times b$ [*Cf.* **P19** in [2], p. 261]

$F36$ $[ab] : a, b \epsilon B . a + a = 0 . \supset . (a \times b) + (a \times b) = 0$ [*Cf.* **P19** in [2], p. 261]

$F37$ $[abc] : a, b, c \epsilon B . a + a = a . b + b = b . c + c = c . \supset . a \times (b \times c) = (a \times b) \times c$ [*Cf.* **P32** in [2], p. 263]

to the set of formulas which are already proven in sections **2.2** and **3.1** of [3].

Moreover, we have

$H1$ $[abc] : a, b, c \epsilon B . \supset . a \times (b + c) = (c \times a) + (b \times a)$[3] [*C1*; *F26*; *F33*]

Then[4]:

$K2$ $[ab] : a, b \epsilon A . a + a = 0 . b + b = 0 . \supset . a \times b = (a \times b) \times b$ [*K1*; *F7*; *D2*]

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3. Formula *H1* is accepted by Croisot, *cf.* [1], p. 27, as an axiom in his axiomatization of distributive lattice, with the constant element *I*.

4. The deductions presented below are also due to Newman, *cf.* [3], p. 265, Theorem 5, but they are given in a very compact way, or even verbally. In order to make this note more clear it was necessary to present these deductions in a formal way.

It is clear that in the field of Newman algebra regardless of its formalization *K1* is inferentially equivalent to *K2*.

K3 $[abc]: a, b, c \epsilon B . a + a = 0 . b + b = 0 . c + c = 0 . \supset . a \times (b \times c)$
$= (a \times b) \times c$

PR $[abc]: \mathrm{Hp}(4) . \supset .$

5. $((b \times a) + (b \times c)) + ((b \times a) + (b \times c)) = (b + b) \times (a + c)$ [1; *C1*; *C2*]
$= 0 \times (a + c) = 0$ [3; *F15*]

6. $(a + c) + (a + c) = (a + a) + (c + c) = 0 + 0 = 0$ [1; *F26*; *F34*; 2; 4; *F12*]

7. $(a \times (b \times c)) + (a \times (b \times c)) = (a + a) \times (b \times c) = 0 \times (b \times c) = 0$
[1; *C2*; 2; *F15*]

8. $((b \times a) + (b \times c) = b \times (a + c) = (b \times (a + c)) \times (a + c)$ [1; *C1*; *K2*; 3; 6]
$= (((b \times a) \times a) + ((b \times c) \times a)) + (((b \times a) \times c)$
$+ ((b \times c) \times c))$ [*C1*; *C2*]
$= ((b \times a) + (a \times (b \times c))) + (((a \times b) \times c) + (b \times c))$
[4; *K2*; *F33*]
$= ((b \times a) + (b \times c)) + ((a \times (b \times c)) + ((a \times b) \times c))$
[*F26*; *F34*]

9. $0 = ((b \times a) + (b \times c)) + ((b \times a) + (b \times c))$ [5]
$= ((b \times a) + (b \times c)) + (((b \times a) + (b \times c)) + ((a \times (b \times c)) + ((a \times b) \times c)))$
[8]
$= 0 + ((a \times (b \times c)) + ((a \times b) \times c))$ [*F26*; *F34*; 5]
$= (a \times (b \times c)) + ((a \times b) \times c)$ [*F17*]

$a \times (b \times c) = (a \times (b \times c)) + 0 = (a \times (b \times c)) + ((a \times (b \times c))$
$+ ((a \times b) \times c))$ [*F12*; 9]
$= (a \times b) \times c$ [*F26*; *F34*; 7; *F17*]

L1 $[abc]: a, b, c \epsilon B . \supset . a \times (b \times c) = (a \times b) \times c$

PR $[abc]: \mathrm{Hp}(1) . \supset .$

$[\exists defgmn] .$

$$\left.\begin{array}{rl}
2. & d, e, f, g, m, n \epsilon B . \\
3. & d + d = d . \\
4. & e + e = 0 . \\
5. & a = d + e . \\
6. & f + f = f . \\
7. & g + g = 0 . \\
8. & b = f + g . \\
9. & m + m = m . \\
10. & n + n = 0 . \\
11. & c = m + n .
\end{array}\right\} \quad [1; F31]$$

12. $(d \times f) + (d \times f) = d \times f .$ [2; *F35*; 3; 6]
13. $(e \times g) + (e \times g) = 0 .$ [2; *F36*; 4; 7]
14. $(f \times m) + (f \times m) = f \times m .$ [2; *F35*; 6; 9]
15. $(g \times n) + (g \times n) = 0 .$ [2; *F36*; 7; 10]
16. $a \times (b \times c) = a \times ((f + g) \times (m + n))$ [1; 2; 8; 11]
$= (d + e) \times ((f \times m) + (g \times n))$ [5; *F32*; 6; 9; 7; 10]
$= (d \times (f \times m)) + (e \times (g \times n))$ [*F32*; 3; 14; 4; 15]
$= ((d \times f) \times m) + ((e \times g) \times n)$
[*F37*; 3; 6; 9; *K3*; 4; 7; 10]

$= ((d \times f) + (e \times g)) \times (m + n)$ [$F32$; 12; 9; 13; 10]
$= ((d + e) \times (f + g)) \times c$ [$F32$; 3; 6; 4; 7; 11]
$= (a \times b) \times c$ [5; 8]
$a \times (b \times c) = (a \times b) \times c$ [16]

Hence, it is shown that the formulas $H1$ and $L1$ are provable in the field of system $\mathfrak{D}$.

3 Now, let us assume, as the axioms, $A1$-$A11$, $F1$, $F2$, $H1$ and $L1$. Then:

$F3$ $[ab]:a,b \epsilon B\,.\supset.\,a = (b + \bar{b}) \times a$
PR $[ab]:\mathrm{Hp}(1)\,.\supset.$
$a = a \times (b + \bar{b}) = (\bar{b} \times a) + (b \times a) = ((\bar{b} \times (b + \bar{b})) \times a) + ((\bar{b} \times (b + \bar{b})) \times a)$ [1; $F2$; $H1$; $F2$]
$= (\bar{b} \times ((b + \bar{b}) \times a)) + (\bar{b} \times ((b + \bar{b}) \times a))$ [$A10$; $L1$]
$= ((b + \bar{b}) \times a) \times (b + \bar{b}) = (b + \bar{b}) \times a$ [$H1$; $F2$]
$F26$ $[ab]:a,b \epsilon B\,.\supset.\,a + b = b + a$ [$H1$; $F2$; $F3$]
$F33$ $[ab]:a,b \epsilon B\,.\supset.\,a \times b = b \times a$
PR $[ab]:\mathrm{Hp}(1)\,.\supset.$
$a \times b = (a \times b) \times (b + \bar{b}) = (\bar{b} \times (a \times b)) + (b \times (a \times b))$ [1; $F2$; $H1$]
$= ((\bar{b} \times a) \times b) + ((b \times a) \times b) = b \times ((\bar{b} \times a) + (b \times a))$ [$L1$; $H1$]
$= b \times (a \times (b + \bar{b})) = b \times a$ [$H1$; $F2$]
$C1$ $[abc]:a,b,c \epsilon B\,.\supset.\,a \times (b + c) = (a \times b) + (a \times c)$ [$H1$; $F26$; $F33$]
$C2$ $[abc]:a,b,c \epsilon B\,.\supset.\,(a + b) \times c = (a \times c) + (b \times c)$ [$C1$; $F33$]

Thus, in the field of the remaining axioms $C1$, $C2$ and $F3$ follow from $F1$, $F2$, $H1$ and $L1$.

4 The proofs given in the sections **2** and **3** above show clearly that in the axiom-system of $\mathfrak{D}$ the formulas $H1$ and $L1$ can be accepted, as the postulates, instead of $C1$, $C2$, $F3$ and $K1$. In [2], p. 271, Example 10, it is proved that $K1$ (and, therefore, $L1$) is not the consequence of $C1$, $C2$, $F1$, $F2$ and $F3$. Matrices $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$, $\mathfrak{M}5$ and $\mathfrak{M}6$, *cf.* section **4** in [3], each of which verifies $K1$ and $L1$ show that the formulas $C1$, $C2$, $F1$, $F2$ and $F3$ are mutually independent. Since $\mathfrak{M}3$ verifies $F2$ and $H1$, but falsifies $F1$, $\mathfrak{M}5$ verifies $F1$ and $H1$, but falsifies $F2$, and $\mathfrak{M}1$ verifies $F1$ and $F2$, but falsifies $H1$ for a/β, $b/0$, c/γ: (i) $\beta \times (0 + \gamma) = \beta \times \gamma = \gamma$ and (ii) $(\gamma \times \beta) + (0 \times \beta) = \beta + 0 = \beta$, we know that the formulas $F1$, $F2$ and $H1$ are also mutually independent.

Thus, it is established that system $\mathfrak{D}$ of an associative Newman algebra can be based either on the set of mutually independent postulates $\{C1; C2; F1; F2; F3; K1\}$ or on the set of mutually independent postulates $\{F1; F2; H1; L1\}$.

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