

VARIOUS SYSTEMS OF SET THEORY
BASED ON COMBINATORY LOGIC

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1. *Introduction* We consider in this paper various systems of set theory which, in their usual formulation, are based on predicate calculus. Here the predicate calculus is an applied predicate calculus, in that alongside the various categories of variables we have constants of various categories; concerning these constants we assume certain additional axioms.

In order to translate this situation into combinatory logic the axioms we assume must be such as to make each of these new constants satisfy the proper grammatical condition (see [2]). If that is the case, such constants can be substituted for the variables with the same grammatical conditions, and these grammatical conditions can be detached from the set of premises of such formulas. Sometimes, these constants can be defined in such a way that the only essentially new axiom is that giving the grammatical condition.

One of these constants is the ob **A**, this is the range of obs over which an ob must be a propositional function before we can apply the deduction theorem (see [3]); we shall also take it as our range of quantification. Thus in this chapter we apply the theorems of [2] with **A** for a . The condition $\mathbf{L}a$ can be dropped in these theorems as we have $\vdash \mathbf{L}a$ by Axiom 8 of [2].

Next we consider the relation ε which is usually taken as primitive in set theory. In our system ε can be defined by **Cl** so that $x \varepsilon y$ is translated as γx . The ob **Cl** can be proved to be a predicate as we have in each system an axiom which gives us

$$\mathbf{A}u, \mathbf{A}v \vdash \mathbf{H}(uv).$$

This rule, as was shown in [4], is inconsistent with **E** for **A**, but with a suitably restricted **A** there should be no problem. The rule gives us $\mathbf{A}u, \mathbf{A}v \vdash \mathbf{H}(\mathbf{Cl}vu)$ which by the deduction theorem gives us,

$$\vdash \mathbf{F}_2 \mathbf{AAH}(\mathbf{Cl}).$$

Other constants that we shall be using are the first order predicates **M** and **M**₁, where **M** is a new primitive for the category of all sets, and **M**₁ is

the category of all classes. In both cases we will need to have the grammatical condition **FAHX** holding, thus we have to have as an axiom:¹

Axiom M. $\vdash \mathbf{FAHM}$.

We do not need to state $\vdash \mathbf{FAHM}_1$ as an axiom as in the Zermelo-Fraenkel system we do not have classes and in the other systems we can define \mathbf{M}_1 so that this property holds.

Bernays in his system [1], where he has both sets and classes, also has two sets of variables, one ranging over sets and one over classes. This is not really necessary as it has been proved that these can be reduced to a single set. In our system we can duplicate Bernay's method of having two sets of variables by considering both \mathbf{M}_1 and \mathbf{M} as subclasses of **FAH**. The grammatical conditions then hold for sets and classes and for example the theorem "if x and y are classes then their intersection is a class" becomes:

$$\mathbf{M}_1x, \mathbf{M}_1y \vdash \mathbf{M}_1(\cap xy).$$

Alternatively we could work without such variables and translate the theorem as

$$\vdash \exists \mathbf{A}[x] : \exists \mathbf{A}[y] : \mathbf{M}_1x \supset \mathbf{M}_1y \supset \mathbf{M}_1(\cap xy).$$

Any of the systems that we consider, in as far as they do not already have them, can be extended to include individuals. There are three ways of handling this problem. First there is Quine's method where an individual is its own unit set; this requires a modification of the axiom of grounding. Second, we could allow XY where X is an individual, always to be a (false) proposition, but this requires a modification of the extensionality axiom. A third solution is not to allow $\vdash \mathbf{H}(XY)$ where X is an individual to be proved in the system. Individuals will in this case not be sets or classes, but they will have to be included in our range of quantification **A**. Modifications are then needed in axioms in which we quantify over sets or classes. We will not consider these modifications for bringing in individuals any further here.

We define the translation of a set theoretical statement and its grammatical conditions exactly as the translation of a predicate calculus statement in [2]. For the predicate constants \mathbf{M} and \mathbf{M}_1 we already have grammatical conditions, we also need them for equality. Set and class equality we define in terms of extensionality, rather than identify them with **Q**. This is because certain problems arise in connection with the extensionality axiom. We can state this axiom as follows.

$$\vdash \exists \Delta[x] : \exists \Delta[y] . (\exists \mathbf{A}[u] . xu \sim yu) \supset \mathbf{Q}xy,$$

1. If we assume $\mathbf{H} = \mathbf{BLK}$ as in [2] we also need to have $\vdash \mathbf{LM}$. If we define \mathbf{L} by **FAH** this property holds trivially.

where in the Zermelo-Fraenkel [7] and Bernays systems Δ is \mathbf{M} and in the Gödel system [8] \mathbf{M}_1 .² As a result of such an axiom, theorems such as the Church-Rosser theorem may fail to hold. To avoid this, or at least to keep the possibility of another interpretation open, we shall define set (or class) equality using the extensionality property and then state an axiom to give the other property of equality. Other solutions, such as defining equality by the Leibnitz property, are less satisfactory than the above; statements such as $\vdash \mathbf{LL}$ are then required as axioms. We could also of course leave set equality undefined and have two axioms.

2. The Zermelo-Fraenkel System We now consider the Zermelo system of set theory as modified by Fraenkel [7]. The modification means, for one thing, that the class of objects we quantify over in the theory is the class of sets itself, rather than an unspecified class. Thus in the work below we identify \mathbf{A} with \mathbf{M} . One further thing we must specify is that every element X of \mathbf{M} must be such that $\vdash \mathbf{H}(XU)$ whenever U is a set. In other words the grammatical condition $\vdash \mathbf{FMH}X$ must hold for all elements X of \mathbf{M} .

Axiom MH. $\vdash \exists \mathbf{M}(\mathbf{FMH})$.

Now we consider the Zermelo-Fraenkel axioms. The first of these is extensionality, which we mentioned in the introduction. The definition of set equality which we use to replace it in this system is as follows.³

Definition \mathbf{Q}_1 . $\mathbf{Q}_1 \equiv [x, y] \wedge (\mathbf{M}x) (\wedge (\mathbf{M}y) (\exists \mathbf{M}[z] xz \sim yz))$.

The required axiom is then:

Axiom \mathbf{Z}'_1 : $\vdash \mathbf{FMH}z \supset z : \exists \mathbf{M}[x]. \exists \mathbf{M}[y] (\mathbf{Q}_1xy \supset . xz \sim zy)$.

Using this definition of equality we now show that we can derive a grammatical condition for \mathbf{Q}_1 .

Theorem 1. $\vdash \mathbf{F}_2\mathbf{MMH}\mathbf{Q}_1$.

Proof. We have

$$\mathbf{M}x \vdash \mathbf{H}(\mathbf{M}x), \quad (1)$$

$$\mathbf{M}y \vdash \mathbf{H}(\mathbf{M}y). \quad (2)$$

Now by Axiom \mathbf{MH} , $\mathbf{M}z$, $\mathbf{M}x \vdash \mathbf{H}(xz)$, and so $\mathbf{M}x$, $\mathbf{M}y$, $\mathbf{M}z \vdash \mathbf{H}(xz \sim yz)$. Thus by Theorem 4 of [2] and $\vdash \mathbf{LM}$,

$$\mathbf{M}x, \mathbf{M}y \vdash \mathbf{H}(\exists \mathbf{M}[z]. xz \sim yz); \quad (3)$$

so using (1), (2), and (3) and Definition \mathbf{Q}_1 we get $\mathbf{M}x, \mathbf{M}y \vdash \mathbf{H}(\mathbf{Q}_1xy)$, and by $\vdash \mathbf{LM}$ and the deduction theorem, $\vdash \mathbf{F}_2\mathbf{MMH}\mathbf{Q}_1$.

The second axiom asserts the existence of a set containing two given objects, this concept can be defined in the system and then specified to be a

2. \sim is used for the equivalence connective.

3. We can prove $\mathbf{M}x, \mathbf{M}y, \mathbf{Q}_1xy \vdash \mathbf{Q}_1xy$.

set by means of an appropriate axiom. For this set containing obs x and y , we take $\mathbf{U}(\mathbf{Q}x)(\mathbf{Q}y)$, where \mathbf{U} is defined as follows.

Definition \mathbf{U} . $\mathbf{U} \equiv \Phi\mathbf{V}$.

Thus $\mathbf{U}xyz = \mathbf{V}(xz)(yz)$, and $\mathbf{U}xy$ can be considered as the union of sets x and y . Similarly \cap can be defined as $\Phi\Lambda$. Now the axiom can be stated.

Axiom $\mathbf{Z2}$. $\vdash \mathbf{F}_2\mathbf{MMM}(\psi\mathbf{UQ}_1)$.

The third axiom states "If the condition Pa is definite for each element a of a set x then there exists a set which contains just those elements a of x for which Pa holds." Fraenkel takes "is definite" to refer to P alone; this we could represent as \mathbf{FMHP} , since then for any set x we have $\mathbf{H}(Px)$. The ob which contains those elements u of x such that yu holds will obviously be $\cap yx$, all that has to be done therefore is to specify that $\cap yx$ must be a set under these conditions.

Axiom $\mathbf{Z3}$. $\vdash \mathbf{FMHy} \supset_y \mathbf{FMM}(\cap y)$.

Here if $\mathbf{M}x$ restricts x to be a set, and \mathbf{FMHy} requires yu to be definite (i.e., a proposition) whenever u is an element of \mathbf{M} , then $\mathbf{M}(\cap xy)$ says that the intersection of x and y is a set.

The next axiom is the power set axiom. Cogan in his thesis [5] gives $\mathbf{C}\exists X$ as the class of all subsets of X and goes on to state an axiom to make any such class a set. He also has an axiom to insure that the intersection of any set and any class is a set, so a subset of a set must be a set. Here we can do without the second axiom by taking $\cap \mathbf{M}(\mathbf{C}\exists X)$ as the set of all subsets of a set X . This leaves open the questions as to whether, for some sets X , there may exist objects Y which are not sets, (i.e., individuals), such that $\exists YX$. Such a Y will not necessarily satisfy the condition of Axiom $\mathbf{Z3}$. Axiom $\mathbf{Z4}$ therefore becomes:

Axiom $\mathbf{Z4}$. $\vdash \mathbf{FMM}([x] \cdot \cap \mathbf{M}(\mathbf{C}\exists x))$.

Note that this can also be written as $\vdash \mathbf{FMM}\{Pw\}$ where $\{Pw\}$ is the "power relation" defined by $[x, y] \cdot \Lambda(\mathbf{M}y)(\exists yx)$.

With Axioms $\mathbf{Z4}$ and $\mathbf{Z2}$ we can prove the two other parts of Zermelo's original Axiom 2.⁴ These asserted the existence of an empty set and a set containing a given element.

Theorem 2. (a) *There exists an empty set.* (b) *For a given set there exists a set containing the given set as sole element.*

Proof. (a) By Theorem 1 we have $\mathbf{M}x \vdash \mathbf{H}(\mathbf{WQ}_1x)$, $\mathbf{M}x \vdash \mathbf{H}(\mathbf{B} - (\mathbf{WQ}_1)x)$, and so $\vdash \mathbf{FMH}(\mathbf{B} - (\mathbf{WQ}_1))$. Now by Axiom $\mathbf{Z3}$ $\mathbf{M}x \vdash \mathbf{M}(\cap (\mathbf{B} - (\mathbf{WQ}_1))x)$. The later axioms such as the axiom of infinity assert the existence of a set, thus substituting any set X for x gives $\vdash \mathbf{M}(\cap (\mathbf{B} - (\mathbf{WQ}_1))X)$. This will be unique by extensionality.

4. This is similar to the proof in Fraenkel and Bar Hillel. For Zermelo's original axioms see [10].

(b) If the set x is the empty set 0 we have by Axiom Z4, $\vdash \mathbf{M}(\cap \mathbf{M}(\mathbf{C} \Xi 0))$.
If we have the extra axiom:

Axiom 12. $\vdash \Lambda xy \supset_{x,y} x$

we can say $\cap \mathbf{M}(\mathbf{C} \Xi 0)x \vdash \mathbf{Q}_1 0x$ so that $\cap \mathbf{M}(\mathbf{C} \Xi 0)$ contains only the given set 0 .

If x is nonempty then by (a) and Axiom 2, $\mathbf{U}(\mathbf{Q}_1 x)(\mathbf{Q}_1 0)$ is a set and Axiom 3 gives us that $\cap (\mathbf{U}(\mathbf{Q}_1 x)(\mathbf{Q}_1 0))(\mathbf{Q}_1 x)$ is a set. This of course contains only x .

Note that we cannot prove that $\mathbf{B} - (\mathbf{W}\mathbf{Q}_1)$ and $\mathbf{Q}_1 x$ are sets given that x is one, as we can only apply extensionality to prove these equal to the above sets if they are already sets. We could simply take $\vdash \mathbf{M}(\mathbf{B} - (\mathbf{W}\mathbf{Q}_1))$ and $\vdash \mathbf{FMM}\mathbf{Q}_1$ as axioms (and grammatical conditions); we would then not need to use Axiom 12.

Axiom Z5 is the sum set axiom. Given a set Y the class of all elements of elements of Y can be given by $\{\mathbf{Un}\}Y$, where,

Definition $\{\mathbf{Un}\}$. $\{\mathbf{Un}\} \equiv [y, x] \Sigma \mathbf{M}[u] (\Lambda (ux)(yu))$.

For this to be a set is needed:

Axiom Z5. $\vdash \mathbf{FMM}\{\mathbf{Un}\}$.

The sixth axiom is the axiom of choice. Zermelo's version of it can be translated into the combinatory system, but its meaning will be clearer if we use a notation which is closer to the classical one. Such a notation will now be defined.

Definition \wedge . $x \wedge y \equiv \Lambda xy$.

Definition \vee . $x \vee y \equiv \mathbf{V}xy$.

We will not introduce $(\forall x)$ for $\Xi \mathbf{M}[x]$ and $(\exists x)$ for $\Sigma \mathbf{M}[x]$, as we may not always want to identify \mathbf{A} with \mathbf{M} . We now state the axiom of choice.

Axiom Z6. $\vdash \Xi \mathbf{M}[x] : . (\Xi \mathbf{M}[u] : \Xi \mathbf{M}[v] . xu \wedge xv \wedge u \neq v \supset \cap uv = 0 \wedge u \neq 0) : \supset : \Sigma \mathbf{M}[t] \Xi \mathbf{M}[w] . tw \supset \{\mathbf{Un}\}xw \wedge (\Xi \mathbf{M}[u] : xu . \supset . \Sigma \mathbf{M}[s] \cap tu = \mathbf{Q}_1 s)$.

Thus if X is a set, the elements of which are nonempty disjoint sets, then their union $\{\mathbf{Un}\}X$ contains at least one subset T which has one and only one element in common with every element of X .

The remaining axiom is the axiom of infinity. This can be treated as Cogan [5] does, by taking a new primitive $\{\mathbf{Inf}\}$ with the following axioms:

Axiom Z7. (a) $\vdash \{\mathbf{Inf}\} 0$.
(b) $\vdash \mathbf{M}\{\mathbf{Inf}\}$.
(c) $\vdash \Xi \mathbf{M}[x] . \{\mathbf{Inf}\}x \supset \{\mathbf{Inf}\}(\mathbf{Q}_1 x)$.

As an alternative to (c) we could take the von Neumann form: $\vdash \Xi \mathbf{M}[x] . \{\mathbf{Inf}\}x \supset \{\mathbf{Inf}\}(\mathbf{U}x(\mathbf{Q}_1 x))$.

The sets $\{\mathbf{Inf}\}$ will be infinite in the usual sense as it will not be possible for an element $\mathbf{Q}_1(\mathbf{Q}_1 \dots (\mathbf{Q}_1 0) \dots)$ with k \mathbf{Q}_1 's to be equal to $\mathbf{Q}_1(\mathbf{Q}_1 \dots (\mathbf{Q}_1 0) \dots)$ with l \mathbf{Q}_1 's. If it were possible then by rules (μ) and

Eq⁵ we would get: $\mathbf{Q}_1(\mathbf{Q}_1 \dots (\mathbf{Q}_1 0) \dots) = 0$, where there are $k - l$ \mathbf{Q}_1 's and this is impossible, as it leads to $\vdash 0(\mathbf{Q}_1 \dots (\mathbf{Q}_1 0) \dots)$ with $k - l - 1$ \mathbf{Q}_1 's.

It is also possible to define $\{\mathbf{CInf}\}$ as follows:

Definition $\{\mathbf{CInf}\}$. $\{\mathbf{CInf}\} \equiv [x] : \mathbf{M}x \cdot \wedge \cdot x0 \cdot \wedge \exists \mathbf{M}[u] \cdot xu \supset x(\mathbf{Q}_1 u)$,

the axiom of infinity then asserts the existence of such a set, i.e.,

Axiom Z7'. $\vdash \Sigma \mathbf{M}\{\mathbf{CInf}\}$.

The above definition of $\{\mathbf{CInf}\}$ is closest to what Zermelo expressed in Axiom Z7 (c), but an alternative with some advantages is von Neumann's definition, with which we can use the same Axiom Z7'.

Definition $\{\mathbf{CInf}\}$ (2). $\{\mathbf{CInf}\} \equiv [x] : \mathbf{M}x \cdot \wedge \cdot x0 \cdot \wedge \exists \mathbf{M}[u] \cdot xu \supset x(\mathbf{U}u(\mathbf{Q}_1 u))$.

The Zermelo system is extended to the Zermelo-Fraenkel system by adding two further axioms, namely the axioms of grounding and replacement. The first of these states that every nonempty set has an element that has no elements in common with it. This can be represented as follows.

Axiom Z8. $\vdash \exists \mathbf{M}[x] : x \neq_1 0 \cdot \supset \cdot \Sigma \mathbf{M}[y] \cdot xy \wedge (\cap xy =_1 0)$.⁶

The replacement axiom requires that given a nonempty set x and a single valued function f (i.e., $\vdash \mathbf{FMM}f$), there is a set y such that $\vdash yu$ if and only if there is a set v contained in x such that $fv =_1 u$.⁷ An ob $\{\mathbf{Rep}\}$ is defined so that this axiom can be expressed.

Definition $\{\mathbf{Rep}\}$. $\{\mathbf{Rep}\} = [f, x, u] \Sigma \mathbf{M}[v] \cdot \wedge (xv)(fv =_1 u)$.

The axiom is then represented by:⁸

Axiom Z9. $\vdash \mathbf{FMM}f \supset_f \cdot \mathbf{FMM}(\{\mathbf{Rep}\}f)$.

Now we can prove the general theorem for the Zermelo-Fraenkel system.

Theorem 3. *If T is a theorem of Zermelo-Fraenkel set theory and N is the set of grammatical conditions for T' , the translation of T into the combinatory system, then $N \vdash T'$.*

Proof. This goes exactly as the proof of Theorem 25 of [2] except that there are some extra axioms to consider, namely those of the Zermelo-Fraenkel set theory itself. It T is the extensionality axiom $\vdash T'$ follows from Definition \mathbf{Q}_1 . T' contains no variables and only the constant predicate \mathbf{Q}_1 for which we have obtained the correct grammatical condition in

5. These are as in [6]

(μ) If $\mathbf{Q}_1 XY$ then $\mathbf{Q}_1 (ZX)(ZY)$

Eq. $\mathbf{Q}_1 XY, X \vdash Y$.

6. $z =_1 w$ stands for $\mathbf{Q}_1 zw$.

7. In the case of an empty set the replacement set is trivially empty.

8. Axiom Z3 becomes provable when Z9 is added (See Fraenkel and Bar Hillel [7]).

Theorem 1. The other property of equality (Z'_1) is in the correct translated form if we apply Rule Ξ once. Axioms Z2, Z4 and Z5 are themselves grammatical conditions for certain constants, similarly Axioms Z3 and Z9 become schemes for grammatical conditions if we apply Rule Ξ with argument z . Axioms Z6, Z7 and Z8 are thus the only axioms not giving grammatical conditions, these assert the existence of certain sets.

Axiom Z6 contains no free variables and the only constant in it we have, as yet, no grammatical condition for is \cap , however $\vdash F_2MMM\cap$ follows by Axioms Z3 and MH. In Axiom Z7 we have the new primitive predicate $\{\text{Inf}\}$, but by Axioms Z7 (b) and MH this has the correct grammatical condition $\vdash FMH\{\text{Inf}\}$. Similarly we can easily prove $\vdash FMH\{\text{CInf}\}$ if we use Axiom Z7'. Axiom Z8 also contains no variables and as constants only 0, \cap and Q_1 for which we have the grammatical conditions. Thus all of the axioms have the correct form and the proof of the theorem goes through as for Theorem 25.

3. *Bernays' System* Now Bernays' system [1] will be considered. It contains not only sets, as Zermelo's system does, but also classes. The basic idea of a class is that one is determined by every predicate over sets. This property is expressed formally by Church's scheme, as given by Bernays, which, in classical notation is given by:

$$x \varepsilon \{x \mid \phi(x)\} \sim \phi(x)$$

whenever x is a set.

As Bernays only has sets as objects, talks of predicate over sets and quantifies only over sets, we can again let A be M in this section. A "predicate over sets" will thus be an ob X such that $\vdash FMHX$, and such that $\vdash LX$. By the use of this a simple combinatory logic version of Church's scheme can be proved. In combinatory logic the concepts " x is an element of y " (or $x \varepsilon y$) and " x has the property y " (or $y(x)$) can both be expressed by yx . Instead of Church's scheme we then get:

$$FMHy, Mx \vdash yx \sim yx.$$

(Note that $FMHy$ and Mx are the grammatical conditions for $yx \sim yx$.)

A class in the Bernays sense (the category of these we will denote by M_1) must therefore have the property FMH . At this point however we have some further alternatives. We can leave M_1 as primitive and have $\vdash M_1x \supset_x FMHx$, as an axiom; we can define M_1 as FMH , which will be an extension of Bernays' system as we will have $\vdash FMH(WQ)$ and thus a class which will contain obs other than sets; or we can define M_1 by $\cap (FMH)(C\Xi M)$, thus restricting classes to obs containing only sets. Here we shall use the second of these approaches. Bernays has no axioms to say that all elements of classes must be sets and FMH also has the advantage of being a grammatical condition.

We now define class equality as we did set equality in the previous section.

Definition Q'_1 . $Q'_1 \equiv [x, y](\Xi M[u]. xu \sim yu . \wedge . M_1x . \wedge . M_1y)$.

Especially if we assume M_1 to be **FMH** it is better not to assume that Q'_1 is the same as Q . In that case it is conceivable that there might be two classes which have all their sets in common, but which are not equal combinatorially. With Q'_1 as it is however, we can prove

$$M_1x, M_1y, x =_1 y \vdash \exists A[u]. xu \sim yu,$$

using our version of Church's scheme.

Next Bernays has two axiom schemes for equality over sets.

E1. $M_1z, Mx, My \vdash x =_1 y \supset . zx \supset zy.$

E2. $Mx, My \vdash \exists M[z]. xz \sim yz \supset . x =_1 y.$

In his system he has a certain class "represented" by a set if it has the same sets as elements as that set. In our system where we can compare sets and classes more easily the "representation" can be replaced by something similar to the notion Q_1 .

Definition Q_0 . $Q_0 \equiv [x, y]. M_1x \wedge My \wedge \exists M[u]. xu \sim yu.$

If we define set equality by Q_1 as in the previous section we have **E2** as a result of the definition and the following assumption.

Axiom M_1M . $\vdash \exists M(\mathbf{FMH}),$

which is equivalent to the grammatical condition $\vdash F_2\mathbf{MMHI}$.

We then can derive **E1** from a generalisation of Axiom Z'_1 , viz.

Axiom $E1'$. $\vdash M_1z \supset_z \exists M[x] \exists M[y] : x =_1 y \supset . zx \supset zy.$

Of the remaining axioms **A1** asserts the existence of an empty set. This we can represent as:

Axiom $BA1$. $\vdash M(\mathbf{B} - (\mathbf{W}Q_1)).$

Note that we write the combinatory versions of Bernays' $A1, \dots$ as $BA1, \dots$. His second says that a set with an extra element added is a set. This can be stated in the following way.

Axiom $BA2$. $\vdash F_2\mathbf{MMM}(\mathbf{C}(\mathbf{BBU}Q_1)).$

(This could be replaced by $\vdash F_2\mathbf{MMM}U$ and $\vdash \mathbf{FMM}Q_1$.)

Axiom $A3$ asserts that if z is a function over sets (i.e., an element of **FMM**) then the union of all the sets zx where $\vdash yx$ and $\vdash My$, is again a set. This can be represented by $\{\mathbf{Rp}\}$ which is defined by,

Definition $\{\mathbf{Rp}\}$. $\{\mathbf{Rp}\} \equiv [z, y, u]. \Sigma M. [x] \wedge (yx)(z xu).$

All that is needed then is an axiom to assert that $\{\mathbf{Rp}\}zy$ is a set, under the required conditions.

Axiom $BA3$. $\vdash F_2(\mathbf{FMM})\mathbf{MM}\{\mathbf{Rp}\}.$

Thus $\mathbf{FMM}z, My \vdash M(\{\mathbf{Rp}\}zy)$. Axioms $BA2$ and $BA3$ in this system replace Axioms $Z2$ and $Z3$ of Zermelo's system. Axioms $A4, A5$ and $A7$ are the power set axiom, the axiom of choice and the axiom of grounding and for these $Z4, Z6$ and $Z8$ can be used.

Axiom A6 is Bernays' axiom of infinity. He defines natural numbers, in terms of a zero ordinal and its successors and then has an axiom to the effect that the class N_n of these is a set. Instead of this we can have Definition $\{CInf\}$ and Axiom Z7'.

Thus we have obtained all the axioms of Bernays' system and we can prove the general theorem for the Bernays' system.

Theorem 4. *If T is a theorem of Bernays' set theory and N is the set of grammatical conditions for T' , the translation of T into the combinatory system, then $N \vdash T'$.*

Proof. This again is proved as Theorem 25 of [2]. We have a theorem with the correct grammatical conditions to correspond to Church's scheme. Also we have Axiom E2 proved from Definition Q_1 . Axiom schemes corresponding to Bernays' Axiom Schemes E1 and A3 follow by one use of Rule \exists and our Axioms E1' and BA3, also with the correct grammatical conditions. Axiom BA2, itself a grammatical condition, is the translation of Bernays' A2. The remaining axioms are handled as they were in the Zermelo-Fraenkel system. Thus we have all the axioms of Bernays in the correct form and the theorem is proved.

4. Gödel's Set Theory As in the other two systems of set theory we shall base the system on the predicate calculus we developed in section 3, taking this time A as the set of all objects. This A cannot be M as it must include M_1 ; but it is possible to replace A by M_1 , the category of all classes, as in the system M is a subclass of M_1 . Below we will use A keeping this possibility in mind, and also leaving open the possibility of A containing individuals and perhaps relations, as well as classes. It is certain in any case that we will need the following as an axiom.

Axiom M_1 . $\vdash \exists M_1 A$.

Gödel's notion of a class is somewhat more restricted than that of Bernays in that he has an axiom to the effect that all members of classes must be sets. For Bernays $X \varepsilon Y$ where both X and Y are classes is not defined, however in the previous section we extended his system to include this possibility. Also Gödel has an axiom to ensure that all sets are also classes, whereas Bernays has certain classes represented by sets, if they have the same elements. The two axioms of the Gödel system that were mentioned above, can be written as follows.⁹

Axiom GA1. $\vdash \exists M[x]. Mx \supset M_1x$.

Axiom GA2. $\vdash \exists A[x] : M_1x \supset : \exists A[y]. xy \supset My$.

Gödel's Axiom A3 is identical to Bernays' axiom of extensionality for

9. Cogan [5] had as a definition $M_1 = FMH$. This gives us obs such as $K(WQS)$ as classes, which have elements which are not necessarily sets. This is Titgemeyer's paradox. This section gives a solution to this paradox. See [9].

classes, for class equality we can use the same definition, with **A** for **M**. Also we need Axiom Z'_1 with $\exists \mathbf{A}[x]. \mathbf{M}_1 x \supset$ instead of $\mathbf{M}_1 x \supset_x$, and \mathbf{Q}'_1 as the equality.

The fourth axiom of the **A** group is the pair set axiom which we can state as follows.

Axiom GA4. $\vdash \exists \mathbf{A}[x]. \exists \mathbf{A}[y]. \mathbf{M}x \supset. \mathbf{M}y \supset \mathbf{M}(\mathbf{U}(\mathbf{Q}'_1 x)(\mathbf{Q}'_1 y))$.

Now we consider the **C**, **D** and **E** axioms. There are four axioms in group **C**, these are the axiom of infinity, the sum set axiom, the power set axiom and the "definiteness axiom"; they can all be formulated as in the first section with $\exists \mathbf{A}[u] \mathbf{M}u \supset$ and $\Sigma \mathbf{A}[u] \mathbf{M}u \wedge$ instead of $\exists \mathbf{A}[u]$ and $\Sigma \mathbf{A}[u]$. This can also be done for Axioms **D** and **E**, the axioms of grounding and choice.

Note that in the other two systems some of the axioms were themselves grammatical conditions for some constant ob. With our replacement of quantification over **M** by quantification over **A** statements such as $\vdash \mathbf{FMM}\{\mathbf{Un}\}$ no longer constitute grammatical conditions, and to fit in the predicate calculus this has to be replaced by $\vdash \exists \mathbf{A}[u]. \mathbf{M}u \supset \mathbf{M}\{\mathbf{Un}\}u$. Also we need a grammatical condition for $\{\mathbf{Un}\}$, viz.

Axiom $\mathbf{A}\{\mathbf{Un}\}$. $\vdash \mathbf{FAA}\{\mathbf{Un}\}$.

Similarly we need:

Axiom $\mathbf{A}\cap$. $\vdash \mathbf{F}_2 \mathbf{AAA}\cap$.

Axiom $\mathbf{A}\{\mathbf{Inf}\}$. $\vdash \mathbf{FAA}\{\mathbf{Inf}\}$.

Axiom $\mathbf{A}\text{subc}$. $\vdash \mathbf{FAA}(\{y\} \cap \mathbf{M}(\mathbf{C}\exists y))$.

Thus we have the **C**, **D** and **E** axioms in their correct translated form with the appropriate grammatical conditions. The **A** axioms are also in the correct form if we have:

Axiom $\mathbf{A}\mathbf{U}$. $\vdash \mathbf{F}_2 \mathbf{AAA}\mathbf{U}$.

Axiom $\mathbf{A}\mathbf{Q}_1$. $\vdash \mathbf{FAAQ}'_1$.

Axiom $\mathbf{H}\mathbf{Q}_1$. $\vdash \mathbf{F}_2 \mathbf{AAHQ}'_1$.

The last of these will be derivable from Definition \mathbf{Q}_1 and Axiom $\mathbf{H}\mathbf{A}$ which we introduce later on. In addition to these we need $\vdash \exists \mathbf{M}_1(\mathbf{FAH})$ and $\vdash \mathbf{FAHM}_1$ but these become derivable when we define \mathbf{M}_1 below.

Gödel's set of axioms **B** deals with relations and classes, using this set of axioms he later proves his "general existence theorem" from which his axioms are then derivable. In our system it seems more natural to define relations as many-placed predicates rather than as classes or ordered n -tuples of sets. We shall therefore prove Gödel's general existence theorem directly using definitions of classes and relations and then show that this method is equivalent to Gödel's.

Definition \mathbf{M}_n . $\mathbf{M}_n \equiv [x]. \mathbf{F}_n \mathbf{A} \dots \mathbf{A} \mathbf{H}x \wedge \exists \mathbf{A}[x_1] \dots \exists \mathbf{A}[x_n]. xx_1 \dots x_n \supset \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n$.

In particular $\mathbf{M}_1 \equiv [x]. \mathbf{FAH}x \wedge \exists \mathbf{A}[x_1]. xx_1 \supset \mathbf{M}x_1$. This definition of \mathbf{M}_1 we

will take for Gödel's primitive notion, the definitions of relations can be regarded as temporary and as a means by which we indicate how Gödel's work could be carried out without the use of ordered pairs. Gödel's definition of relations can then be used as he does, in terms of \mathbf{M} and \mathbf{M}_1 . Note that Axiom GA2 is now derivable.

Before proving the theorem we require a number of assumptions. First we need Axiom 12¹⁰ which gives

$$\Lambda xy \vdash x, \tag{1}$$

$$\Lambda xy \vdash y, \tag{2}$$

for which we could take $\vdash \Lambda xy \supset_{x,y} y$ as an extra axiom. Also the universe \mathbf{A} has to be restricted in such a way that we have:

Axiom HA. $\vdash \exists \mathbf{A}(\mathbf{FAH})$.

This together with Definition \mathbf{M}_1 gives us $\vdash \mathbf{FAHM}_1$. This property would seem most natural if we identified \mathbf{A} and \mathbf{M}_1 , but then our definition of \mathbf{M}_1 would circular, as \mathbf{M} is defined in terms of \mathbf{A} . We could make the identification and then have what is in Definition \mathbf{M}_1 in the form of two axioms. In the work below we leave this question open.

Before stating his general existence theorem Gödel had an inductive definition; this we shall repeat writing " xy " for " $y \varepsilon x$," " $\Sigma \mathbf{A}[u]. \mathbf{M}u \wedge$ " for " $(\exists u)$ " etc.

Definition ppf.¹¹ (i) If X and Y are sets or special classes¹² then XY is a ppf.

(ii) If X and Y are ppfs, then so are $\neg X$ and ΛXY .

(iii) If X is a ppf then so is $\Sigma \mathbf{A}[u]. \mathbf{M}u \wedge X$, and if provided $\vdash \mathbf{M}x$, X is a ppf then provided $\vdash \mathbf{M}u$, $[x]Xu$ is a ppf.¹³

Theorem 5. (The general existence theorem for ppfs.) If $\phi(x_1, \dots, x_n)$ is a ppf containing no free variables other than at most x_1, \dots, x_n ; then there exists an ob X such that $\vdash \mathbf{M}_n X$ and $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash Xx_1 \dots x_n \sim \phi(x_1, \dots, x_n)$.

Proof. By induction on the structure of $\phi(x_1, \dots, x_n)$.

Case 1. ϕ has no logical operators. Then $\phi(x_1, \dots, x_n)$ is $x_s x_r$ or $\mathbf{A}_k x_r$ (where \mathbf{A}_k is a special class). The case \mathbf{GA}_k (where \mathbf{G} is a set or class) can be represented as $\Sigma \mathbf{A}[x]. \mathbf{M}x \supset \wedge (\mathbf{Q}'_1 x \mathbf{A}_k)(\mathbf{G}x)$ by Axiom GA2, and $\mathbf{Q}'_1 x \mathbf{A}_k$ by $\exists \mathbf{A}[u]: \mathbf{M}u \supset .xu \sim \mathbf{A}_i u$ by Axiom GA3. Now let

10. It is possible to avoid the use of this axiom if we introduce a number of extra primitives and further axioms.

11. This notion Gödel calls a "primitive propositional function." It is not a function in our sense; it is one only in the sense that it may involve parameters.

12. These Gödel leaves unspecified.

13. Gödel had a fourth clause which said that ppf's could only be formed as in clauses (i), (ii) and (iii). This we take to be understood in inductive definitions. In our system, where we can do without variables and our functions do not involve them, the last part of condition (iii) is really irrelevant.

$$\mathbf{X} = [x_1, \dots, x_n] \cdot x_s x_r \wedge \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n.$$

Then by propositional calculus and $\mathbf{M}x_s \vdash \mathbf{F}Mx_s$, we know $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{X}x_1 \dots x_n \sim x_s x_r$. As

$$\mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{H}(\mathbf{X}x_1 \dots x_n) \tag{3}$$

by Axioms M and HA, we have

$$\vdash \mathbf{F}_n \mathbf{A} \dots \mathbf{A} \mathbf{H} \mathbf{X}. \tag{4}$$

By (2) we have $\mathbf{X}x_1 \dots x_n \vdash \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n$; so by (3) $\mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{X}x_1 \dots x_n \supset \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n$, and thus $\vdash \exists \mathbf{A}[x_1] \dots \exists \mathbf{A}[x_n] \cdot \mathbf{X}x_1 \dots x_n \supset \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n$. Thus by (4) we have $\vdash \mathbf{M}_n \mathbf{X}$. Similarly we prove this for

$$\mathbf{X} = [x, \dots x_n] \mathbf{A}k x_r \wedge \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n.$$

Case 2. Here we assume that ϕ has m of the logical connectives \neg, \wedge and $(\Sigma \mathbf{A}[u] \mathbf{M}u \wedge)$ and that the theorem holds for ϕ 's with fewer logical connectives. ϕ can take three forms.

Form (i): $\phi(x_1, \dots, x_n) = \neg(\psi(x_1, \dots, x_n))$

where $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{Y}x_1 \dots x_n \sim \psi(x_1, \dots, x_n)$, and $\vdash \mathbf{M}_n \mathbf{Y}$. Now let

$$\mathbf{X} = [x_1, \dots, x_n] \neg(\mathbf{Y}x_1 \dots x_n \wedge \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n).$$

Then as $\mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{H}(\mathbf{Y}x_1 \dots x_n)$, we have $\mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{H}(\mathbf{X}x_1 \dots x_n)$, and thus $\vdash \mathbf{F}_n \mathbf{M} \dots \mathbf{A} \mathbf{H} \mathbf{X}$. The other properties of \mathbf{X} we obtain as in Case 1, so $\vdash \mathbf{M}_n \mathbf{X}$. Also $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{X}x_1 \dots x_n \sim \neg(\mathbf{Y}x_1 \dots x_n)$ so $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{X}x_1 \dots x_n \sim \phi(x_1, \dots, x_n)$.

Form (ii): $\phi(x_1, \dots, x_n) = \psi(x_1, \dots, x_n) \wedge \chi(x_1, \dots, x_n)$,

where by the inductive hypothesis there is a \mathbf{Y} and a \mathbf{Z} such that, $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{Y}x_1 \dots x_n \sim \psi(x_1, \dots, x_n)$, $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{Z}x_1 \dots x_n \sim \chi(x_1, \dots, x_n)$, and $\vdash \mathbf{M}_n \mathbf{Y}$ and $\vdash \mathbf{M}_n \mathbf{Z}$. Let

$$\mathbf{X} = [x_1, \dots, x_n] \mathbf{Y}x_1 \dots x_n \wedge \mathbf{Z}x_1 \dots x_n \wedge \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n.$$

Then similarly all the necessary properties hold for $\vdash \mathbf{M}_n \mathbf{X}$.

Form (iii): $\phi(x_1, \dots, x_n) = \Sigma \mathbf{A}[u] \mathbf{M}u \wedge \theta(u, x_1, \dots, x_n)$

where $\mathbf{M}x_1, \dots, \mathbf{M}x_n, \mathbf{M}u \vdash \theta(u, x_1, \dots, x_n) \sim \mathbf{T}u x_1 \dots x_n$ and $\vdash \mathbf{M}_{n+1} \mathbf{T}$. Then let

$$\mathbf{X} = [x_1, \dots, x_n] \Sigma \mathbf{A}([u] \mathbf{M}u \wedge \mathbf{T}u x_1 \dots x_n) \wedge \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n$$

and again all properties hold. Thus the theorem holds in all cases.

Gödel also states a generalisation of this to "normal functions," this proof however does not depend on any of his assumptions concerning relations so it would be trivial to repeat it here.

To prove our method of dealing with relations equivalent to that of Gödel we have to show that for every one of our n -ary relations X we have a class Y such that $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash Xx_1 \dots x_n \sim Y \langle x_1, \dots, x_n \rangle$ where the ordered n -tuple $\langle x_1, \dots, x_n \rangle$ is defined as follows.

Definition $\langle x_1, \dots, x_n \rangle$.

$\langle x_1, x_2 \rangle \equiv \mathbf{U}(\mathbf{Q}'_1(\mathbf{U}(\mathbf{Q}'_1 x_1)(\mathbf{Q}'_1 x_2)))$,

$\langle x_1, \dots, x_n \rangle \equiv \mathbf{U}(\mathbf{Q}'_1 \langle x_1, \dots, x_{n-1} \rangle)(\mathbf{Q}'_1(\mathbf{U}(\mathbf{Q}'_1 \langle x_1, \dots, x_{n-1} \rangle)(\mathbf{Q}'_1 x_n)))$.

We can prove that there is such a class. First we shall show that any ordered n -tuple of sets is a set.

Lemma 1. $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{M}\langle x_1, \dots, x_n \rangle$.

Proof. From Axioms GA4, GA1 and \mathbf{M}_1 we get, $\mathbf{M}x_1 \vdash \mathbf{M}(\mathbf{U}(\mathbf{Q}'_1 x_1)(\mathbf{Q}'_1 x_1))$ and also $\mathbf{M}x_1, \mathbf{M}x_2 \vdash \mathbf{M}(\mathbf{U}(\mathbf{Q}'_1 x_1)(\mathbf{Q}'_1 x_2))$. Thus again by Axiom GA4,

$$\mathbf{M}x_1 \mathbf{M}x_2 \vdash \mathbf{M}(\mathbf{U}(\mathbf{Q}'_1(\mathbf{U}(\mathbf{Q}'_1 x_1)(\mathbf{Q}'_1 x_2))) (\mathbf{Q}'_1(\mathbf{U}(\mathbf{Q}'_1 x_1)(\mathbf{Q}'_1 x_2))))$$

which is, $\mathbf{M}x_1, \mathbf{M}x_2 \vdash \mathbf{M}\langle x_1, x_2 \rangle$. Thus we have the theorem for $n = 2$. We now assume the case of $n - 1$, then $\mathbf{M}x_1, \dots, \mathbf{M}x_{n-1} \vdash \mathbf{M}\langle x_1, \dots, x_{n-1} \rangle$. Then by Axiom GA4,

$$\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{M}(\mathbf{Q}'_1(\mathbf{U}(\mathbf{Q}'_1 \langle x_1, \dots, x_{n-1} \rangle)(\mathbf{Q}'_1 x_n)))$$

and using the inductive hypothesis, $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{M}\langle x_1, \dots, x_n \rangle$. Now we can state the theorem.

Theorem 6. *If $\vdash \mathbf{M}_n \mathbf{X}$ then there exists a \mathbf{Y} such that $\vdash \mathbf{M}_1 \mathbf{Y}$ and $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{X}x_1 \dots x_n \sim \mathbf{Y}\langle x_1, \dots, x_n \rangle$.*

Proof. Let

$$\mathbf{Y} = [t] \Sigma \mathbf{A}[x_1] \dots \Sigma \mathbf{A}[x_n] \cdot \mathbf{Q}'_1 t \langle x_1, \dots, x_n \rangle \wedge \mathbf{X}x_1 \dots x_n.$$

Now by Axioms $\mathbf{A}\mathbf{Q}'_1$ and $\mathbf{A}\mathbf{U}$, $\mathbf{A}t, \mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{H}(\mathbf{Q}'_1 t \langle x_1, \dots, x_n \rangle)$, and by Definition \mathbf{M}_n , $\mathbf{M}_n \mathbf{X}$, $\mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{H}(\mathbf{X}x_1 \dots x_n)$. Therefore,

$$\mathbf{M}_n \mathbf{X}, \mathbf{A}t, \mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{H}(\mathbf{Q}'_1 t \langle x_1, \dots, x_n \rangle \wedge \mathbf{X}x_1 \dots x_n), \quad (5)$$

and so

$$\mathbf{M}_n \mathbf{X}, \mathbf{A}t \vdash \mathbf{H}(\Sigma \mathbf{A}[x_1] \dots \Sigma \mathbf{A}[x_n] \mathbf{Q}'_1 t \langle x_1, \dots, x_n \rangle \wedge \mathbf{X}x_1 \dots x_n), \quad (6)$$

and by the deduction theorem, $\mathbf{M}_n \mathbf{X} \vdash \mathbf{F}\mathbf{A}\mathbf{H}\mathbf{Y}$. Now let

$$\mathbf{Z} = [t, x_1, \dots, x_n] \mathbf{Q}'_1 t \langle x_1, \dots, x_n \rangle \wedge \mathbf{X}x_1 \dots x_n,$$

then, as $\mathbf{A}x_1, \dots, \mathbf{A}x_n, \mathbf{M}_n \mathbf{X}, \mathbf{X}x_1 \dots x_n \vdash \mathbf{M}x_1 \wedge \dots \wedge \mathbf{M}x_n$, we have by the lemma and the axioms for equality, $\mathbf{A}t, \mathbf{A}x_1, \dots, \mathbf{A}x_n, \mathbf{M}_n \mathbf{X}, \mathbf{Z}tx_1 \dots x_n \vdash \mathbf{M}t$. Then by (5) and the deduction theorem for \mathbf{P} , $\mathbf{M}_n \mathbf{X}, \mathbf{A}t, \mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \mathbf{Z}tx_1 \dots x_n \supset \mathbf{M}t$, and by $\vdash \mathbf{L}\mathbf{A}$ and the deduction theorem for \exists , $\mathbf{M}_n \mathbf{X}, \mathbf{A}t \vdash \exists \mathbf{A}[x_1] \dots \exists \mathbf{A}[x_n] \cdot \mathbf{Z}tx_1 \dots x_n \supset \mathbf{M}t$. Then by (5), $\mathbf{A}t \vdash \mathbf{H}(\mathbf{M}t)$ and by Theorem 30 of [2],

$$\mathbf{M}_n \mathbf{X}, \mathbf{A}t \vdash \mathbf{Y}t \supset \mathbf{M}t. \quad (7)$$

Therefore by (6) $\mathbf{M}_n \mathbf{X} \vdash \exists \mathbf{A}[t] \cdot \mathbf{Y}t \supset \mathbf{M}t$. Thus by (6) again,

$$\mathbf{M}_n \mathbf{X} \vdash \mathbf{M}_1 \mathbf{Y}. \quad (8)$$

Now $\mathbf{A}x_1, \dots, \mathbf{A}x_n, \mathbf{Y}\langle x_1, \dots, x_n \rangle \vdash \Sigma \mathbf{A}[u_1] \dots \Sigma \mathbf{A}[u_m] \cdot \mathbf{Q}'_1 \langle x_1, \dots, x_n \rangle$

$\langle u_1, \dots, u_n \rangle \wedge \mathbf{X}u_1 \dots u_n$. Also $\mathbf{A}x_1, \dots, \mathbf{A}x_n, \mathbf{A}u_1, \dots, \mathbf{A}u_n, \mathbf{Q}'_1 \langle x_1, \dots, x_n \rangle \langle u_1, \dots, u_n \rangle \vdash \mathbf{Q}'_1 x_1 u_1 \wedge \dots \wedge \mathbf{Q}'_1 x_n u_n$, so since we have the correct grammatical conditions as in the proof of (7),

$$\mathbf{A}x_1, \dots, \mathbf{A}x_n, \mathbf{A}u_1, \dots, \mathbf{A}u_n, \mathbf{Q}'_1 \langle x_1, \dots, x_n \rangle \langle u_1, \dots, u_n \rangle \wedge \mathbf{X}u_1 \dots u_n \vdash \mathbf{X}x_1 \dots x_n$$

and thus $\mathbf{A}x_1, \dots, \mathbf{A}x_n, \mathbf{A}u_1, \dots, \mathbf{A}u_n \vdash \mathbf{Q}'_1 \langle x_1, \dots, x_n \rangle \langle u_1, \dots, u_n \rangle \wedge \mathbf{X}u_1 \dots u_n \supset \mathbf{X}x_1 \dots x_n$. Therefore by Theorem 26 of [2]

$$\mathbf{A}x_1, \dots, \mathbf{A}x_n \vdash \Sigma \mathbf{A}[u_1] \dots \Sigma \mathbf{A}[u_n]. \mathbf{Q}'_1 \langle x_1, \dots, x_n \rangle \langle u_1, \dots, u_n \rangle \wedge \mathbf{X}u_1 \dots u_n \supset \mathbf{X}x_1 \dots x_n$$

and so

$$\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{Y} \langle x_1, \dots, x_n \rangle \supset \mathbf{X}x_1 \dots x_n. \tag{9}$$

Now $\mathbf{X}x_1 \dots x_n, \mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{Q}'_1 \langle x_1, \dots, x_n \rangle \langle x_1, \dots, x_n \rangle \wedge \mathbf{X}x_1 \dots x_n$, so $\mathbf{X}x_1 \dots x_n, \mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \Sigma \mathbf{A}[u_1] \dots \Sigma \mathbf{A}[u_n]. \mathbf{Q}'_1 \langle x_1, \dots, x_n \rangle \langle u_1, \dots, u_n \rangle \wedge \mathbf{X}x_1 \dots x_n$. Therefore $\mathbf{X}x_1 \dots x_n, \mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{Y} \langle x_1, \dots, x_n \rangle$, and by $\mathbf{M}_n \mathbf{X}, \mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{X}x_1 \dots x_n \supset \mathbf{Y} \langle x_1, \dots, x_n \rangle$. Thus by (9),

$$\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{X}x_1 \dots x_n \sim \mathbf{Y} \langle x_1, \dots, x_n \rangle.$$

Now we combine Theorems 5 and 6 to give us:

Theorem 7. *If $\phi(x_1, \dots, x_n)$ is a ppf containing no free variables other than at most x_1, \dots, x_n then there exists a class \mathbf{Y} such that $\mathbf{M}x_1, \dots, \mathbf{M}x_n \vdash \mathbf{Y} \langle x_1, \dots, x_n \rangle \sim \phi(x_1, \dots, x_n)$.*

This is the general existence theorem exactly as Gödel has it. If we apply the deduction theorem for \mathbf{P} to the formula in Theorem 7 several times we get:

$$\vdash \exists \mathbf{A}[x_1] \dots \exists \mathbf{A}[x_n]. \mathbf{M}x_1 \supset \dots \mathbf{M}x_n \supset \mathbf{Y} \langle x_1, \dots, x_n \rangle \sim \phi(x_1, \dots, x_n).$$

This theorem from which we derive all the axioms of the B group is also in the correct translated form. We therefore have proved the general theorem for the Gödel system.

Theorem 8. *If T is a theorem of the Gödel system of set theory and N is the set of grammatical conditions for T' , the translation of T into the combinatory system, then $N \vdash T'$.*

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