Notre Dame Journal of Formal Logic
Volume XIV, Number 4, October 1973
NDJFAM

# INFINITE SERIES OF T-REGRESSIVE ISOLS

JUDITH L. GERSTING

1 Introduction.* Let $E$ denote the collection of all non-negative integers (numbers), $\Lambda$ the collection of all isols, $\Lambda_R$ the collection of all regressive isols, and $\Lambda_{ZR}$ the collection of all cosimple regressive isols. Infinite series of regressive isols were defined by J. C. E. Dekker in [4]; A. Nerode in [14] associated with every recursive function $f(x)$ an extension of $f(x)$ to a mapping $D_f(X)$ on $\Lambda$. In [1], J. Barback showed that $D_f(X)$ for $f$ an increasing recursive function and $X \epsilon \Lambda_R$ can be represented as an infinite series. Universal isols were introduced by E. Ellentuck in [6].

The collection $\Lambda_{TR}$ of T-regressive isols was introduced in [8]. There a result was proved concerning an equality between infinite series of T-regressive isols; viewing the extension of a recursive combinatorial function to $\Lambda_R$ in terms of infinite series, this result led to a proof that T-regressive isols are universal. In the present paper, three further results are obtained concerning equalities and inequalities between infinite series of isols when T-regressive isols are involved. As applications of Theorem 1 below, we obtain new proofs of several previously known results concerning extensions of recursive functions to $\Lambda_R$. Theorem 3 below is used by M. Hassett in obtaining his main result of [10].

2 Preliminaries. We recall from [4] the definition of an infinite series of isols, $\sum_T a_n$, where T denotes an infinite regressive isol and $a_n$ denotes a function from $E$ into $E$:

$$\sum\nolimits_T a_n = \mathrm{Req} \sum_0^\infty j(t_n, \nu(a_n))$$

where $j(x, y)$ is a recursive function mapping $E^2$ one-to-one onto $E$, $t_n$ is any regressive function ranging over a set in T, and for any number $n$, $\nu(n) = \{x \mid x < n\}$. By results in [4], $\sum_T a_n$ is an isol and is independent of the choice of the regressive function whose range is in T. In [2], J. Barback studied infinite series of the form $\sum_T a_n$ where $T \leqslant^* a_{n-1}$. The relation

*The author wishes to express appreciation to Professor J. Barback for some very helpful suggestions concerning some of the topics presented here.

*Received May 5, 1972*

$\mathrm{T} \leqslant^* a_{n-1}$ for $\mathrm{T}$ an infinite regressive isol and $a_n$ a function from $E$ into $E$ implies that for every regressive function $t_n$ ranging over a set in $\mathrm{T}$, $t_n \leqslant^* a_{n\dot{-}1}$, that is, the mapping $t_n \to a_{n\dot{-}1}$ has a partial recursive extension. It was established in [2] that

(1) *if* $\mathrm{T} \leqslant^* a_n$, *then* $\mathrm{T} \leqslant^* a_{n-1}$

and

(2) *if* $\mathrm{T} \leqslant^* a_{n-1}$, *then* $\sum_{\mathrm{T}} a_n$ *is a regressive isol, where*

$$j(t_0, 0), \ldots, j(t_0, a_0 - 1), j(t_1, 0), \ldots, j(t_1, a_1 - 1), j(t_2, 0), \ldots,$$

*represents a regressive enumeration of a set belonging to* $\sum_{\mathrm{T}} a_n$.

For $f$ an increasing recursive function, the *e-difference* function of $f$, $e_f$, is defined by

$$e_f(0) = f(0)$$
$$e_f(n+1) = f(n+1) - f(n).$$

Since $f$ is increasing and recursive, $e_f$ is a recursive function, and it follows that for $\mathrm{T}$ an infinite regressive isol, $\mathrm{T} + 1 \leqslant^* e_f(n)$. The following result is Proposition 2 of [1]:

Lemma 1. *Let* $f(x)$ *be an increasing recursive function. Then for any infinite regressive isol* $\mathrm{T}$,

$$\mathrm{D}_f(\mathrm{T}) = \sum_{\mathrm{T}+1} e_f(n).$$

A property of numbers is said to hold *eventually* if there is an $n \in E$ such that $x$ has the property for every $x > n$. In [8] a retraceable function $a_n$ is called $\mathrm{T}$-*retraceable* if it has the property that for each partial recursive function $p(x)$, $p(a_n) < a_{n+1}$ eventually. An infinite regressive isol is $\mathrm{T}$-*regressive* if it contains a set which is the range of a $\mathrm{T}$-retraceable function. $\Lambda_{\mathrm{TR}}$ denotes the collection of all $\mathrm{T}$-regressive isols. It is known that cosimple $\mathrm{T}$-regressive isols exist and that if $\mathrm{T} \in \Lambda_{\mathrm{TR}}$, then $\mathrm{T} + 1 \in \Lambda_{\mathrm{TR}}$.

3 An Inequality Between Infinite Series. We use the following two lemmas, stated here without proof, in the proof of Theorem 1 below.

Lemma 2 (Corollary 1 of [8]). *Let* $\mathrm{T} \in \Lambda_{\mathrm{TR}}$ *and let* $a_n$ *and* $b_n$ *be any functions such that both* $\mathrm{T} \leqslant^* a_n$ *and* $\mathrm{T} \leqslant^* b_n$. *Then*

$$\sum_{\mathrm{T}} a_n = \sum_{\mathrm{T}} b_n \Longrightarrow a_n = b_n \textit{ eventually}.$$

Lemma 3 (Theorem 1 of [9]). *Let* $\mathrm{T} \in \Lambda_{\mathrm{R}} - E$ *and* $b_n$ *be a function such that* $\mathrm{T} \leqslant^* b_n$. *Let* $A$ *be an isol such that* $A \leqslant \sum_{\mathrm{T}} b_n$.

(Since $\sum_{\mathrm{T}} b_n \in \Lambda_{\mathrm{R}}$, it follows from results in [5] that $A \in \Lambda_{\mathrm{R}}$.) *Then there exists a function* $c_n$ *such that*

$$\mathrm{T} \leqslant^* c_n,$$
$$c_n \leqslant b_n \textit{ for all } n,$$
$$A = \sum_{\mathrm{T}} c_n.$$

Theorem 1. *Let* $\mathsf{T} \in \Lambda_{\mathsf{TR}}$ *and let* $a_n$ *and* $b_n$ *be functions such that both* $\mathsf{T} \leq^* a_n$ *and* $\mathsf{T} \leq^* b_n$. *Let*

$$\sum\nolimits_{\mathsf{T}} a_n \leq \sum\nolimits_{\mathsf{T}} b_n.$$

*Then* $a_n \leq b_n$ *eventually.*

*Proof:* Denote $\sum_{\mathsf{T}} a_n$ by $A$. Now using Lemma 3 there exists a function $c_n$ such that $\mathsf{T} \leq^* c_n$, $c_n \leq b_n$ for all $n$, and $A = \sum_{\mathsf{T}} c_n$. Thus $\sum_{\mathsf{T}} a_n = \sum_{\mathsf{T}} c_n$. By Lemma 2, we have

$$a_n = c_n \textit{ eventually}$$

and thus

$$a_n \leq b_n \textit{ eventually}.$$

Corollary 1. *Let* $\mathsf{T} \in \Lambda_{\mathsf{TR}}$ *and let* $f$ *and* $g$ *be increasing recursive functions. Let* $\mathsf{D}_f(\mathsf{T}) \leq \mathsf{D}_g(\mathsf{T})$. *Then* $f \leq g$ *eventually.*

*Proof:* Letting $e_f$ and $e_g$ denote the $e$-difference functions of $f$ and $g$ respectively, we have from Lemma 1 that

$$\mathsf{D}_f(\mathsf{T}) = \sum\nolimits_{\mathsf{T}+1} e_f(n) \qquad \mathsf{D}_g(\mathsf{T}) = \sum\nolimits_{\mathsf{T}+1} e_g(n)$$

and thus

(3) $\sum_{\mathsf{T}+1} e_f(n) \leq \sum_{\mathsf{T}+1} e_g(n)$.

Since $\mathsf{T}+1 \in \Lambda_{\mathsf{TR}}$ and $\mathsf{T}+1 \leq^* e_f(n)$, $\mathsf{T}+1 \leq^* e_g(n)$, it follows from the theorem that

(4) $e_f(n) \leq e_g(n)$ *eventually.*

It is then easy to see, using (3) and (4), that $f \leq g$ eventually.

We remark here that Corollary 1 has been shown by J. Barback to be true for $\mathsf{T}$ any universal regressive isol; however, it is the stronger result of Theorem 1 that is needed for the four applications below.

Theorem A (Barback, [1]). *Let* $f$ *be a recursive function such that* $\mathsf{D}_f(X)$ *maps* $\Lambda_{\mathsf{R}}$ *into* $\Lambda_{\mathsf{R}}$. *Then* $f$ *is eventually increasing.*

*Proof:* Let $f^+$ and $f^-$ denote recursive combinatorial functions such that $f(x) = f^+(x) - f^-(x)$ for all $x \in E$. Then $f^+$ and $f^-$ are increasing recursive functions; let $e_{f^+}$ and $e_{f^-}$ denote their respective $e$-difference functions. Let $\mathsf{T} \in \Lambda_{\mathsf{TR}}$. By Corollary 3 of [1],

$$\mathsf{D}_f(\mathsf{T}) = \sum\nolimits_{\mathsf{T}+1} e_{f^+}(n) - \sum\nolimits_{\mathsf{T}+1} e_{f^-}(n).$$

Since $\mathsf{D}_f(\mathsf{T})$ is a member of $\Lambda_{\mathsf{R}}$, it follows that

$$\sum\nolimits_{\mathsf{T}+1} e_{f^-}(n) \leq \sum\nolimits_{\mathsf{T}+1} e_{f^+}(n).$$

Now by Theorem 1 we have

$$e_{f^-}(n) \leq e_{f^+}(n) \textit{ eventually}$$

which implies

$e_{f^+}(n) - e_{f^-}(n) \geq 0$ *eventually*, $e_f(n) \geq 0$ *eventually*, *f is eventually increasing.*

The proof of Theorem B will be omitted; it follows that of Theorem A, with T taken to be a cosimple T-regressive isol.

Theorem B (Catlin, [3]). *Let f be a recursive function such that* $D_f(X)$ *maps* $\Lambda_{ZR}$ *into* $\Lambda_{ZR}$. *Then f is eventually increasing.*

Theorem C (Sansone [15]). *Let f be an increasing recursive function such that* $D_f(X)$ *is ultimately order-preserving on* $\Lambda_R$. *Then* $e_f$ *is eventually increasing.*

*Proof:* Let $T \in \Lambda_{TR}$. Then $T - 1 \leq T$, so that, since $D_f(X)$ is ultimately order-preserving,

$$D_f(T - 1) \leq D_f(T).$$

By Lemma 1,

$$\textstyle\sum_T e_f(n) \leq \sum_{T+1} e_f(n).$$

Let the recursive function $d_n$ be defined by

$$d(0) = 0,$$
$$d(n + 1) = e_f(n).$$

Then

$$\textstyle\sum_T e_f(n) = \sum_{T+1} d(n)$$

and thus

$$\textstyle\sum_{T+1} d(n) \leq \sum_{T+1} e_f(n).$$

Applying Theorem 1,

$$d(n) \leq e_f(n) \text{ eventually}$$

or

$$e_f(n - 1) \leq e_f(n) \text{ eventually}$$

which says that the function $e_f$ is eventually increasing.

Again by taking T to be a cosimple T-regressive isol, the proof of Theorem C yields the following result:

Theorem D. *Let f be an increasing recursive function such that* $D_f(X)$ *is ultimately order-preserving on* $\Lambda_{ZR}$. *Then* $e_f$ *is eventually increasing.*

We note here that the proofs of these four theorems actually yield stronger results than those stated. For example, in the proof of Theorem A, the hypothesis may be weakened to $f$ being a recursive function such that $D_f(T) \in \Lambda$ for some T-regressive isol T. Theorems B, C, and D may be similarly strengthened. These strengthened forms of the theorems may also be obtained by using the property that every T-regressive isol is *strongly universal* (see Ellentuck, [7]). We note also that in the cited references for Theorems A, B, and C the results given are both necessary and sufficient conditions, so it is only one direction of each of these results which is obtained here.

## 4 Two Equalities Between Infinite Series.

Theorem 2. *Let* $\mathrm{T}, \mathrm{S} \in \Lambda_{\mathrm{TR}}$ *and let* $a_n$ *and* $b_n$ *be functions such that* $1 \leqslant a_n$ *and* $1 \leqslant b_n$ *for all* $n \in E$, *and also* $\mathrm{T} \leqslant^* a_n$ *and* $\mathrm{S} \leqslant^* b_n$. *Let* $\sum_{\mathrm{T}} a_n = \sum_{\mathrm{S}} b_n$. *Then there exists a number* $m \in E$ *and an integer* $k \geqslant 1 - m$ *such that*

$$n \geqslant m \Longrightarrow a_n = b_{n+k}.$$

*Proof:* Let $t_n$ and $s_n$ be T-retraceable functions ranging over sets in T and S, respectively. By (2),

$$j(t_0, 0), \ldots, j(t_0, a_0 - 1), j(t_1, 0), \ldots, j(t_1, a_1 - 1), j(t_2, 0), \ldots,$$
$$j(s_0, 0), \ldots, j(s_0, b_0 - 1), j(s_1, 0), \ldots, j(s_1, b_1 - 1), j(s_2, 0), \ldots,$$

represent regressive enumerations of sets belonging to $\sum_{\mathrm{T}} a_n$ and $\sum_{\mathrm{S}} b_n$, respectively. Let $g_n$ and $\tilde{g}_n$ denote the respective regressive enumerations determined above. Since $\sum_{\mathrm{T}} a_n = \sum_{\mathrm{S}} b_n$, it follows from results in [5] that there exists a one-to-one partial recursive function $p(x)$ such that $(\forall n)[p(g_n) = \tilde{g}_n]$. Because $\mathrm{T} \leqslant^* a_n$ and $\mathrm{S} \leqslant^* b_n$, there will be partial recursive functions $f_a$ and $f_b$ such that $(\forall n)[f_a(t_n) = a_n - 1]$ and $(\forall n)[f_b(s_n) = b_n - 1]$. It follows that the mapping

$$q(x) = kp^{-1}j(kpj(x, f_a(x)), \; lpj(x, f_a(x)) + 1)$$

is a partial recursive function. Because $t_n$ is a T-retraceable function, there exists a number $\overline{n}$ such that for $n \geqslant \overline{n}$, $q(t_n) < t_{n+1}$. Consider a number $n \geqslant \overline{n}$ and let $pj(t_n, a_n - 1)$ be denoted by $j(s_x, y)$. If $y \neq b_x - 1$, then $q(t_n) = t_{n+1}$, which is a contradiction. Thus for every $n \geqslant \overline{n}$, $pj(t_n, a_n - 1)$ is a number of the form $j(s_x, b_x - 1)$. Because $s_n$ is a T-retraceable function, we can use a similar argument to prove that there exists a number $\overline{\overline{n}}$ such that for every $n \geqslant \overline{\overline{n}}$, $p^{-1}j(s_n, b_n - 1)$ is a number of the form $j(t_x, a_x - 1)$. Let $m$ be a number such that

$$m > \overline{n} \text{ and } (\forall n)(n \geqslant m \text{ and } pj(t_n, 0) = j(s_x, 0) \Longrightarrow x \geqslant \overline{\overline{n}}).$$

Thus for $n \geqslant m$, the "blocks" of length $a_n$ in the enumeration $g_n$ will be mapped by $p$ into "blocks" of length $b_{n+k}$ in the enumeration $\tilde{g}_n$, where $k \geqslant 1 - m$ since $a_m = b_{m+k}$ with $m + k \geqslant 1$. This completes the proof.

Corollary 2.1. *Let* $\mathrm{T}, \mathrm{S} \in \Lambda_{\mathrm{TR}}$ *and let* $f$ *and* $g$ *be strictly increasing recursive functions. Let* $\mathrm{D}_f(\mathrm{T}) = \mathrm{D}_g(\mathrm{S})$. *Then there exists a number* $m \in E$ *and an integer* $k \geqslant 1 - m$ *such that*

$$n \geqslant m \Longrightarrow e_f(n) = e_g(n + k),$$

i.e., *the rate of growth of* $f$ *and* $g$ *is "parallel."*

*Proof:* By Lemma 1,

$$\mathrm{D}_f(\mathrm{T}) = \mathrm{D}_g(\mathrm{S}) \Rightarrow \textstyle\sum_{\mathrm{T}+1} e_f(n) = \sum_{\mathrm{S}+1} e_g(n).$$

By Theorem 2, the result holds.

Corollary 2.2. *Let* $\mathrm{T}, \mathrm{S} \in \Lambda_{\mathrm{TR}}$ *and let* $f$ *and* $g$ *be strictly increasing recursive functions. Let*

(5) $\mathrm{D}_f(\mathrm{T}) = \mathrm{D}_g(\mathrm{S})$.

*Then there exists a number* $u \in E$ *such that* $\mathrm{T} = \mathrm{S} \pm u$.

*Proof:* From Corollary 2.1 there exists a number $m \in E$ and an integer $k$ such that

$$n \geq m \Rightarrow e_f(n) = e_g(n+k)$$

or

$$n \geq m \Rightarrow f(n) - f(n-1) = g(n+k) - g(n+k-1)$$

from which

$$n \geq m \Rightarrow f(n) = g(n+k) + \overline{m},\ \overline{m} \text{ an integer}$$

or

$$(\forall n)(f(n+m) = g(n+m+k) + \overline{m}).$$

Thus for any $A \in \Lambda$ we have

$$\mathrm{D}_{f(n+m)}(A) = \mathrm{D}_{g(n+m+k)+\overline{m}}(A)$$

which implies (by a result of A. Nerode)

$$\mathrm{D}_f(A+m) = \mathrm{D}_g(A+m+k) + \overline{m}.$$

In particular,

$$\mathrm{D}_f(\mathrm{T}) = \mathrm{D}_f(\mathrm{T} - m + m) = \mathrm{D}_g(\mathrm{T} - m + m + k) + \overline{m} = \mathrm{D}_g(\mathrm{T} + k) + \overline{m}.$$

Using (5),

(6) $\mathrm{D}_g(\mathrm{S}) = \mathrm{D}_g(\mathrm{T}+k) + \overline{m}$.

By writing the extension mappings as infinite series and using a proof similar to that of Theorem 2, it is not difficult to show that for $h$ a strictly increasing recursive function, $A, B \in \Lambda_{\mathrm{TR}}$, and $p$ some number $\geq 1$, we have

$$\mathrm{D}_h(A) = \mathrm{D}_h(B) + p \Rightarrow A = B + q \text{ for some } q \in E,\ q \geq 1.$$

It also becomes clear here that $e_h$ is eventually a cyclic function of period $q$. Applying this to (6) we obtain the desired result; in addition, if $\overline{m} \neq 0$, we see that $e_g$ (and hence $e_f$) is eventually cyclic.

Theorem 3. *Let* $\mathrm{T} \in \Lambda_{\mathrm{TR}}$, $\mathrm{S} \in \Lambda_{\mathrm{R}} - E$, *and let* $a_n$ *and* $b_n$ *be functions such that* $1 \leq a_n$ *and* $1 \leq b_n$ *for all* $n \in E$, *and also* $\mathrm{T} \leq^* a_n$, $\mathrm{S} \leq^* b_{n-1}$. *Let* $\sum_{\mathrm{T}} a_n = \sum_{\mathrm{S}} b_n$. *Then there exists a number* $k \in E$ *and a strictly increasing function* $h(n)$ *such that*

$$\sum_{i=0}^{k} a_i = \sum_{i=0}^{h(0)} b_i$$

*and*

$$a_{k+n+1} = \sum_{i=h(n)+1}^{h(n+1)} b_i \textit{ for all } n \in E.$$

*Proof:* Let $t_n$ be a T-retraceable function ranging over a set in T and $s_n$ a regressive function ranging over a set in S. By (2),

$$j(t_0, 0), \ldots, j(t_0, a_0 - 1), j(t_1, 0), \ldots, j(t_1, a_1 - 1), j(t_2, 0), \ldots,$$
$$j(s_0, 0), \ldots, j(s_0, b_0 - 1), j(s_1, 0), \ldots, j(s_1, b_1 - 1), j(s_2, 0), \ldots,$$

represent regressive enumerations of sets belonging to $\sum_{\mathrm{T}} a_n$ and $\sum_{\mathrm{S}} b_n$, respectively. Let $g_n$ and $\tilde{g}_n$ denote the respective regressive enumerations determined above, and, since $\sum_{\mathrm{T}} a_n = \sum_{\mathrm{S}} b_n$, let $p(x)$ be the one-to-one partial recursive function such that $(\forall n)(p(g_n) = \tilde{g}_n)$. An argument similar to that in the proof of Theorem 2 proves the existence of a number $k$ such that for every $n \geq k$, $pj(t_n, a_n - 1)$ is a number of the form $j(s_x, b_x - 1)$. Then for every $n \geq k + 1$, every "$a$-block" in the enumeration $g_n$ will be mapped by $p$ into the sum of a number of "$b$-blocks" in the enumeration $\tilde{g}_n$. This completes the proof.

Corollary 3.1. *Let* $\mathrm{T} \in \Lambda_{\mathrm{TR}}$, $\mathrm{S} \in \Lambda_{\mathrm{R}} - E$, *and let* $f$ *and* $g$ *be strictly increasing recursive functions. Let* $\mathrm{D}_f(\mathrm{T}) = \mathrm{D}_g(\mathrm{S})$. *Then there exists a number* $k \in E$ *and a strictly increasing recursive function* $h(n)$ *such that*

$$f(n + k) = g(h(n)) \textit{ for all } n \in E,$$

i.e., *$f$ eventually takes on only values of $g$.*

*Proof:* The result follows at once from the Theorem by applying Lemma 1.

Corollary 3.2. *Let* $\mathrm{T} \in \Lambda_{\mathrm{TR}}$, $\mathrm{S} \in \Lambda_{\mathrm{R}} - E$, *and let* $f$ *and* $g$ *be strictly increasing recursive functions. Let* $\mathrm{D}_f(\mathrm{T}) = \mathrm{D}_g(\mathrm{S})$. *Then there exists a number* $k \in E$ *and a strictly increasing recursive function* $h(n)$ *such that*

$$\mathrm{S} = \mathrm{D}_h(\mathrm{T} - k).$$

*Proof:* By Corollary 3.1, there exists a number $k \in E$ and a strictly increasing recursive function $h(n)$ such that

$$(\forall n)\,[f(n + k) = g(h(n))].$$

Thus

$$\mathrm{D}_g(\mathrm{S}) = \mathrm{D}_f(\mathrm{T}) = \mathrm{D}_{f(n+k)}(\mathrm{T} - k) = \mathrm{D}_{g(h(n))}(\mathrm{T} - k) = \mathrm{D}_g(\mathrm{D}_h(\mathrm{T} - k)).$$

Since $h$ is a strictly increasing recursive function, by results in [1], $\mathrm{D}_h(\mathrm{T} - k) \in \Lambda_{\mathrm{R}}$. Also, by a result of A. Nerode, if $g$ is a strictly increasing recursive function, then $\mathrm{D}_g$ is one-to-one on $\Lambda_{\mathrm{R}}$ and hence

$$\mathrm{D}_g(\mathrm{S}) = \mathrm{D}_g(\mathrm{D}_h(\mathrm{T} - k)) \Longrightarrow \mathrm{S} = \mathrm{D}_h(\mathrm{T} - k).$$

This completes the proof.

## REFERENCES

[1] Barback, J., "Recursive functions and regressive isols," *Mathematica Scandinavica*, vol. 15 (1964), pp. 29-42.

[2] Barback, J., "Regressive upper bounds," *Seminario Matematico*, vol. 39 (1967), pp. 248-272.

[3] Catlin, S., "Some theorems concerning regressive sets and regressive isols of order greater than or equal to one," Ph.D. Thesis, Arizona State University (1969).

[4] Dekker, J. C. E., "Infinite series of isols," in *Recursive Function Theory*, American Mathematical Society Proceedings of Symposia in Pure Mathematics, vol. 5 (1962), pp. 77-96.

[5] Dekker, J. C. E., "The minimum of two regressive isols," *Mathematische Zeitschrift*, vol. 83 (1964), pp. 345-366.

[6] Ellentuck, E., "Universal isols," *Mathematische Zeitschrift*, vol. 98 (1967), pp. 1-8.

[7] Ellentuck, E., "On the degrees of universal regressive isols," to appear.

[8] Gersting, J., "A rate of growth criterion for universality of regressive isols," *Pacific Journal of Mathematics*, vol. 31 (1969), pp. 669-677.

[9] Gersting, J., "A note on infinite series of isols," *Rocky Mountain Journal of Mathematics*, vol. 2 (1972), pp. 661-666.

[10] Hassett, M., "Extension pathology in regressive isols," to appear.

[11] McLaughlin, T. G., "Hereditarily retraceable isols," *Bulletin of the American Mathematical Society*, vol. 73 (1967), pp. 113-115.

[12] McLaughlin, T. G., "Thinning the branches of a semicomputable tree," Special Session on Recursion Theory, AMS Regional Meeting, Milwaukee, November, 1971.

[13] Myhill, J., "Recursive equivalence types and combinatorial functions," *Bulletin of the American Mathematical Society*, vol. 64 (1958), pp. 373-376.

[14] Nerode, A., "Extensions to isols," *Annals of Mathematics*, vol. 73 (1961), pp. 362-403.

[15] Sansone, F. J., "On order-preserving extensions to regressive isols," *Michigan Mathematical Journal*, vol. 13 (1966), pp. 353-355.

*Purdue University at Indianapolis*
*Indianapolis, Indiana*