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INCOMPLETE MODELS

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Given a relational structure $\mathfrak{A}$ and an atomic sentence ψ in $\mathcal{L}(A)$,

(*) $$\text{either } \mathfrak{A} \models \psi \text{ or } \mathfrak{A} \models \sim\psi.$$

In this paper we consider structures for which (*) does not hold. We may think of these structures as relational structures about which we do not have complete information. It would be possible to set up a 3-valued logic such that if $||\psi||_{\mathfrak{A}} \neq 1$ (i.e., not $\mathfrak{A} \models \psi$) and $||\psi||_{\mathfrak{A}} \neq 0$ (i.e., not $\mathfrak{A} \models \sim\psi$), then $||\psi||_{\mathfrak{A}} = \frac{1}{2}$. Truth-values could then be defined for all the sentences of $\mathcal{L}(A)$ if the truth-tables are given for the connectives and quantifiers. A general situation, where the set of truth-values is a compact Hausdorff space, was investigated in [2].

We also set up a 3-valued logic such that for every sentence of $\mathcal{L}(A)$ either $||\psi||_{\mathfrak{A}} = 1$ or $||\psi||_{\mathfrak{A}} = \frac{1}{2}$ or $||\psi||_{\mathfrak{A}} = 0$. Since we do not have complete information about our structures we say that $\mathfrak{A} \models \psi$ only if ψ holds in every relational structure that we obtain from $\mathfrak{A}$ by arbitrarily assigning the truth-values 0 or 1 to the atomic sentences of $\mathcal{L}(A)$ whose actual truth-value is $\frac{1}{2}$. In this case some of the connectives are no longer truth-functional. For example, in any structure $\mathfrak{A}$ if $||\psi||_{\mathfrak{A}} = \frac{1}{2}$ then $||\sim\psi||_{\mathfrak{A}} = \frac{1}{2}$, $||\psi \vee \psi||_{\mathfrak{A}} = \frac{1}{2}$ but $||\psi \vee \sim\psi||_{\mathfrak{A}} = 1$.

1 Notations and Terminology If $\mu = \langle n_\theta \rangle_{\theta<\xi}$ with at least one $n_\theta > 0$, then μ is a type. If μ_1 and μ_2 are types, $\mu_1 = \langle n_\theta \rangle_{\theta<\xi_1}$, $\mu_2 = \langle n'_\theta \rangle_{\theta<\xi_2}$, then $\mu_1 * \mu_2 = \langle n''_\theta \rangle_{\theta<\xi_1+\xi_2}$ where $n''_\theta = n_\theta$ for $\theta < \xi_1$ and $n''_{\xi_1+\theta} = n'_\theta$. The appropriate language for μ is $\mathcal{L}_\mu$ and what we call a formula of μ is a formula of $\mathcal{L}_\mu$; ψ, χ stand for formulas. We also use α, β, γ, δ, κ for cardinals, and m, n, r for natural numbers. Subscripts and superscripts may be omitted or inserted for clarity of notation.

A relational structure of type μ is $\mathfrak{A} = \langle A, R_0^{\mathfrak{A}}, \ldots, R_\theta^{\mathfrak{A}}, \ldots \rangle_{\theta<\xi}$ where each $R_\theta^{\mathfrak{A}}$ is an n_θ-ary relation on A, i.e., if $n_\theta = 0$ then $R_\theta = c_\theta$ where $c_\theta^{\mathfrak{A}} = a_\theta$, an element of A; and if $n_\theta > 0$ then

(*) $$\left\{\begin{array}{l}\text{for every } n_\theta\text{-tuple, } \langle a_1, \ldots, a_{n_\theta}\rangle \text{, of } A \text{, either} \\ \mathfrak{A} \models R_\theta(a_1, \ldots, a_{n_\theta}) \text{ or } \mathfrak{A} \models \sim R_\theta(a_1, \ldots, a_{n_\theta})\end{array}\right\}.$$

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If (*) does not hold for R_θ we call it an incomplete relation (on A). Then if at least one R_θ is incomplete we call $\mathfrak{A}$ an incomplete structure. Note that if R_θ is an incomplete relation it has all the other properties of a relation except for (*).

Although we rely mostly on [1] for our notations and terminology, we use the same notation for various notions concerning relational structures and incomplete structures. Unless otherwise specified $\mathfrak{A}$ and $\mathfrak{B}$ are infinite incomplete structures of type μ and Σ is a theory (set of sentences) of μ. Elements of the domain of $\mathfrak{A}$, A, are denoted by a and we assume that A can be well ordered as $a = \langle a_\iota \rangle_{\iota < |A|}$. Then the type of $(\mathfrak{A}, a)$ is μ. The cardinal of a set S is $|S|$ and $|\mu|$ is $|\xi|$.

We write $||R(a_1, \ldots, a_n)||_{\mathfrak{A}} = 1$ if $\mathfrak{A} \models R(a_1, \ldots, a_n)$, $||R(a_1, \ldots, a_n)||_{\mathfrak{A}} = 0$ if $\mathfrak{A} \models \sim R(a_1, \ldots, a_n)$ and $||R(a_1, \ldots, a_n)||_{\mathfrak{A}} = \frac{1}{2}$ otherwise. $\overline{\mathfrak{A}}$ is a completion of $\mathfrak{A}$ and when we say for every $\overline{\mathfrak{A}}$ we mean for every completion of $\mathfrak{A}$.

2 Definitions and Examples In this section we propose a number of model-theoretic definitions for incomplete structures.

1) $\overline{\mathfrak{A}}$ is a completion of $\mathfrak{A}$ if a), b), and c) hold:

a) $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ have the same type μ and the same domain A;
b) $\overline{\mathfrak{A}}$ is a relational structure;
c) For every n-ary relation R, $n > 0$, and n-tuple $\langle a_1, \ldots, a_n \rangle$ of A, if $\mathfrak{A} \models R(a_1, \ldots, a_n)$ (resp. $\mathfrak{A} \models \sim R(a_1, \ldots, a_n)$) then $\overline{\mathfrak{A}} \models R(a_1, \ldots, a_n)$ (resp. $\overline{\mathfrak{A}} \models \sim R(a_1, \ldots, a_n)$).

Note that the number of completions is always a power of 2.

2) $\mathfrak{A} \cong \mathfrak{B}$ if there is a bijective map $f: A \to B$ such that for every relation R, if R is n-ary for $n > 0$, then $||R(a_1, \ldots, a_n)||_{\mathfrak{A}} = ||R(f(a_1), \ldots, f(a_n))||_{\mathfrak{B}}$, and if $n = 0$ then $f(a) = b$.
3) $\mathfrak{A} \subseteq \mathfrak{B}$ if for every $\overline{\mathfrak{A}}$ there is a $\overline{\mathfrak{B}}$ such that $\overline{\mathfrak{A}} \subseteq \overline{\mathfrak{B}}$. Such a pair, $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{B}}$, is then called a pair of corresponding completions of $\mathfrak{A}$ and $\mathfrak{B}$.
4) $\mathfrak{A} \prec \mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and for every pair of corresponding completions $\overline{\mathfrak{A}}$ and $\overline{\mathfrak{B}}$, $\overline{\mathfrak{A}} \prec \overline{\mathfrak{B}}$.
5) $\mathfrak{A} \models \psi$ if for every $\overline{\mathfrak{A}}$, $\overline{\mathfrak{A}} \models \psi$. In this case $\mathfrak{A}$ is said to be an incomplete model of ψ. If $\mathfrak{A} \models \psi$ for every $\psi \in \Sigma$ then $\mathfrak{A}$ is said to be an incomplete model of Σ. $\mathrm{Th}(\mathfrak{A}) = \{\psi \mid \mathfrak{A} \models \psi\}$.
6) $\mathfrak{A} \equiv \mathfrak{B}$ if for every sentence ψ of $\mathcal{L}$, $\mathfrak{A} \models \psi$ iff $\mathfrak{B} \models \psi$.
7) $\mathfrak{C}^*$ is an incompletion of the relational structure $\mathfrak{C}$ if $\mathfrak{C}$ is a completion of $\mathfrak{C}^*$.
8) $\mathfrak{A}$ is R-incomplete if R is an incomplete relation on A. $\mathfrak{A}$ is κ-incomplete if there are κ completions of $\mathfrak{A}$. If κ is finite we say that $\mathfrak{A}$ is finitely incomplete.
9) $\mathfrak{A}$ is totally incomplete if for every n-ary R for $n > 0$ and every n-tuple, $\langle a_1, \ldots, a_n \rangle$, $||R(a_1, \ldots, a_n)|| = \frac{1}{2}$.
10) A chain of incomplete structures, $\langle \mathfrak{A}_\zeta \rangle_{\zeta < \eta}$ is an elementary chain if $\mathfrak{A}_\zeta \prec \mathfrak{A}_{\zeta'}$ whenever $\zeta < \zeta'$.

Note that an algebraic theory, i.e., where every relation is a function, cannot have any incomplete models.

Given a theory Σ in type μ we can obtain a theory Σ' in a type μ' such that $\Sigma \subseteq \Sigma'$, $\mu' = \mu * \mu$ and Σ' possesses no incomplete models although Σ and Σ' are in a sense equivalent. Just let

$$\Sigma' = \Sigma \cup \{c_\theta = c_{\xi+\theta} | n_\theta = 0\} \cup \{(\forall x_1, \ldots, x_{n_\theta})\ (R_\theta(x_1, \ldots, x_{n_\theta}) \leftrightarrow R_{\xi+\theta}(x_1, \ldots, x_{n_\theta})) \,|\, n_\theta > 0\}.$$

Conversely, given any consistent theory Σ in type μ if we let Σ' be Σ in type μ', where $\mu' = \mu * \langle 1 \rangle$, then Σ' has at least one incomplete model.

For every cardinal κ there is a theory Σ_κ in type μ_κ with κ incomplete models of power κ. Namely, let $\mu_\kappa = \langle n_\theta \rangle_{\theta < \kappa}$ where $n_0 = 1$ and $n_\theta = 0$ for $0 < \theta < \kappa$, and let $\Sigma_\kappa = \{(\exists^{\leq 1} x)\, R_0(x)\} \cup \{c_\theta \neq c_{\theta'} \,|\, 0 < \theta < \theta' < \kappa\}$.

Note also that if $\mathfrak{A}$ is totally incomplete then $\mathfrak{A} \models \psi$ iff ψ is valid in $|A|$. In particular, any two totally incomplete infinite structures of the same type are elementarily equivalent. Thus the valid sentences for incomplete structures are the same as the valid sentences for relational structures.

Next we have 2 examples.

1) Let $\mu = \langle 1 \rangle$ and $\Sigma = \{(\exists^{\geq n} x)\, R_0(x),\ (\exists^{\geq n} x) \sim R_0(x) \,|\, n \in \omega\}$. Σ is a complete theory since it is ω-categorical; it has ω incomplete models of power ω and every model of Σ has infinitely many incompletions.
2) Let $\mu = \langle n_\theta \rangle_{\theta < \omega}$ where $n_0 = 1$, $n_\theta = 0$ for $0 < \theta < \omega$, and $\Sigma = \{R_0(c_m) \,|\, m \text{ odd}\} \cup \{\sim R_0(c_m) \,|\, m \text{ even}\}$. Then the canonical model of Σ has no incompletions but any other model of Σ possesses at least one incompletion.

Some results of model theory concerning elementary extensions and elementary equivalence remain true for incomplete models. For example:

1) If $\mathfrak{A} \prec \mathfrak{B}$ then $(\mathfrak{A}, a) \equiv (\mathfrak{B}, a)$.
2) $(\mathfrak{A}, a) \equiv (\mathfrak{B}, a)$ and $\mathfrak{A} \subseteq \mathfrak{B}$ do not imply $\mathfrak{A} \prec \mathfrak{B}$. Just let $\mu = \langle 1 \rangle$, A = an infinite set, $B = A \cup \{c\}$ where $c \notin A$, both $\mathfrak{A}$ and $\mathfrak{B}$ totally incomplete.
3) If $\mathfrak{A} \prec \mathfrak{C}$, $\mathfrak{B} \prec \mathfrak{C}$, and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.
4) The union of an elementary chain of incomplete structures is an elementary extension of each element of the chain.

3 Analogs of the Löwenheim-Skolem Theorems First we prove an analog of the downward Löwenheim-Skolem theorem by introducing Skolem functions ([4], page 112) for incomplete structures. Let us fix a cardinal κ (for the number of completions) and then instead of introducing 1 new n-ary function symbol f_ψ for each formula ψ of $n + 1$ free variables, let us add κ distinct new n-ary function symbols $\langle f_\psi^\iota \rangle_{\iota < \kappa}$ to $\mathcal{L}$ to form $\mathcal{L}^\kappa$. Then we define the languages $\mathcal{L}_n^\kappa$ and $\mathcal{L}\kappa$ by induction as follows: $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{L}_{n+1} = (\mathcal{L}_n)^\kappa$, and $\mathcal{L}\kappa = \bigcup \mathcal{L}_n^\kappa$ for $n \in \omega$. Let $\mu\kappa$ be the type appropriate for $\mathcal{L}\kappa$. The f_ψ^ι are called Skolem functions. For each Skolem function f_ψ^ι there is a defining sentence S_ψ^ι:

$$(\forall x_1, \ldots, x_n)\ [(\exists x_0)\, \psi(x_0, x_1, \ldots, x_n) \rightarrow \psi(f_\psi^\iota(x_1, \ldots, x_n), x_1, \ldots, x_n)].$$

A κ-incomplete structure $\mathfrak{A}\kappa$ of type $\mu\kappa$ is a Skolem structure if there is a well ordering of the completions of A, $\langle\overline{\mathfrak{A}}_\iota\rangle_{\iota<\kappa}$, such that $\overline{\mathfrak{A}}_\iota \models S_\psi^\iota$ for every $\iota < \kappa$ and every formula ψ of $n+1$ free variables in $\mathcal{L}\kappa$.

Now observe the following:

1) $|\mu\kappa| = \max\{|\mu|, \omega, \kappa\}$.
2) Every κ-incomplete structure $\mathfrak{A}$ of type μ can be expanded to a κ-incomplete Skolem structure $\mathfrak{A}\kappa$ of type $\mu\kappa$ such that if ψ is a sentence of $\mathcal{L}$ then $\mathfrak{A}\models\psi$ iff $\mathfrak{A}\kappa\models\psi$.
3) If $\mathfrak{A}\kappa \subseteq \mathfrak{B}\kappa$ and for each $\iota < \kappa$ $\overline{\mathfrak{A}}_\iota \subseteq \overline{\mathfrak{B}}_\iota$, then $\mathfrak{A}\kappa \prec \mathfrak{B}\kappa$.
4) We say that an element a in $\mathfrak{A}$ is incomplete if there is an n-ary relation R in μ and an n-tuple, $\langle a_1, \ldots, a_n\rangle$, such that $||R(a_1, \ldots, a_n)|| = \frac{1}{2}$ and $a = a_i$ for some i, $1 \leqslant i \leqslant n$. We let $E_{\mathfrak{A}}$ be the set of incomplete elements of $\mathfrak{A}$. Note that if $Z \subseteq A\kappa$ then there is a smallest κ-incomplete substructure of $\mathfrak{A}\kappa$, denoted by $\mathfrak{S}(Z, \mathfrak{A}\kappa)$, which contains $Z \cup E$. $\mathfrak{S}(Z, \mathfrak{A}\kappa) \prec \mathfrak{A}\kappa$ and $|S(Z, \mathfrak{A}\kappa)| \leqslant \max(\kappa, |Z|, |\mu\kappa|)$.

Now we can prove our

Theorem 1 *If $\mathfrak{A}$ is κ-incomplete, $\kappa \leqslant \beta$, β infinite, and $|\mu| \leqslant \beta \leqslant |A|$ then $\mathfrak{A}$ has a κ-incomplete elementary substructure of cardinal β.*

Proof: Obtain $\mathfrak{A}\kappa$ from $\mathfrak{A}$ as in 2) above. Let $Z \subseteq A$, $|Z| = \beta$. By 4) above $\mathfrak{S}(Z, \mathfrak{A}\kappa) \upharpoonright \mu$ is an elementary substructure of $\mathfrak{A}$ of cardinal β.

Corollary *If $|\mu| \leqslant \beta \leqslant |A|$ then $\mathfrak{A}$ has an elementary substructure of cardinal β.*

The compactness theorem is a basic result in model theory. However the analog of the compactness theorem does not hold for incomplete models. We may obtain a counterexample as follows:

Let $\mu = \langle n_\theta\rangle_{\theta<\omega}$ where $n_\theta = 1$ for each $\theta < \omega$; σ_0: $(\exists x)\, R_0(x)$, σ_i: $(\forall x)(R_{i-1}(x) \leftrightarrow R_i(x)) \wedge (\exists x)\, R_{i+1}(x)$ for $i \geqslant 1$; $\Sigma = \{\sigma_i\}_{i<\omega}$. Now every finite subset of Σ has an incomplete model but Σ has no incomplete models. Actually in this case every finite subset of Σ has an incomplete model even if the type is restricted to the relations which appear in that finite set.

Observe that we may define ultraproducts of incomplete structures as follows:

$\Pi A_I/F = \{f/F \mid f \in \Pi A_i\}$ and if R is an n-ary relation then $||R(f_1/F, \ldots, f_n/F)||_{\Pi\mathfrak{A}_i/F} = v$ iff $\{i \in I \mid\ ||R(f_1(i), \ldots, f_n(i))||_{\mathfrak{A}_i} = v\} \in F$.

We write $\mathfrak{A}^I/F$ for an ultrapower. Now we list some results concerning ultrapowers.

1) If $\mathfrak{A}$ is R-incomplete then so is $\mathfrak{A}^I/F$.
2) If $\mathfrak{A}$ is α-incomplete then $\mathfrak{A}^I/F$ is β-incomplete for some $\beta \geqslant \alpha$. If α is finite then $\alpha = \beta$.
3) If $\mathfrak{A}$ is n-incomplete then $\mathfrak{A} \prec \mathfrak{A}^I/F$.

Let us now consider the proof of the compactness theorem using ultraproducts ([1], page 102). This proof does not carry over to incomplete models because an ultraproduct of incomplete structures need not be incomplete. Note that $\Pi\mathfrak{A}_\Delta/F$ is incomplete iff there is a $K \in F$ and a relation R in μ such that if $k \in K$ then $\mathfrak{A}_k$ is R-incomplete. In particular if each $\mathfrak{A}_\Delta$ is R-incomplete for some specific R then $\Pi\mathfrak{A}_\Delta/F$ is incomplete. This enables us to prove various results for incomplete models whose proofs in ordinary model theory use the compactness theorem. Note also that the analog of the compactness theorem holds for incomplete models if $|\mu| < \omega$.

To illustrate the remarks above we now prove an analog of the upward Löwenheim-Skolem theorem as our

Theorem 2 *If* $\mathfrak{A}$ *is n-incomplete and* $\beta \geq \max(|A|, \min(|\mu|, |A|^\omega))$ *then* $\mathfrak{A}$ *has an n-incomplete elementary extension of power* β.

Proof: Case 1. $|\mu| \leq |A|^\omega$. Let $\mu' = \langle r_\theta\rangle_{\theta<\xi+\beta}$ where $r_\theta = n_\theta$ for $\theta < \xi$ and $r_\theta = 0$ for $\xi \leq \theta < \xi + \beta$. Let $\Sigma = \mathrm{Th}(\mathfrak{A}) \cup \{c_\theta \neq c_{\theta'} \mid \xi \leq \theta < \theta' < \xi + \beta\}$. By the remarks above Σ has an incomplete model $\mathfrak{B}$ in μ' such that $\mathfrak{B} \upharpoonright \mu$ is an ultrapower of $\mathfrak{A}$ and it is n-incomplete. Since $|B \upharpoonright \mu| \geq \beta$ and $\mathfrak{A} \prec \mathfrak{B} \upharpoonright \mu$, we may apply Theorem 1 to $\mathfrak{B} \upharpoonright \mu$.

Case 2. $|A|^\omega < |\mu|$. Let $Y \subseteq A$, $|Y| = \omega$, and obtain an n-incomplete Skolem structure $\mathfrak{A}n$ from $\mathfrak{A}$. Each term $t(x_1, \ldots, x_n)$ of μn defines a map $t*: Y^n \to A$. We say that t_1 and t_2 are equivalent if $t_1^* = t_2^*$. The number of equivalence classes is $\leq |A|^\omega$. Now let $\mu' = \mu n * \langle r_\theta\rangle_{\theta<\beta}$ where $r_\theta = 0$ for $\theta < \beta$. Let

$\Sigma' = \mathrm{Th}(\mathfrak{A}n) \cup \{c_\theta \neq c_{\theta'} \mid c_\theta, c_{\theta'},$ new distinct constants added to $\mu n\} \cup \{t_i(c_1, \ldots, c_n) = t_j(c_1, \ldots, c_n) \mid t_i$ and t_j equivalent, $\langle c_1, \ldots, c_n\rangle$ an n-tuple of new distinct constants added to $\mu n\}$.

By the remarks above Σ' has an n-incomplete model $\mathfrak{B}$ in μ' such that $\mathfrak{B} \upharpoonright \mu$ is an ultrapower of $\mathfrak{A}$ and it is n-incomplete. Since $|\mathfrak{B} \upharpoonright \mu| \geq \beta$ and $\mathfrak{A} \prec \mathfrak{B} \upharpoonright \mu$, $\mathfrak{S}(A, \mathfrak{B}) \upharpoonright \mu$ satisfies the conclusion of the theorem.

Corollary *If* Σ *is a theory which has an infinite incomplete model then* Σ *has incomplete models in every cardinal* $\geq \max(\omega, |\mu|)$.

4 Further Results In this section we state several analogs of results of model theory for incomplete models. Our proofs are based on the proofs given in [1] and [4] with modifications using sections **2** and **3**.

Some two-cardinal theorems hold for incomplete models. We assume that R_0 is a unary relation and then say that Σ admits $\langle\alpha, \beta\rangle$ if Σ has an incomplete model $\mathfrak{A}$ such that $|A| = \alpha$ and for every $\overline{\mathfrak{A}}$, $|R_0^{\overline{\mathfrak{A}}}| = \beta$. We write $\langle\alpha, \beta\rangle \to \langle\gamma, \delta\rangle$ if every theory which admits $\langle\alpha, \beta\rangle$ also admits $\langle\gamma, \delta\rangle$.

1) If $\gamma \geq |\mu|$ then $\langle\alpha, \beta\rangle \to \langle\gamma, \gamma\rangle$. If also $\beta \leq \gamma \leq \alpha$ then $\langle\alpha, \beta\rangle \to \langle\gamma, \beta\rangle$.
2) $\langle\alpha, \beta\rangle \to \langle\alpha^\gamma, \beta^\gamma\rangle$ for any γ.
3) $\langle\alpha, \beta\rangle \to \langle\alpha^{<\gamma}, \beta^{<\gamma}\rangle$ for any infinite γ.
4) $\langle\alpha, \beta\rangle \to \langle\gamma, \delta\rangle$ if $\alpha \geq \gamma \geq \delta \geq \beta^\omega$ and $|\mu| \leq \alpha$.

If Y is an infinite ordered set, $\langle Y, <\rangle$, $Y \subseteq A$, and for every formula ψ with free variables, say $\psi = \psi(x_1, \ldots, x_n)$, and for every pair of increasing sequences $y_1^1 < \ldots < y_n^1$ $y_1^2 < \ldots < y_n^2$ of Y, $\mathfrak{A} \models \psi(y_1^1, \ldots, y_n^1) \leftrightarrow \psi(y_1^2, \ldots, y_n^2)$, then Y is called a set of indiscernibles of $\mathfrak{A}$. We can prove the following analog of the Ehrenfeucht-Mostowski theorem:

Theorem 3 *If $\mathfrak{A}$ is n-incomplete and Y is an ordered set, $\langle Y, <\rangle$, such that $Y \cap A = \emptyset$, then $\mathfrak{A}$ has an n-incomplete elementary extension $\mathfrak{B}$ such that $Y \subseteq B$ and Y is a set of indiscernibles of $\mathfrak{B}$.*

Corollary 1 *Every theory which has an infinite incomplete model possesses incomplete models with large automorphism groups, for example if $|\mu| \leqslant \omega$ a countable incomplete model with 2^ω automorphisms.*

Corollary 2 *Every theory which has an infinite incomplete model possesses an incomplete model which can be mapped elementarily onto a proper submodel.*

Next we assume that Ψ is a set of formulas of one free variable x in $\boldsymbol{\mu}$. We say that $\mathfrak{A}$ realizes Ψ if there is an $a \in A$ such that $\mathfrak{A} \models \psi(a)$ for every $\psi \in \Psi$ and $\mathfrak{A}$ finitely realizes Ψ if $\mathfrak{A}$ realizes every finite subset of Ψ. We call $\mathfrak{A}$ α-saturated if whenever $\mathfrak{A}$ finitely realizes Ψ for $|\Psi| < \alpha$ then $\mathfrak{A}$ realizes Ψ. We can now prove

Theorem 4 *Suppose that $\mathfrak{A}$ is n-incomplete, $|\mu| \leqslant \alpha$, and $|A| \leqslant 2^\alpha$. Then there is an α^+-saturated n-incomplete structure $\mathfrak{B}$ such that $|B| = 2^\alpha$ and $\mathfrak{A} \prec \mathfrak{B}$.*

So far we have considered only the case where $\mathcal{L}$ is first-order logic. However we can also consider infinitary languages. For example a result analogous to the downward Löwenheim-Skolem theorem for $\mathcal{L}_{\kappa\kappa}$ ([3], pages 280-282) can be proved for incomplete models. We can also introduce incomplete continuous models and then prove results analogous to the Löwenheim-Skolem theorems for continuous models ([2], pages 64-65 and 72).

It was pointed out in section 2 that consistent theories need not have incomplete models. However it is possible to alter the definitions in such a way that the notion of models with incomplete information be extended to this case. This and other applications of incomplete models will be considered in a forthcoming paper.

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