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SYNTACTICALLY FREE, SEMANTICALLY BOUND (A NOTE ON VARIABLES)

HUGUES LEBLANC

> *Apparent variables*. The symbol "$(x).\phi x$" denotes one definite proposition, and there is no distinction in meaning between "$(x).\phi x$" and "$(y).\phi y$" when they occur in the same context.
> *Principia Mathematica*, Introduction, Ch. I.

The old distinction between an *apparent variable* and a *real* one was never too clearly drawn. Passages from *Principia Mathematica* and earlier logic treatises suggest, though, that an individual variable X *apparently* occurs in a formula A if X occurs in A just for form, i.e., if X can be replaced *salvo sensu* in A by some individual variable foreign to A, and that X *really* occurs in A if X does not *apparently* occur in A. Thus, 'x' *apparently* occurs in Russell's '$(\forall x) f(x)$' (= '$(x).\phi x$'), since 'x' can be replaced *salvo sensu* in '$(\forall x) f(x)$' by 'y', whereas 'x' *really* occurs in '$f(x)$'. The distinction between a bound variable and a free one, which eventually displaced that between an apparent variable and a real one, does not match it, all assertions to the contrary notwithstanding. An individual variable may—by current standards—occur free in a given formula, and yet not *really* occur therein by the above critierion. 'x', for example, though it occurs free in '$(\forall x) f(x)$ & $f(x)$', does not *really* occur in '$(\forall x) f(x)$ & $f(x)$', since it can be replaced *salvo sensu* in '$(\forall x) f(x)$ & $f(x)$' by any one of 'y', 'z', 'x'', 'y'', 'z'', and so on, or—as we prefer to put it—since '$(\forall x) f(x)$ & $f(x)$' is semantically equivalent to any one of '$(\forall y) f(y)$ & $f(y)$', '$(\forall z) f(z)$ & $f(z)$', and so on.[1]

Because of this discrepancy we would urge that an individual variable X, when it occurs bound (free) in a formula A by current standards, be said to occur *syntactically bound* (*syntactically free*) in A, and that a fresh distinction be introduced according to which: (i) X is said to occur *semantically bound* in A if A is semantically equivalent to any (hence, to

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every) formula that is like A except for exhibiting an individual variable foreign to A wherever A exhibits X, and (ii) X is said to occur *semantically free* in A if X does not occur semantically bound in A. It is readily verified that under this understanding of things any individual variable that does not occur syntactically free in a formula A, does not occur semantically free either in A. On many occasions, though, an individual variable that occurs syntactically free in a formula A, occurs semantically bound rather than semantically free in A, and those are possibly occasions when according to *Principia Mathematica* and its forerunners the variable in question only *apparently* occurs in A.

The distinction we urge, besides being of historical interest, may also be of theoretical value.[2] A case in point is the following. In his paper "On proper quantifiers" Ludwik Borkowski takes a first-order quantifier, say, (Q_iX), to permit definition of another such quantifier, say, (Q_jX) if—'f' being a monadic predicate variable—there exists a formula A of QC_i, a first-order quantificational calculus without identity having 'Q_i' as its one primitive quantifier symbol, such that: (1) A is semantically equivalent to $(Q_jX)\,f(X)$, and (2) no individual variable of QC_i occurs free in A.[3] It is readily shown that if condition (2) is amended to read: no individual variable of QC_i occurs semantically free in A, then (2) is satisfied if (1) is, and hence can be dispensed with. For suppose a given individual variable Y occurs syntactically free in A, and Z is any individual variable of QC_i that is foreign to A. Then $(Q_jX)\,f(X) \equiv A(Z/Y)$, where $A(Z/Y)$ is the result of replacing every occurrence of Y in A by an occurrence of Z, is sure to be valid if $(Q_jX)\,f(X) \equiv A$ is valid. Hence, if there exists a formula A of QC_i that meets condition (1) only, that very formula A meets our amendment of condition (2) as well. Borkowski's criterion thereby becomes more palatable: so long indeed as $(Q_jX)\,f(X)$ and A can be interchanged at will, A should clearly count as a definiens for $(Q_jX)\,f(X)$ (and, more generally, the formula B', where B' is the result of replacing every occurrence of $f(X)$ in A by one of B, count as a definiens for $(Q_jX)B$). And, interestingly enough, the quantifier (Q_5X), to be interpreted: For all X or for none, now permits definition of the familiar universal quantifier. There exists indeed a formula A of QC_5, a first-order quantificational calculus without identity having 'Q_5' as its one primitive quantifier symbol, such that A is semantically equivalent to $(\forall X)\,f(X)$, to wit: $(Q_5X)\,f(X)\ \&\ f(X)$. Hence $(Q_1X)B$ can be defined in QC_5 by means of $(Q_5X)B\ \&\ B$.[4]

The definition, by the way, cannot be improved upon, a matter well worth looking into.

Let A be a formula in which no individual variable occurs semantically free, but in which one individual variable, say, X, occurs syntactically free. So long as the first-order quantificational calculus without identity of which A is a formula, has one of the four quantifier symbols 'Q_1' (= 'For all'), 'Q_2' (= 'For none'), 'Q_3' (= 'For some'), and 'Q_4' (= 'Not for all') as a primitive quantifier symbol, then there does exist a formula of the same calculus that is semantically equivalent to A and in which no individual variable occurs syntactically free. For, by definition, $A \equiv A(Y/X)$ is valid, where Y is any individual variable foreign to A; hence so is $(Q_1Y)\ (A \equiv A(Y/X))$; hence so is $(Q_1Y)A \equiv (Q_1Y)A(Y/X)$; hence so is $A \equiv (Q_1Y)A(Y/X)$; and hence so is $A \equiv (Q_1X)A$. Hence $(Q_1X)A$ is semantically equivalent to A. Hence, clearly, so are $(Q_2X) \sim A$, $\sim(Q_3X) \sim A$, and $\sim(Q_4X)A$.

The trick fails, however, when the first-order quantificational calculus without identity of which A is a formula, has the above 'Q_5' or any one of the remaining three first-order quantifier symbols 'Q_6' (= 'For some only'), 'Q_7' (= 'For all or for some only'), and 'Q_8' (= 'For all and for some only') as a primitive quantifier symbol. None of $(Q_5X)A$, $(Q_6X) \sim A$, $\sim(Q_7X) \sim A$, and $\sim(Q_8X)A$ is semantically equivalent to A. Nor will any substitute trick do, where this one fails. Examination of all relevant possibilities quickly reveals that no formula of QC_5, for example, in which no individual variable occurs syntactically free is semantically equivalent to $(Q_5X)\ f(X)\ \&\ f(X)$. Hence there is no hope of our defining $(Q_1X)B$ in QC_5 by means of a formula in which no individual variable occurs syntactically free.

As the reader will have noticed, we have talked so far of semantical bondage and freedom in connection with individual variables only, not in connection with occurrences of individual variables; and our treatment of things has been such that no individual variable can occur both semantically bound and semantically free in the same formula. One might perhaps favor a different account, under which: (i) each occurrence of an individual variable is itself declared either semantically bound or semantically free, and (ii) an individual variable X that occurs in a formula A is said to occur semantically bound in A if at least one occurrence of X in A is semantically bound, and to occur semantically free in A if at least one occurrence of X in A is semantically free. Examples readily come to mind which argue for such an approach. As things now stand, 'x' occurs (only) semantically free in '$(\forall x)\ f(x)\ \&\ g(x)$', in spite of the fact that it occurs (only) semantically bound in the first component '$(\forall x)\ f(x)$' of '$(\forall x) f(x)\ \&\ g(x)$'.

We accordingly submit the following alternative:

(1) Declare semantically bound in a formula A every occurrence of X in A that is syntactically bound;

(2) Declare semantically bound in A every syntactically free occurrence of X in A if A is semantically equivalent to a formula that is like A except for exhibiting occurrences of an individual variable foreign to A wherever A exhibits free occurrences of X; otherwise, declare semantically free in A every syntactically free occurrence of X in A;

and

(3) Declare (as already suggested) X semantically bound in A if at least one occurrence of X in A is semantically bound, semantically free in A if at least one occurrence of X in A is semantically free.

Under this revised understanding of things, any individual variable that in the parlance of *Principia Mathematica* and like-minded classics does not really occur in a formula A, will occur semantically bound, and semantically bound only, in A (if the variable occurs at all in A), and the purpose of the old distinction between apparent and real variables will still be served.[5]

NOTES

1. We take two formulas A and B to be semantically equivalent if the biconditional $A \equiv B$ is valid.

2. It is not meant to supersede the distinction between a syntactically bound variable and a syntactically free one, which is needed to specify which formulas of the first-order quantificational calculus without identity count as axioms of the calculus, (frequently, though not always) which ones follow from which, and so on. Whether our distinction is effective seems to be an open question.

3. Borkowski further requires that no sentence variable and no predicate variable other than 'f' occur in A, see "On proper quantifiers I," *Studia Logica*, vol. 8 (1958), pp. 65-130. The condition can be dispensed with, however, as R. H. Thomason and I showed in "All or none: A novel choice of primitives for elementary logic," *The Journal of Symbolic Logic*, vol. 22 (1967), pp. 345-351. Exactly eight quantifiers, the 'Q_1' - 'Q_8' discussed further in the text, are definable in a first-order quantificational calculus without identity that has '$\forall$' (here 'Q_1') as a primitive.

4. Axioms and rules for QC_5 are supplied in "All or none: A novel choice of primitives for elementary logic."

5. My thanks go to R. H. Thomason, who read an earlier draft of the present note and contributed much to it.

Temple University
Philadelphia, Pennsylvania