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SOME RESULTS CONCERNING FINITE MODELS FOR SENTENTIAL CALCULI

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Terminology and notation. Let $S_{\aleph_0}$ be the set of wffs built up in the usual way from denumerably many *letters* $p_1, p_2, \ldots$ and finitely many *connectives* $F_1, \ldots, F_n$ (each F_i a k_i-place connective for some positive integer k_i): letters are wffs, and $F_i\alpha_1 \ldots \alpha_{k_i}$ is a wff if $\alpha_1, \ldots, \alpha_{k_i}$ are wffs. A *rule of inference* is an s-tuple of wffs; and a set of wffs T is *closed under* a rule of inference $\langle\beta_1, \ldots, \beta_{s-1}, \beta_s\rangle$ just in case $\gamma_s \epsilon T$ whenever $\gamma_1, \ldots, \gamma_{s-1}, \gamma_s$ result from $\beta_1, \ldots, \beta_{s-1}, \beta_s$, respectively, by a uniform substitution of wffs for letters, and $\gamma_1, \ldots, \gamma_{s-1} \epsilon T$.

$\mathsf{P} = \langle T, A, R_1, \ldots, R_r\rangle$ is a *sentential calculus* if and only if A, the set of *axioms* of P, is a set of wffs, $R_1, \ldots, R_r$ are rules of inference, and T, the set of *theorems* of P, is the least set containing A and closed under substitution and each of $R_1, \ldots, R_r$. (Where $r = 0$, T is simply the set of substitution instances of members of A.) For each such P define an equivalence relation, $\cong_P$, on $S_{\aleph_0}$ by letting $\alpha \cong_P \beta$ just in case replacement of zero or more occurrences of α by β in each wff in T (respectively, not in T) results in a wff in T (respectively, not in T). For $\alpha \epsilon S \subset S_{\aleph_0}$, let $[\alpha] \cong_{P|S}$ be the set of β's in S such that $\alpha \cong_P \beta$ and let $S/\cong_P$ be the set of $[\alpha] \cong_{P|S}$'s such that $\alpha \epsilon S$.

$\mathfrak{M} = \langle V, D, f_1, \ldots, f_n\rangle$ is a *matrix* if and only if V is a non-empty set, $D \subset V$, and each f_i is a k_i-ary operation in V. A function h: $S_{\aleph_0} \rightarrow V$ is a *value function of* $\mathfrak{M}$ just in case $h(F_i\alpha_1 \ldots \alpha_{k_i}) = f_i(h(\alpha_1), \ldots, h(\alpha_{k_i}))$ for all $\alpha_1, \ldots, \alpha_{k_i} \epsilon S_{\aleph_0}$, and α is an $\mathfrak{M}$-*tautology* just in case $h(\alpha) \epsilon D$ for every value function h of $\mathfrak{M}$. We denote the set of $\mathfrak{M}$-tautologies by '$\mathrm{E}(\mathfrak{M})$'. Where $\mathfrak{M} = \langle V, D, f_1, \ldots, f_n\rangle$ and $\mathfrak{M}' = \langle V', D', f_1', \ldots, f_n'\rangle$ are matrices the matrix $\mathfrak{M} \times \mathfrak{M}' = \langle V \times V', D \times D', f_1^X, \ldots, f_n^X\rangle$, where $f_i^X(\langle v_1, v_1'\rangle, \ldots, \langle v_{k_i}, v_{k_i}'\rangle) = \langle f_i(v_1, \ldots, v_{k_i}), f_i'(v_1', \ldots, v_{k_i}')\rangle$, is called the *product* of $\mathfrak{M}$ and $\mathfrak{M}'$. Evidently (cf. [5]), $\mathrm{E}(\mathfrak{M} \times \mathfrak{M}') = \mathrm{E}(\mathfrak{M}) \cap \mathrm{E}(\mathfrak{M}')$.

The matrix $\mathfrak{M} = \langle V, D, f_1, \ldots, f_n\rangle$ is a *model* of the sentential calculus $\mathsf{P} = \langle T, A, R_1, \ldots, R_r\rangle$ if $T \subset \mathrm{E}(\mathfrak{M})$ and for each value function h of $\mathfrak{M}$ and each rule $\langle\beta_1, \ldots, \beta_{s-1}, \beta_s\rangle$ of P, if $h(\beta_1), \ldots, h(\beta_{s-1}) \epsilon D$ then $h(\beta_s) \epsilon D$. If $\mathfrak{M}$ is a model of P with $\mathrm{E}(\mathfrak{M}) = T$, we call $\mathfrak{M}$ a *characteristic* matrix for P.

For each set of letters L we let S_L be the set of wffs in which the only

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letters occurring are those in L; following Lindenbaum and Łoś (cf. [7]) we let $\mathsf{Ld}^L(\mathsf{P})$ be the matrix $\langle S_L/\cong_P, (S_L \cap T)/\cong_P, f_1^L, \ldots, f_n^L\rangle$, where $f_i^L([\alpha_1]\cong_{P|S_L}, \ldots, [\alpha_{k_i}]\cong_{P|S_L}) = [F_i\alpha_1 \ldots \alpha_{k_i}]\cong_{P|S_L}$ for all $\alpha_1, \ldots, \alpha_{k_i} \in S_L$.

A general result concerning the finite model property, with three applications. Generalizing theorem 13 of [7] and a remark following theorem 19 in an obvious way, we obtain:

Lemma 1. $\mathsf{Ld}^L(\mathsf{P})$ *is a model of* P, *for each sentential calculus* P.

For the proof, assume first that $\alpha \notin \mathrm{E}(\mathsf{Ld}^L(\mathsf{P}))$. Then there exists a value function h of $\mathsf{Ld}^L(\mathsf{P})$ such that $h(\alpha) \in S_L/\cong_P - (S_L \cap T)/\cong_P$. Pick $\gamma_1 \in h(p_1), \ldots, \gamma_k \in h(p_k)$, where the letters in α are among $p_1, \ldots, p_k$, and let α^* result from α by substitution of γ_1 for $p_1, \ldots,$ and γ_k for p_k, throughout. It follows (induce on the length of α) that $h(\alpha) = [\alpha^*]\cong_{P|S_L}$. Then $\alpha^* \notin T$; and since T is closed under substitution, $\alpha \notin T$.

Now let $\langle\beta_1, \ldots, \beta_{s-1}, \beta_s\rangle$ be a rule of P and h a value function of $\mathsf{Ld}^L(\mathsf{P})$ with $h(\beta_1), \ldots, h(\beta_{s-1}) \in (S_L \cap T)/\cong_P$. As before, pick $\gamma_1 \in h(p_1), \ldots, \gamma_k \in h(p_k)$, where the letters in $\beta_1, \ldots, \beta_{s-1}, \beta_s$ are among $p_1, \ldots, p_k$, and let $\beta_1^*, \ldots, \beta_{s-1}^*, \beta_s^*$ result, respectively, from $\beta_1, \ldots, \beta_{s-1}, \beta_s$ by a uniform substitution of γ_1 for $p_1, \ldots,$ and γ_k for p_k. Since $\beta_1^* \in h(\beta_1), \ldots, \beta_{s-1}^* \in h(\beta_{s-1})$ and $h(\beta_1), \ldots, h(\beta_{s-1}) \in (S_L \cap T)/\cong_P$, it follows that $\beta_1^*, \ldots, \beta_{s-1}^* \in T$. Since T is closed under the rule in question, then, $\beta_s^* \in T$. But $\beta_s^* \in h(\beta_s)$, so $h(\beta_s) \in (S_L \cap T)/\cong_P$ and our proof is complete.

A sentential calculus P is said to have the *finite model property* just in case each non-theorem of P can be rejected by a finite model of P. If $S_L/\cong_P$ is finite for each finite set of letters L, it will follow from lemma 1, and the observation (cf. theorem 14 of [7] or the proof of theorem 1 below) that $\mathrm{E}(\mathsf{Ld}^L(\mathsf{P})) \cap S_L = T \cap S_L$ for each such L, that P has the finite model property. For a number of calculi, however, a somewhat better result can be obtained. Let us call $\mathsf{P}' = \langle T', A', R_1, \ldots, R_r, \ldots, R_{r+t}\rangle$ an *extension* of the sentential calculus $\mathsf{P} = \langle T, A, R_1, \ldots, R_r\rangle$ if P' is a sentential calculus with $T \subset T'$; and let's call P *standard* if there exist wffs $\phi_1(p_1, p_2), \ldots, \phi_m(p_1, p_2)$ such that for all wffs α and β, and for each extension P' of P, $\alpha \cong_{P'} \beta$ if and only if $\phi_1(\alpha, \beta), \ldots, \phi_m(\alpha, \beta) \in T'$. Then we have:

Theorem 1. *If* P *is a standard sentential calculus with* $S_L/\cong_P$ *finite for each finite set of letters L then every extension of* P *has the finite model property.*[1]

For the proof let $\mathsf{P} = \langle T, A, R_1, \ldots, R_r\rangle$ satisfy the hypothesis of the

1. Theorem 1, its three corollaries, and lemma 2 were announced in [14]. All but the third corollary were obtained, along with lemma 3, in the author's 1967 Wayne State University doctoral dissertation, *Matrices for sentential calculi*. The author would like to thank J. Michael Dunn for our many conversations concerning these matters; he would like also to thank Jerzy Łoś for his [7], to which this paper seems but a series of footnotes.

theorem, let $\mathbf{P}' = \langle T', A', R_1, \ldots, R_r, \ldots, R_{r+t}\rangle$ be any extension of $\mathbf{P}$, and assume $\alpha \notin T'$. Then $\alpha \notin S_L \cap T'$, where L is the set of letters occurring in α. Let h be a value function of $\mathbf{Ld}^L(\mathbf{P}')$ with $h(p_i) = [p_i]{\cong_{P'|S_L}}$ for each letter $p_i \in L$. It follows by a straightforward induction on the length of α that $h(\alpha) = [\alpha]{\cong_{P'|S_L}}$. But $[\alpha]{\cong_{P'|S_L}} \in S_L/{\cong_{P'}} - (S_L \cap T')/{\cong_{P'}}$, so $\alpha \notin \mathrm{E}(\mathbf{Ld}^L(\mathbf{P}'))$. Since $\mathbf{Ld}^L(\mathbf{P}')$ is a model of $\mathbf{P}'$ by lemma 1, we have only to show that it is finite. If not, there must exist an infinite sequence of wffs $\alpha_1, \alpha_2, \ldots \in S_L$ such that $\alpha_i \cong_{P'} \alpha_j$ only if $i = j$. But $S_L/{\cong_P}$ is finite since L is, so there exist distinct i and j such that $\alpha_i \cong_P \alpha_j$. Since $\mathbf{P}$ is standard, $\phi_1(\alpha_i, \alpha_j), \ldots, \phi_m(\alpha_i, \alpha_j) \in T \subset T'$ for the appropriate $\phi_1(\alpha_i, \alpha_j), \ldots, \phi_m(\alpha_i, \alpha_j)$, whence $\alpha_i \cong_{P'} \alpha_j$ and we are done.

Scroggs [13] and McKay [9] argue along similar lines in connection with their work on the special cases of S5 and certain proper fragments of the intuitionistic sentential calculus. Our general theorem gives us such additional results as:

Corollary 1. *Let* $\mathbf{LC}_X$ *be any (not necessarily proper) fragment of* **LC** *which includes the implicational fragment of* **LC**. *Then all extensions of* $\mathbf{LC}_X$ *have the finite model property.*

Corollary 2. *Let* $\mathbf{RM}_X$ *be any (not necessarily proper) fragment of* **R**-*Mingle which includes the implicational fragment of* **R**-*Mingle. Then all extensions of* $\mathbf{RM}_X$ *have the finite model property.*

Proofs that these calculi are standard are straightforward with $m = 2$, $\phi_1(p_1, p_2) = Cp_1p_2$ and $\phi_2(p_1, p_2) = Cp_2p_1$. That $S_L/{\cong_{\mathbf{LC}_X}}$ is finite for each finite set of letters L was originally established by Dummett [2] and the corresponding result for $\mathbf{RM}_X$ is due to Meyer (cf. [11]).[2]

Corollary 3. *Let* $\mathrm{E5}_X$ *be any (not necessarily proper) fragment of* E5 *which includes the implicational fragment of* E5. *Then all extensions of* $\mathrm{E5}_X$ *have the finite model property.*

We sketch the proof for E5, drawing on Lemmon's work in [6] on his systems E5 and E; obvious modifications extend the result to appropriate fragments of E5.

E5 is standard since $\vdash_{\mathrm{E5+}} C\alpha\beta, C\beta\alpha$ if and only if $\alpha \cong_{\mathrm{E5+}} \beta$, for each extension E5+ of E5. With theorem 1 we have only to show that $S_L/{\cong_{\mathrm{E5}}}$ is finite for each finite set of letters L, so let L be any such set and let $\alpha_1, \alpha_2, \ldots$ be any infinite sequence of wffs in S_L. According to [6] and [13], respectively, $S_L/{\cong_{\mathrm{E}}}$ and $S_L/{\cong_{\mathrm{S5}}}$ are finite, so $\mathbf{Ld}^L(\mathrm{E})$ and $\mathbf{Ld}^L(\mathrm{S5})$ are finite. Then the product of these two matrices, $\mathbf{Ld}^L(\mathrm{E}) \times \mathbf{Ld}^L(\mathrm{S5})$, is also finite. There must then exist distinct i and j such that $h(\alpha_i) = h(\alpha_j)$ for each value function h of the product matrix. Then $C\alpha_i\alpha_j$ and $C\alpha_j\alpha_i$ are tautologies of this matrix and so of $\mathbf{Ld}^L(\mathrm{E})$ and of $\mathbf{Ld}^L(\mathrm{S5})$ as well. So $\vdash_{\mathrm{E}} C\alpha_i\alpha_j$, $C\alpha_j\alpha_i$ and

2. The results for **LC** and **R**-Mingle, though not for their fragments, have been improved by Dunn [3]: all extensions of these two calculi have finite characteristic matrices.

$\vdash_{S5} C\alpha_i\alpha_j$, $C\alpha_j\alpha_i$. But (cf. [6]) the set of theorems of E5 is the intersection of the set of theorems of E with the set of theorems of S5, so $\vdash_{E5} C\alpha_i\alpha_j$ and $\vdash_{E5} C\alpha_j\alpha_i$. Since E5 is standard, then, $\alpha_i \cong_{E5} \alpha_j$.

An undecidable sentential calculus with the finite model property. The significance of the corollaries obtained above derives in part from a general result of Harrop's [4] with which they provide solutions to the decision problems for all finitely axiomatizable extensions of **LC**, **R**-Mingle, E5 and various fragments of these calculi:

(H) *Every finitely axiomatizable sentential calculus with the finite model property is decidable.*[3]

Harrop's proof of (H) makes important use of the assumption of finite axiomatizability, but he leaves open the question whether this assumption can be dropped, or at least weakened. Would it do, for example, to require only that the calculus in question have a recursively enumerable set of axioms, or that it be recursively axiomatizable? To show that none of these weakenings are possible, we first establish a lemma of independent interest:

Lemma 2. *Let* $\mathsf{P} = \langle T, A, R_1, \ldots, R_r \rangle$ *be a sentential calculus. Then* P *has a finite characteristic matrix if and only if* (i) $S_L/\cong_{\mathsf{P}}$ *is finite for each finite set of letters L and* (ii) *there exists a finite set of letters M such that for each wff* α, $\alpha \in T$ *if every substitution instance of* α *in* S_M *is in* T.

The necessity of the two conditions is well known, and obvious in any case. To see that they are jointly sufficient, assume that P satisfies both of them. Then $\mathsf{Ld}^M(\mathsf{P})$ is finite, by condition (i), and a model of P, by lemma 1, so it suffices to show that $\mathrm{E}(\mathsf{Ld}^M(\mathsf{P})) \subset T$. But if $\alpha \in \mathrm{E}(\mathsf{Ld}^M(\mathsf{P}))$ then evidently each substitution instance of α in S_M is in $\mathrm{E}(\mathsf{Ld}^M(\mathsf{P}))$ and hence in T; by condition (ii), then, $\alpha \in T$.

The two conditions are, incidentally, independent. The calculus P^* of [4], which has no finite models at all except the trivial ones of which all wffs are tautologies, has at least one Post-consistent, Post-complete extension, P^{**}. But every Post-complete calculus satisfies condition (ii) with $M = \{p_1\}$; so P^{**} cannot satisfy condition (i) else $\mathsf{Ld}^M(\mathsf{P}^{**})$ would be a finite model of P^*.[4] Dummett's **LC**, on the other hand, satisfies (i) but has no finite characteristic matrix [2] and so fails to satisfy condition (ii).

Lemma 3. *Let U be any set of wffs with maximum length m and let* $T = \bigcup\{V\colon V \cap U = \Lambda$ *and V is closed under substitution*$\}$. *Then* $\mathsf{P} = \langle T, T\rangle$ *is a sentential calculus with a finite characteristic matrix.*

3. The result appears to have been known, though perhaps not in its full generality, to McKinsey (cf. [10]) and Łoś (cf. [7], especially pp. 19-20).

4. McCall and Nat have recently asked ([8], p. 214) whether or not there exists a Post-complete C-N-K system with no finite characteristic matrix. P^{**} of course provides an affirmative answer to this question.

T is clearly closed under substitution, so **P** is a sentential calculus. To see that **P** satisfies condition (i) of lemma 2, suppose to the contrary that for some finite set of letters, L, S_L contains infinitely many distinct wffs α_1, α_2, . . . such that $\alpha_i \cong_P \alpha_j$ only if $i = j$. Notice that $\alpha \cong_P \beta$ if the lengths of α and β each exceed m, since in that case no $\gamma \epsilon U$ can be a substitution instance of any wff in which either α or β occurs. Then there must exist, among $\alpha_1, \alpha_2, \ldots,$ infinitely many distinct wffs whose lengths do not exceed m. Since L is finite, though, there can be only finitely many such wffs.

Finally, if every substitution instance of α involving at most the letters $p_1, \ldots, p_m$ is in T, then evidently no $\gamma \epsilon U$ can be a substitution instance of α, so $\alpha \epsilon T$. **P**, then, must satisfy condition (ii) of lemma 2, completing our proof.

If we restrict the membership of U to a single wff we get theorem 16 of [7], recently rediscovered by Pahi and Applebee [12], as a special case. Either result may be used to establish:

Theorem 2. *There exist recursively axiomatizable, undecidable sentential calculi with the finite model property.*

Let $A_0 = Cp_1p_1$ and $A_{i+1} = Cp_1A_i$, let J be an r.e. but not recursive set of natural numbers, let $A = \{A_j : j \epsilon J\}$ and let **P** be the sentential calculus with no rules of inference (in our sense) whose set of axioms is A. Then the set of theorems of **P** is the set of substitution instances of members of A, and since no two members of A have a substitution instance in common **P** is a sentential calculus with a recursively enumerable set of axioms which is (recursively) undecidable. By lemma 3, **P** has the finite model property; and from Craig's [1] it follows that **P** is in fact recursively axiomatizable.

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